02—Traditional Logic I

The Importance of Being Formal

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The Importance of Being Formal

First Agenda

Find out *in detail* how formal systems work

Goal

Thorough understanding of formal logic as an example *par excellence* for formal methods

Approach

Study a series of logics: traditional, propositional, predicate logic

The Importance of Being Formal

Second Agenda

Explore fundamental boundaries of formal reasoning

Goal

Appreciate Undecidability and Gödel's incompleteness results

Approach

Study predicate logic deep enough to understand his formal arguments

The Importance of Being Formal

Third Agenda

Explore formal methods across fields

Approach

Students write essays and present their findings

Goal

Overview of formal methods and their limitations in our civilization

Hallmarks of Formal Methods

- Discreteness
- Naming
- Abstraction (classification)
- Reification
- Self-reference
- **Form vs content**
- **Syntax vs semantics**

Traditional Logic

Origins

Greek philosopher Aristotle (384–322 BCE) wrote treatise *Prior Analytics*; considered the earliest study in formal logic; widely accepted as the definite approach to deductive reasoning until the 19$^{\text{th}}$century.

Goal

Formalize relationships between sets; allow reasoning about set membership

Example 1

All humans are mortal.
All Greeks are humans.
Therefore, all Greeks are mortal.

Makes “sense”, right?

Why?

Example 2

All cats are predators.
Some animals are cats.
Therefore, all animals are predators.

Does not make sense!

Why not?

Example 3

All slack track systems are caterpillar systems.
All Christie suspension systems are slack track systems.
Therefore, all Christie suspension systems are caterpillar systems.

Makes sense, even if you do not know anything about suspension systems.

Form, not content

In logic, we are interested in the form of valid arguments, irrespective of any particular domain of discourse.

Categorical Terms

Terms refer to sets

Term `animals` refers to the set of animals,
term `brave` refers to the set of brave persons, etc

`Term`

The set `Term` contains all terms under consideration

Examples

$$\texttt{animals} \in \texttt{Term}$$

$$\texttt{brave} \in \texttt{Term}$$

Models

Meaning

A model $\mathcal{M}$ fixes what elements we are interested in, and what we mean by each term

Fix universe

For a particular $\mathcal{M}$, the universe $U^{\mathcal{M}}$ contains all elements that we are interested in.

Meaning of terms

For a particular $\mathcal{M}$ and a particular term t, the meaning of t in $\mathcal{M}$, denoted $t^{\mathcal{M}}$, is a particular subset of $U^{\mathcal{M}}$.

Example 1A

For our examples, we have
$\texttt{Term} = \{\texttt{cats}, \texttt{humans}, \texttt{Greeks}, \ldots\}$.

First meaning $\mathcal{M}$

- $U^{\mathcal{M}}$: the set of all living beings,
- $\texttt{cat}^{\mathcal{M}}$ the set of all cats,
- $\texttt{humans}^{\mathcal{M}}$ the set of all humans,
- $\ldots$

Example 1B

Consider the same $\texttt{Term} = \{\texttt{cats}, \texttt{humans}, \texttt{Greeks}, \ldots\}$.

Second meaning $\mathcal{M}'$

- $U^{\mathcal{M}'}$: A set of 100 playing cards, *depicting* living beings,
- $\texttt{cat}^{\mathcal{M}'}$: all cards that show cats,
- $\texttt{humans}^{\mathcal{M}'}$: all cards that show humans,
- $\ldots$

Example 2A

Consider the following set of terms:

$\texttt{Term} = \{\texttt{even}, \texttt{odd}, \texttt{belowfour}\}$

First meaning $\mathcal{M}_1$

- $U^{\mathcal{M}_1} = \{0, 1, 2, 3, \ldots\}$,
- $\texttt{even}^{\mathcal{M}_1} = \{0, 2, 4, \ldots\}$,
- $\texttt{odd}^{\mathcal{M}_1} = \{1, 3, 5, \ldots\}$, and
- $\texttt{belowfour}^{\mathcal{M}_1} = \{0, 1, 2, 3\}$.

