## 03—Propositional Logic II

### The Importance of Being Formal

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### Review: Syntax and Semantics of Propositional Logic

- 2 Natural Deduction
- 3 Soundness and Completeness

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Propositional Atoms and Propositions Semantics of Formulas Validity, Satisfiability, Truth Tables

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### Review: Syntax and Semantics of Propositional Logic

- Propositional Atoms and Propositions
- Semantics of Formulas
- Validity, Satisfiability, Truth Tables

### 2 Natural Deduction

3 Soundness and Completeness

Review: Syntax and Semantics of Propositional Logic

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### Atoms

### Anything goes

We allow any kind of proposition, for example "The sun is shining today".

**Review: Syntax and Semantics of Propositional Logic** 

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## Atoms

### Anything goes

We allow any kind of proposition, for example "The sun is shining today".

### Convention

We usually use p, q,  $p_1$ , etc, instead of sentences like "The sun is shining today".

**Review: Syntax and Semantics of Propositional Logic** 

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## Atoms

### Anything goes

We allow any kind of proposition, for example "The sun is shining today".

### Convention

We usually use p, q,  $p_1$ , etc, instead of sentences like "The sun is shining today".

#### Atoms

More formally, we fix a set *A* of propositional atoms.

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## Meaning of Atoms

### Models assign truth values

A model assigns truth values (F or T) to each atom.

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# Meaning of Atoms

#### Models assign truth values

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### More formally

A model for a propositional logic for the set A of atoms is a mapping from A to  $\{T, F\}$ .

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# Meaning of Atoms

#### Models assign truth values

A model assigns truth values (F or T) to each atom.

### More formally

A model for a propositional logic for the set A of atoms is a mapping from A to  $\{T, F\}$ .

#### How do you call them?

Models for propositional logic are called valuations.

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# Syntax of Propositions

### Definition

For a given set A of propositional atoms, the set of *well-formed* formulas in propositional logic is the smallest set  $\mathcal{F}$  that fulfills the following rules:

- The constant symbols  $\perp$  and  $\top$  are in  $\mathcal{F}$ .
- Every element of A is in  $\mathcal{F}$ .
- If  $\phi$  is in  $\mathcal{F}$ , then  $(\neg \phi)$  is also in  $\mathcal{F}$ .
- If  $\phi$  and  $\psi$  are in  $\mathcal{F}$ , then  $(\phi \land \psi)$  is also in  $\mathcal{F}$ .
- If  $\phi$  and  $\psi$  are in  $\mathcal{F}$ , then  $(\phi \lor \psi)$  is also in  $\mathcal{F}$ .
- If  $\phi$  and  $\psi$  are in  $\mathcal{F}$ , then  $(\phi \rightarrow \psi)$  is also in  $\mathcal{F}$ .

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 $(((\neg p) \land q) \rightarrow (\top \land (q \lor (\neg r))))$ 

is a well-formed formula in propositional logic.

If the train arrives late and there are no taxis at the station, then John is late for his meeting.

can be represented by  $(p \land q) \rightarrow r$ , with the respective abbreviations p, q, r.

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# **Evaluation of Formulas**

### Definition

The result of *evaluating* a well-formed propositional formula  $\phi$  with respect to a valuation v, denoted  $v(\phi)$  is defined as follows:

- If  $\phi$  is the constant  $\bot$ , then  $v(\phi) = F$ .
- If  $\phi$  is the constant  $\top$ , then  $v(\phi) = T$ .
- If  $\phi$  is an propositional atom p, then  $v(\phi) = p^{v}$ .
- If  $\phi$  has the form  $(\neg \psi)$ , then  $v(\phi) = \setminus v(\psi)$ .
- If  $\phi$  has the form  $(\psi \wedge \tau)$ , then  $v(\phi) = v(\psi) \& v(\tau)$ .
- If  $\phi$  has the form  $(\psi \lor \tau)$ , then  $v(\phi) = v(\psi) | v(\tau)$ .
- If  $\phi$  has the form  $(\psi \to \tau)$ , then  $v(\phi) = v(\psi) \Rightarrow v(\tau)$ .

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# Validity and Satisfiability

#### Validity

A formula is *valid* if it computes T for all its valuations.

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# Validity and Satisfiability

#### Validity

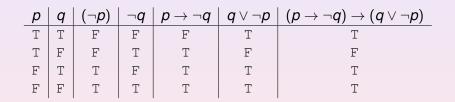
A formula is *valid* if it computes T for all its valuations.

### Satisfiability

A formula is *satisfiable* if it computes T for at least one of its valuations.

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### Example of Truth Table



The Problem with Truth Tables Sequents Basic Rules Basic and Derived Rules

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### Review: Syntax and Semantics of Propositional Logic

### 2 Natural Deduction

- The Problem with Truth Tables
- Sequents
- Basic Rules
- Basic and Derived Rules

### 3 Soundness and Completeness

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## The Problem with Truth Tables

### The problem

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# The Problem with Truth Tables

### The problem

Truth tables get very large.

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# The Problem with Truth Tables

### The problem

Truth tables get very large. How large?

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# The Problem with Truth Tables

### The problem

Truth tables get very large. How large?

