M/G/1 and Priority Queueing

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Outline

- PASTA
- $M/G/1$ Workload and FIFO Delay
- Pollaczek-Khinchine Formula
- $M/G/1$ Priority Queueing Delay
- $M/G/1$ Conservation Law
Steady State: A Revisit

- Steady state distribution $\pi = (\pi_0, \pi_1 \ldots)$

- $\pi_i$ is the limiting probability that the steady-state $X \overset{\text{def}}{=} \lim_{t \to \infty} X(t)$ is in state $i$:

  $$\pi_i = P\{X = i\} = \lim_{t \to \infty} P\{X(t) = i\}$$

- $\pi_i$ is also the limiting proportion of time that the system is in state $i$:

  $$\pi_i = \lim_{t \to \infty} \frac{1}{t} \int_0^t 1_{\{X(s)=i\}} \, ds$$
System State Seen by Arrivals

- A point process: $\psi = \{t_n: n \geq 0\}$, where $t_i$ is the arrival time of the $i$th customer
- $\{X(t_n^-): n \geq 0\}$ defines the system states seen by the customers upon their arrivals
- Denote $\pi_i' = \lim_{n \to \infty} P\{X(t_n^-) = i\}$ or $\pi_i' = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n} 1_{\{X(t_j^-) = i\}}$ Does $\pi_i' = \pi_i$ hold?
- In a D/D/1 system with $T = 4$, and $S = 3$:
  - $\pi_0 = 1/4$ and $\pi_0' = 1$
**PASTA**

- Poisson Arrivals See Time Averages

**Explanation:**
- If arrivals are Poisson, then the proportion of time a queueing system spends in a given state ($\pi_i$) is equal to the proportion ($\pi_i'$) of arrivals who find the system in that state.

**Notation**
- State process: $X = \{X(t): t \geq 0\}$
- Poisson point process: $\psi = \{t_n: n \geq 0\}$ at rate $\lambda$
  - with counting process $\{N(t): t \geq 0\}$
Assumption: Lack of Anticipation (LAA)

For any $t \geq 0$, the future (Poisson) increments $\{N(t+s) - N(t): s \geq 0\}$ are independent of the joint past $\{\{X(u): u \leq t\}, \{N(u): u \leq t\}\}$.

Theorem: Any stochastic process $\{X(t): t \geq 0\}$ (with cadlag sample paths) jointly with a Poisson point process $\{t_n: n \geq 0\}$ that satisfies LAA, and let $f$ be any real bounded function. Then with probability 1,

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t f(X(s)) ds = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^n f(X(t_j^-))$$
Intuition/Informal “Proof”

- Let \( \pi_i(t) \overset{\text{def}}{=} P\{X(t) = i\} \) and define an arrival event \( E_a(t, dt) \overset{\text{def}}{=} \{N(t + dt) - N(t) > 0\} \)

- The corresponding \( \pi_i'(t) \) can be defined as
  \[
  \pi_i'(t) \overset{\text{def}}{=} \lim_{dt \to 0} P\{X(t) = i \mid E_a(t, dt)\}
  \]

\[
\Rightarrow \pi_i'(t) = \lim_{dt \to 0} \frac{P\{X(t) = i, E_a(t, dt)\}}{P\{E_a(t, dt)\}}
\]

\[
= \lim_{dt \to 0} \frac{P\{E_a(t, dt) \mid X(t) = i\}P\{X(t) = i\}}{P\{E_a(t, dt)\}}
\]

- \( P\{E_a(t, dt) \mid X(t) = i\} = P\{E_a(t, dt)\} \) (LAA)

\[
\Rightarrow \pi_i'(t) = P\{X(t) = i\} = \pi_i(t)
\]
M/G/1 Workload & FIFO Delay

Take $X(t) = V(t)$, check LAA & use PASTA

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t 1_{\{V(s) \leq x\}} ds = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^n 1_{\{V(t^-_j) \leq x\}}$$

$$\lim_{t \to \infty} \frac{1}{t} \int_0^t P(V(s) \leq x) ds = \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^n P(D(t^-_j) \leq x)$$

$$\Rightarrow P\{V \leq x\} = P\{D \leq x\}$$

$$\Rightarrow V \cong D \Rightarrow E[V] = E[D] \overset{\text{def}}{=} d_F \ (\text{FIFO delay})$$

What is the value of $E[V]$?
Brumelle’s Formula: A Revisit

- With FIFO service discipline
  
  \[ E[V] = \lambda E[SD] + \frac{\lambda}{2} E[S^2] \]