Example 2B

Consider the same $\texttt{Term} = \{\texttt{even}, \texttt{odd}, \texttt{belowfour}\}$

Second meaning $\mathcal{M}_2$

- $U^{\mathcal{M}_2} = \{a, b, c, \ldots, z\}$,
- $\texttt{even}^{\mathcal{M}_2} = \{a, e, i, o, u\}$,
- $\texttt{odd}^{\mathcal{M}_2} = \{b, c, d, \ldots\}$, and
- $\texttt{belowfour}^{\mathcal{M}_2} = \emptyset$.

Categorical Propositions

`All cats are predators`

expresses a relationship between the terms `cats` (subject) and `predators` (object).

Intended *meaning*

Every *thing* that is included in the class represented by `cats` is also included in the class represented by `predators`.

Four Kinds of Categorical Propositions

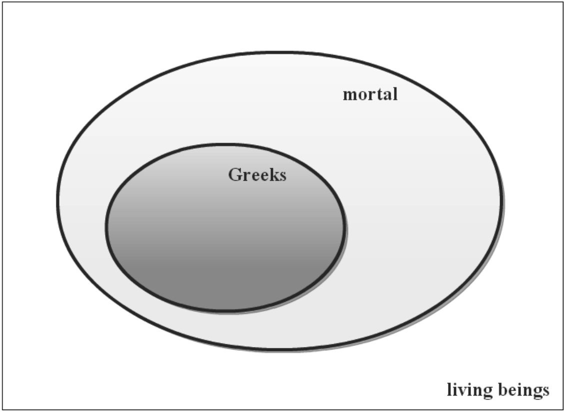
		Quantity	
		universal	particular
Quality	affirmative	`All` t_1 `are` t_2	`Some` t_1 `are` t_2
	negative	`No` t_1 `are` t_2	`Some` t_1 `are not` t_2

Example

`Some cats are not brave` is a *particular*, *negative* proposition.

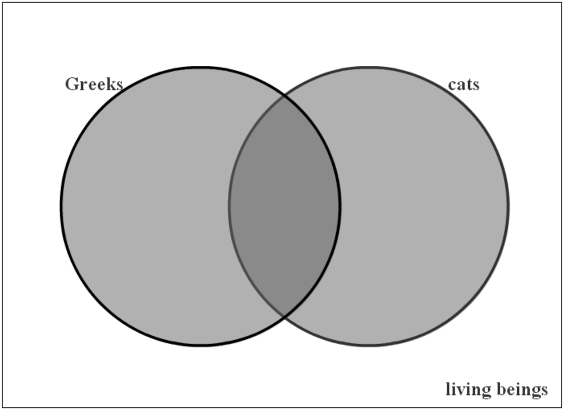
Meaning of Universal Affirmative Propositions

In a particular model $\mathcal{M}$, `All Greeks are mortal` means that $\texttt{Greeks}^{\mathcal{M}}$ is a subset of $\texttt{mortal}^{\mathcal{M}}$



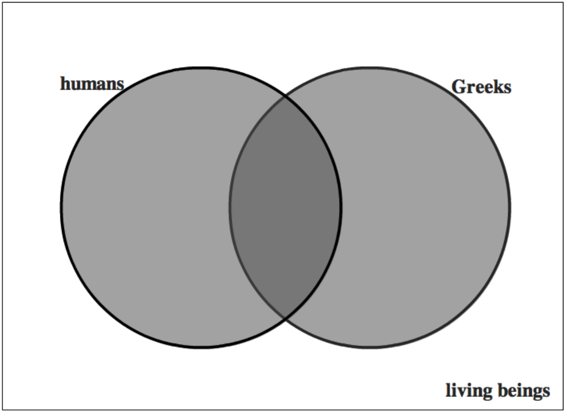
Meaning of Universal Negative Propositions

In a particular model $\mathcal{M}$, `No Greeks are cats` means that the intersection of $\mathtt{Greeks}^{\mathcal{M}}$ and $\mathtt{cats}^{\mathcal{M}}$ is empty.



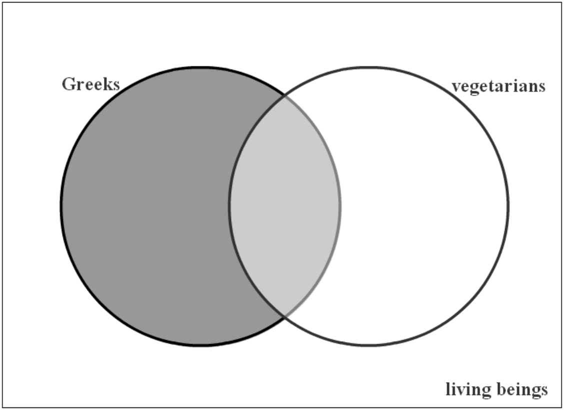
Meaning of Particular Affirmative Propositions

In a particular model $\mathcal{M}$, `Some humans are Greeks` means that the intersection of $\texttt{humans}^{\mathcal{M}}$ and $\texttt{Greeks}^{\mathcal{M}}$ is not empty.



Meaning of Particular Negative Propositions

In model $\mathcal{M}$, `Some Greeks are not vegetarians` means the difference of $\texttt{Greeks}^{\mathcal{M}}$ and $\texttt{vegetarians}^{\mathcal{M}}$ is not empty.



Axioms

Axioms are propositions that are assumed to hold.

Axiom (HM)

The proposition `All humans are mortal` *holds.*

Axiom (GH)

The proposition `All Greeks are humans` *holds.*

Graphical Notation

$$\overline{}\ [\text{HumansMortality}]$$
$$\texttt{All humans are mortal}$$

Lemmas

Lemmas are affirmations that follow from all known facts.

Proof obligation

A lemma must be followed by a proof that demonstrates how it follows from known facts.

Trivial Example of Proof

Lemma

The proposition `All humans are mortal` *holds.*

Proof.

$$\frac{}{\texttt{All humans are mortal}}\,[\mathrm{HM}]$$

□

Unusual Models

We can choose any model for our terms, also “unusual” ones.

Example

$U^{\mathcal{M}} = \{0, 1\}$, $\texttt{humans}^{\mathcal{M}} = \{0\}$, $\texttt{mortal}^{\mathcal{M}} = \{1\}$

Here

`All humans are mortal`

does not hold.

Asserting Axioms

Purpose of axioms

By asserting an axiom A, we are focusing our attention to only those models $\mathcal{M}$ for which $A^{\mathcal{M}} = T$.

Consequence

The lemmas that we prove while utilizing an axiom only hold in the models in which the axiom holds.

Validity

A proposition is called *valid*, if it holds in all models.

Complement

We allow ourselves to put non in front of a term.

Meaning of complement

In a model $\mathcal{M}$, the meaning of $\mathrm{non}\ t$ is the complement of the meaning of t

More formally

In a model $\mathcal{M}$, $(\mathrm{non}\ t)^{\mathcal{M}} = U^{\mathcal{M}}/t^{\mathcal{M}}$

Double Complement

Axiom (NonNon)

For any term t, the term $\texttt{non}\ \ \texttt{non}\ t$ *is considered equal to t.*

$$\frac{\cdots t\ \cdots}{\cdots \texttt{non}\ \ \texttt{non}\ t \cdots}\,[\text{NNI}]$$

$$\frac{\cdots \texttt{non}\ \ \texttt{non}\ t\ \cdots}{\cdots t \cdots}\,[\text{NNE}]$$

Rule Schema

$$\frac{\cdots t \cdots}{\cdots \texttt{non non } t \cdots}\text{[NNI]}$$

is a rule schema. An instance is:

$$\frac{\texttt{Some } t_1 \texttt{ are } t_2}{\texttt{Some non non } t_1 \texttt{ are } t_2}$$

Definitions

We allow ourselves to state definitions that may be convenient. Definitions are similar to axioms; they fix the properties of a particular item for the purpose of a discussion.

Definition (ImmDef)

The term `immortal` is considered equal to the term `non mortal`.