- Let \( \rho = \frac{\lambda}{E[S]} \), with G/G/1 mean workload
  
  - Service time is independent of arrivals
    
    \[ E[V] = \lambda E[S]E[D] + \frac{\lambda}{2} E[S^2] = \rho d_F + \rho E[R] \]
  
  - Avg. work in queue + avg. work in service

- M/G/1 mean workload and FIFO delay
  
  \[ E[V] = d_F = \rho d_F + \rho E[R] \text{ (by PASTA)} \]
M/G/1 Workload & FIFO Delay

- M/G/1 mean FIFO delay has a closed-form

\[ d_F = \rho d_F + \rho E[R] \Rightarrow d_F = \frac{\rho}{1 - \rho} E[R] = \frac{\lambda E[S^2]}{2(1 - \rho)} \]

- So as the workload under any work-conserving service discipline

\[ E[V] = d_F = \frac{\rho E[R]}{1 - \rho} = \frac{\lambda E[S^2]}{2(1 - \rho)} \]

- Sanity check for M/M/1, \( E[V] = \frac{\rho}{\mu(1-\rho)} \).
Pollaczek-Khinchine Mean Formula

- For an M/G/1 system with FIFO:
  - Mean Queueing Delay (P-K mean formula)
    \[ E[D] = \frac{\rho E[R]}{1 - \rho} = \frac{\lambda E[S^2]}{2(1 - \rho)} \]
  - Mean Sojourn Time
    \[ E[W] = E[D] + E[S] \]
  - Mean Customers in Queue
    \[ E[Q] = \lambda E[D] \]
  - Mean Customers in System
    \[ E[L] = \lambda E[W] = \lambda (E[D] + E[S]) = E[Q] + \rho \]
*Pollaczek-Khinchine Formula*

\[ D \cong \sum_{i=1}^{G} R_i \]

- \( \{R_i\} \) are i.i.d. distributed as \( R \)
- \( G \) is geometric, independent of \( \{R_i\} \), with
  \[ P\{G = i\} = (1 - \rho)\rho^i \]
- If \( G = 0 \), then \( D = 0 \), which shows
  \[ P\{D = 0\} = P\{G = 0\} = 1 - \rho \]
- \( E[D] = E[\sum_{i=1}^{G} R_i] = E[G]E[R] = \frac{\rho}{1-\rho}E[R] \)
- Can be derived by RCL & Laplace transform
M/G/1 Priority Queueing

Settings:
- $I$ service classes, indexed by $i = 1, \ldots, I$
- Lower index $\rightarrow$ Higher priority
- Define $\lambda_i$ as the Poisson arrival rate, $S_i$ as the service time of a class $i$ job

Assumptions:
- Work-serving
- Non-preemptive or preemptive
- FIFO within each service class
Priority Queueing: Illustration

Non-preemptive:
- Arrive at $t = 1$
- Arrive at $t = 3$
- Arrive at $t = 5$

Preempt-resume:
- Arrive at $t = 1$
- Arrive at $t = 3$
- Arrive at $t = 5$

$S = 5$

$S = 3$

$S = 4$
Definitions of Some Entities

- Define $i$th service class's utilization as
  \[ \rho_i \overset{\text{def}}{=} \lambda_i E[S_i] \]
  - Mean arrival rate of workload from class $i$

- System arrival rate and mean service time
  \[
  \lambda \overset{\text{def}}{=} \sum_{i=1}^{I} \lambda_i, \quad \bar{S} \overset{\text{def}}{=} \sum_{i=1}^{I} (\lambda_i/\lambda)E[S_i].
  \]

- System utilization
  \[
  \rho \overset{\text{def}}{=} \lambda \bar{S} = \sum_{i=1}^{I} \rho_i \Rightarrow P[\text{server idle}] = 1 - \rho
  \]
Mean Queueing Delay

First, consider a non-preemptive system

- \( E[D_i] \overset{\text{def}}{=} E[\text{waiting time for jobs from class } i] \)
- \( E[W_i] \overset{\text{def}}{=} E[\text{class } i \text{ sojourn time}] = E[D_i] + E[S_i] \)

Average waiting time \( E[D_i] = \overline{V_R} + \overline{D_i^1} + \overline{D_i^2} \)

- \( \overline{V_R} \): mean residual workload in the server
- \( \overline{D_i^1} \): delay due to jobs in the queue upon arrival
- \( \overline{D_i^2} \): delay due to jobs who arrive afterwards
Mean Residual Workload $\overline{V_R}$

- Apply PASTA again

$$\overline{V_R} = \sum_{j=1}^{I} \rho_j E[R_j] = \sum_{j=1}^{I} \rho_j \frac{E[S_j^2]}{2E[S_j]} = \sum_{j=1}^{I} \frac{\lambda_j E[S_j^2]}{2}$$

where, $E[R_j] = \frac{E[S_j^2]}{2E[S_j]}$ is the mean residual time

- Under non-preemptive system, $\overline{V_R}$ is independent of priority classes.
For Single-Class M/G/1

- Average workload in queue is
  \[ E \left[ \sum_{i=1}^{Q} S_i \right] = E[Q]E[S] = \lambda E[D]E[S] = \rho E[D] \]
  - by Wald’s equation
  - \( E[Q] = \lambda E[D] \) (by Little’s law)

- Compare with Brumelle’s formula:
  \[ E[V] = \lambda E[S]E[D] + \frac{\lambda}{2} E[S^2] = \rho d_F + \rho E[R] \]
\( D_1^1 \) and \( D_1^2 \)

- \( D_1^1 \) can also be considered as
  - average workload in queue upon arrival of a class \( i \) customer that needs to be finished before serving the customer

\[
D_1^1 = \sum_{j=1}^{i} \mathbb{E}[S_j] \mathbb{E}[Q_j] = \sum_{j=1}^{i} \mathbb{E}[S_j] \lambda_j \mathbb{E}[D_j] = \sum_{j=1}^{i} \rho_j \mathbb{E}[D_j]
\]

\[
D_1^2 = \sum_{j=1}^{i-1} \mathbb{E}[S_j] \mathbb{E}[M_j] = \sum_{j=1}^{i-1} \mathbb{E}[S_j] \lambda_j \mathbb{E}[D_i] = \sum_{j=1}^{i-1} \rho_j \mathbb{E}[D_i]
\]
Triangular Equations

\[
E[D_i] = \overline{V}_R + \sum_{j=1}^{i} \rho_j E[D_j] + \sum_{j=1}^{i-1} \rho_j E[D_i]
\]

\[
= \frac{\overline{V}_R + \sum_{j=1}^{i} \rho_j E[D_j]}{1 - \sum_{j=1}^{i} \rho_j}
\]

Compute starting from \( i = 1 \)

\[
E[D_i] = \frac{\overline{V}_R}{(1 - \sum_{j=1}^{i} \rho_j)(1 - \sum_{j=1}^{i-1} \rho_j)}
\]
Priority Class $i$

- **Mean Queueing Delay**
  \[ E[D_i] = \frac{\bar{V}_R}{(1 - \sum_{j=1}^{i} \rho_j)(1 - \sum_{j=1}^{i-1} \rho_j)} \]

- **Mean Sojourn Time**
  \[ E[W_i] = E[D_i] + E[S_i] \]

- **Mean Customers in Queue**
  \[ E[Q_i] = \lambda_i E[D_i] \]

- **Mean Customers in System**
  \[ E[L_i] = E[Q_i] + \rho_i \]
How about Preemptive-Resume?

- Without preemption:
  \[
  E[D_i] = \frac{V_R}{(1 - \sum_{j=1}^{i} \rho_j)(1 - \sum_{j=1}^{i-1} \rho_j)} \\
  E[W_i] = E[D_i] + E[S_i]
  \]

- With preemption:
  \[
  E[\bar{D}_i] = \frac{\sum_{j=1}^{i} \rho_j E[R_j]}{(1 - \sum_{j=1}^{i} \rho_j)(1 - \sum_{j=1}^{i-1} \rho_j)} \\
  E[\bar{W}_i] = E[\bar{D}_i] + E[\bar{S}_i] = E[\bar{D}_i] + \frac{E[S_i]}{1 - \sum_{j=1}^{i-1} \rho_j}
  \]
Service Time with Preemption

- Consider a customer of class 2, starting service at time $t_0$, during service time, it might be preempted by class 1 customers.

- Time until it finishes is $\tilde{S}_2 \overset{\text{def}}{=} \sum_{j=1}^{\infty} S_2^{(j)}$
Deriving $E\left[ S_{i}^{(j)} \right]$ 

- For any class $i$ customer, $\tilde{S}_{i} \overset{\text{def}}{=} \sum_{j=1}^{\infty} S_{i}^{(j)}$
- Condition on the length of the previous cycle

$$
E \left[ S_{i}^{(j+1)} \right] = \int_{x=0}^{\infty} E \left[ S_{i}^{(j+1)} \mid S_{i}^{(j)} = x \right] f_{j}(x) \, dx
$$

$$
= \int_{x=0}^{\infty} \left( \lambda_{1} x E[S_{1}] + \cdots + \lambda_{i-1} x E[S_{i-1}] \right) f_{j}(x) \, dx
$$

$$
= \int_{x=0}^{\infty} \left( \rho_{1} + \cdots + \rho_{i-1} \right) x f_{j}(x) \, dx
$$

$$
= (\rho_{1} + \cdots + \rho_{i-1}) E \left[ S_{i}^{(j)} \right]
$$
Deriving $E[\widetilde{S}_i]$