Writing a Proof Graphically

Lemma

The proposition `All humans are non immortal` *holds.*

Proof.

$$\dfrac{\dfrac{\dfrac{}{\texttt{All humans are mortal}}\,[\mathrm{HM}]}{\texttt{All humans are non non mortal}}\,[\mathrm{NNI}]}{\texttt{All humans are non immortal}}\,[\mathrm{ImmDef}]$$

Writing a Text-based Proof

Lemma

The proposition `All humans are non immortal` *holds.*

Proof.

1	`All humans are mortal`	HM
2	`All humans are non non mortal`	NNI 1
3	`All humans are non immortal`	ImmDef 2

□

Conversion switches subject and object

Definition (ConvDef)

For all terms t_1 and t_2, we define

$$\begin{aligned}
\texttt{convert}(\texttt{All } t_1 \texttt{ are } t_2) &= \texttt{All } t_2 \texttt{ are } t_1 \\
\texttt{convert}(\texttt{Some } t_1 \texttt{ are } t_2) &= \texttt{Some } t_2 \texttt{ are } t_1 \\
\texttt{convert}(\texttt{No } t_1 \texttt{ are } t_2) &= \texttt{No } t_2 \texttt{ are } t_1 \\
\texttt{convert}(\texttt{Some } t_1 \texttt{ are not } t_2) &= \texttt{Some } t_2 \texttt{ are not } t_1
\end{aligned}$$

Which Conversions Hold?

If

```
All Greeks are humans
```

holds in a model, then does

```
All humans are Greeks
```

hold?

Valid Conversions

Axiom (ConvE1)

If, for some terms t_1 and t_2, the proposition

$$\texttt{convert}(\texttt{Some } t_1 \texttt{ are } t_2)$$

holds, then the proposition

$$\texttt{Some } t_1 \texttt{ are } t_2$$

also holds.

Valid Conversions

Axiom (ConvE2)

If, for some terms t_1 and t_2, the proposition

$$\texttt{convert}(\texttt{No}\ t_1\ \texttt{are}\ t_2)$$

holds, then the proposition

$$\texttt{No}\ t_1\ \texttt{are}\ t_2$$

also holds.

In Graphical Notation

In graphical notation, two rules correspond to the two cases.

$$\frac{\mathit{convert}(\texttt{Some } t_1 \texttt{ are } t_2)}{\texttt{Some } t_1 \texttt{ are } t_2}\,[\mathrm{ConvE}_1]$$

$$\frac{\mathit{convert}(\texttt{No } t_1 \texttt{ are } t_2)}{\texttt{No } t_1 \texttt{ are } t_2}\,[\mathrm{ConvE}_2]$$

Example

Axiom (AC)

The proposition `Some animals are cats` *holds.*

Lemma

The proposition `Some cats are animals` *holds.*

Proof

$$\dfrac{\dfrac{\dfrac{}{\texttt{Some animals are cats}}\,[\mathrm{AC}]}{\texttt{convert(Some cats are animals)}}\,[\mathrm{ConvDef}]}{\texttt{Some cats are animals}}\,[\mathrm{ConvE}_1]$$

Example (text-based proof)

Proof.

1	`Some animals are cats`	AC
2	`convert(Some cats are animals)`	ConvDef 1
3	`Some cats are animals`	ConvE_1 2

□

Contraposition switches and complements

Definition (ContrDef)

For all terms t_1 and t_2, we define

$$
\begin{aligned}
& \texttt{contrapose}(\texttt{All } t_1 \texttt{ are } t_2) \\
= \; & \texttt{All non } t_2 \texttt{ are non } t_1 \\
& \texttt{contrapose}(\texttt{Some } t_1 \texttt{ are } t_2) \\
= \; & \texttt{Some non } t_2 \texttt{ are non } t_1 \\
& \texttt{contrapose}(\texttt{No } t_1 \texttt{ are } t_2) \\
= \; & \texttt{No non } t_2 \texttt{ are non } t_1 \\
& \texttt{contrapose}(\texttt{Some } t_1 \texttt{ are not } t_2) \\
= \; & \texttt{Some non } t_2 \texttt{ are not non } t_1
\end{aligned}
$$