- $E \left[ S_i^{(j+1)} \right] = (\rho_1 + \cdots + \rho_{i-1})^j E \left[ S_i^{(1)} \right]$

- Let $\gamma \overset{\text{def}}{=} \rho_1 + \cdots + \rho_{i-1}$

$$E[\widetilde{S}_i] = \sum_{j=1}^{\infty} E \left[ S_i^{(j)} \right] = (1 + \gamma + \gamma^2 + \cdots) E \left[ S_i^{(1)} \right]$$

$$= (1 + \gamma + \gamma^2 + \cdots) E[S_i] = \frac{1}{1 - \gamma} E[S_i]$$

$$\Rightarrow E[\widetilde{S}_i] = \frac{E[S_i]}{1 - \sum_{j=1}^{i-1} \rho_j}$$
Conservation Law

- “You don’t get something for nothing.”
  - Shorter delay for higher class \( \rightarrow \) longer delay for lower class

- What is unchanged/conserved?  workload
  - In busy periods, independent of order of service
  - Require no work is created/destroyed in system
  - No user leaves the system before finishing
  - No idle time when facing non-empty queue
  - Only consider work-conserving service disciplines
M/G/1 Conservation Law

- Under M/G/1, if the service discipline is
  1. non-preemptive and work conserving
  2. independent of the service times

  then the following must hold:

  \[
  \sum_{j=1}^{l} \rho_j E[D_j] = \begin{cases} \frac{\rho}{1 - \rho} \bar{V}_R & \rho < 1 \\ \infty & \rho \geq 1 \end{cases}
  \]

- Weighted sum of the queueing delay $E[D_j]$ can NEVER CHANGE regardless how sophisticated the service discipline is.
Conservation Law: Derivation

- Avg. workload in queue (or $D_I^1$) is conserved

$$E[V] - \overline{V_R} = \sum_{j=1}^{l} E[Q_j]E[S_j] = \sum_{j=1}^{l} \lambda_j E[D_j]E[S_j] = \sum_{j=1}^{l} \rho_j E[D_j]$$

$\rho_j E[D_j]$ is the mean workload from class $j$ in queue.

- From the P-K mean formula for $M/G/1$

$$E[V] = d_F = \frac{\lambda E[S^2]}{2(1 - \rho)} = \frac{\overline{V_R}}{1 - \rho}$$

because $E[S^2] = \sum_{j=1}^{l} \frac{\lambda_j}{\lambda} E[S_j^2] = \sum_{j=1}^{l} \frac{2 \lambda_j E[S_j^2]}{\lambda} = \frac{2}{\lambda} \overline{V_R}$
Class-homogenous Service Time

- When $E[S_j] = E[S]$,

$$\sum_{j=1}^{l} \rho_j E[D_j] = \frac{\rho}{1 - \rho} \overline{V_R} \Rightarrow \sum_{j=1}^{l} \lambda_j E[D_j] = \frac{\lambda}{1 - \rho} \overline{V_R}$$

- The average # of customers is a constant.

- By Little’s Law, the average queueing delay (“averaged” over all classes) is a constant:

$$\sum_{j=1}^{l} (\lambda_j/\lambda) E[D_j] = \frac{1}{1 - \rho} \overline{V_R}$$
Workload in Queue Conserved

Workload in queue:

$S = 3$
$S = 4$

M/G/1 FIFO
Workload in Queue Conserved

Workload in queue:

\[ S = 3 \]
\[ S = 4 \]
*G/G/1 Conservation Law*

- If select customers in a way that is independent of their service time, then
  - distribution of the number of customers in the system will be invariant of the order of service.
  - Avg. queueing delay is also invariant. (by LL)

- In particular, for non-preemptive systems:

\[
\sum_{j=1}^{I} \rho_j E[D_j] = E[V] - \overline{V_R} = E[V] - \sum_{j=1}^{I} \frac{\lambda_j E[S_j^2]}{2}
\]
Service-Time Dependency

- The mean delay with FIFO is a tight lower bound for work conserving and service time independent service disciplines.

- Service time dependent scheduling:
  - SPT: shortest processing time first
  - SRPT: shortest remaining processing time first
    \[ E[D]_{FIFO} \geq E[D]_{SPT} \geq E[D]_{SRPT} \]
  - However, uncommon in packet switching because the packet ordering will be modified and delay for large packets increases.
References