For which propositions is contraposition valid?

$$\frac{\textit{contrapose}(\texttt{All } t_1 \texttt{ are } t_2)}{\texttt{All } t_1 \texttt{ are } t_2}\,[\mathrm{ContrE}_1]$$

$$\frac{\textit{contrapose}(\texttt{Some } t_1 \texttt{ are not } t_2)}{\texttt{Some } t_1 \texttt{ are not } t_2}\,[\mathrm{ContrE}_2]$$

Obversion switches quality and complements object

Definition (ObvDef)

For all terms t_1 and t_2, we define

$$
\begin{aligned}
\texttt{obvert}(\texttt{All } t_1 \texttt{ are } t_2) &= \texttt{No } t_1 \texttt{ are non } t_2 \\
\texttt{obvert}(\texttt{Some } t_1 \texttt{ are } t_2) &= \texttt{Some } t_1 \texttt{ are not non } t_2 \\
\texttt{obvert}(\texttt{No } t_1 \texttt{ are } t_2) &= \texttt{All } t_1 \texttt{ are non } t_2 \\
\texttt{obvert}(\texttt{Some } t_1 \texttt{ are not } t_2) &= \texttt{Some } t_1 \texttt{ are non } t_2
\end{aligned}
$$

Examples

Obversion switches quality and complements object

Example 1

```
obvert(All Greeks are humans)
 = No Greeks are non humans
```

Example 2

```
obvert(Some animals are cats)
 = Some animals are not non cats
```

Validity of Obversion

Obversion is valid for all kinds of propositions.

Axiom (ObvE)

If, for some proposition p

$$\mathtt{obvert}(p)$$

holds, then the proposition p also holds.

$$\frac{obvert(p)}{p}\;[\mathrm{ObvE}]$$

Example

Axiom (SHV)

The proposition `Some humans are vegans` *holds.*

Lemma (NNVeg)

The proposition `Some humans are not non vegans` *holds.*

Proof

[SHV]

`Some humans are vegans`

[NNI]

`Some humans are non non vegans`

[ObvDef]

`obvert(Some humans are not non vegans)`

[

`Some humans are not non vegans`

Proof (text-based)

Proof.

1	`Some humans are vegans`	SHV
2	`Some humans are non non vegans`	NNI 1
3	`obvert(Some humans are not non vegans)`	ObvDef 2
4	`Some humans are not non vegans`	ObvE 3

□

Another Lemma

Lemma (SomeNon)

For all terms t_1 and t_2, if the proposition `Some non` t_1 `are non` t_2 *holds, then the proposition* `Some non` t_2 `are not` t_1 *also holds.*

A lemma of the form “If p_1 then p_2” is valid, if in every model in which the proposition p_1 holds, the proposition p_2 also holds.

Proof

Lemma (SomeNon)

For all terms t_1 and t_2, if the proposition `Some non` t_1 `are non` t_2 *holds, then the proposition* `Some non` t_2 `are not` t_1 *also holds.*

Proof.

1	`Some non` t_1 `are non` t_2	premise
2	`convert(Some non` t_2 `are non` t_1`)`	ConvDef 1
3	`Some non` t_2 `are non` t_1	ConvE_1 2
4	`obvert(Some non` t_2 `are not` t_1`)`	ObvDef 3
5	`Some non` t_2 `are not` t_1	ObvE 4

□

“iff” means “if and only if”

Lemma (AllNonNon)

For any terms t_1 and t_2, the proposition `All non` t_1 `are non` t_2 *holds iff the proposition* `All` t_2 `are` t_1 *holds.*

$$\frac{\texttt{All non } t_1 \texttt{ are non } t_2}{\texttt{All } t_2 \texttt{ are } t_1}$$

$$\frac{\texttt{All } t_2 \texttt{ are } t_1}{\texttt{All non } t_1 \texttt{ are non } t_2}$$