### Consider for example

prop.txt from verifying the correctness of a railroad station

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## **Practical Examples**

### Timetabling and scheduling

Satisfiability of propositional formulas is used for solving scheduling problems (timetabling, sports tournaments etc)

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# **Practical Examples**

### Timetabling and scheduling

Satisfiability of propositional formulas is used for solving scheduling problems (timetabling, sports tournaments etc) (hundreds or thousands of atoms, thousands of operators)

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# **Practical Examples**

### Timetabling and scheduling

Satisfiability of propositional formulas is used for solving scheduling problems (timetabling, sports tournaments etc) (hundreds or thousands of atoms, thousands of operators)

### Computer hardware design

Satisfiability of propositional formulas is used for designing and verifying the correctness of the hardware (millions of atoms, millions of operators)

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# **Practical Examples**

### Timetabling and scheduling

Satisfiability of propositional formulas is used for solving scheduling problems (timetabling, sports tournaments etc) (hundreds or thousands of atoms, thousands of operators)

### Computer hardware design

Satisfiability of propositional formulas is used for designing and verifying the correctness of the hardware (millions of atoms, millions of operators)

### Software verification using model checking

Satisfiability of propositional formulas is used for verifying the correctness of software (embedded systems etc) (hundreds of thousands of atoms and operators)

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# Introduction

#### Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as "If the train arrives late...".

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# Introduction

#### Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as "If the train arrives late...".

#### Idea

We introduce *proof rules* that allow us to derive a formula  $\psi$  from a number of other formulas  $\phi_1, \phi_2, \dots \phi_n$ .

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# Introduction

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#### Idea

We introduce *proof rules* that allow us to derive a formula  $\psi$  from a number of other formulas  $\phi_1, \phi_2, \dots \phi_n$ .

#### Notation

We write a *sequent*  $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$  to denote that we can derive  $\psi$  from  $\phi_1, \phi_2, \ldots, \phi_n$ .

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### **Example**—Revisited

### English

If the train arrives late and there are no taxis at the station, then John is late for his meeting.

John is not late for his meeting. The train did arrive late.

Therefore, there were taxis at the station.

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## **Example**—Revisited

### English

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Therefore, there were taxis at the station.

### Sequent

$$p \land \neg q \to r, \neg r, p \vdash q$$

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## **Example**—Revisited

### English

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Therefore, there were taxis at the station.

#### Sequent

$$p \land \neg q \to r, \neg r, p \vdash q$$

### Remaining task

Find proof rules that allow us to establish such sequents.

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# **Rules for Conjunction**

### Introduction of Conjunction

$$\frac{\phi \quad \psi}{\phi \land \psi} [\land i]$$

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# **Rules for Conjunction**

### Introduction of Conjunction

$$\begin{array}{ccc}
\phi & \psi \\
\hline
\phi \wedge \psi
\end{array} [\wedge i]$$

#### Elimination of Conjunction

$$\begin{array}{ccc}
\phi \land \psi & \phi \land \psi \\
\hline
& & & \\
\phi & & & \\
\phi & & & \psi
\end{array}$$

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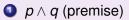
### **Example of Proof**

To show		
	$p \land q, r \vdash q \land r$	
How to start?		
	p∧q r	
	$q \wedge r$	

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# Proof Step-by-Step



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# Proof Step-by-Step

•  $p \wedge q$  (premise)

2 r (premise)

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# Proof Step-by-Step

- $p \land q$  (premise)
- 2 r (premise)
- **3** q (by using Rule  $\wedge e_2$  and Item 1)

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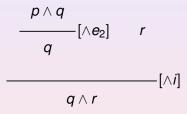
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## Proof Step-by-Step

- $p \land q$  (premise)
- Image: Provide the second s
- **3** q (by using Rule  $\wedge e_2$  and Item 1)
- $q \wedge r$  (by using Rule  $\wedge i$  and Items 3 and 2)

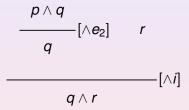
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### **Graphical Representation of Proof**



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## Graphical Representation of Proof



Find the parts of the corresponding sequent:

 $p \land q, r \vdash q \land r$ 

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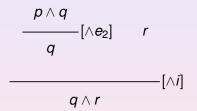
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### **Graphical Representation of Proof**

$$\frac{p \land q}{q} [\land e_2] \qquad r$$
$$\frac{q \land r}{q} [\land i]$$

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## Graphical Representation of Proof



Find the parts of the corresponding proof:

- $p \land q$  (premise)
- I (premise)
- q (by using Rule  $\wedge e_2$  and Item 1)
- $q \wedge r$  (by using Rule  $\wedge i$  and Items 3 and 2)

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### Where are we heading with this?

 We would like to prove sequents of the form φ<sub>1</sub>, φ<sub>2</sub>, ..., φ<sub>n</sub> ⊢ ψ

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- We would like to prove sequents of the form  $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$
- We introduce rules that allow us to form "legal" proofs

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- What does it mean to be meaningful?

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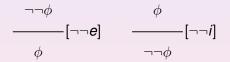
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- Can we say that sequents with a correct proof are somehow "valid", or "meaningful"?
- What does it mean to be meaningful?
- Can we say that any meaningful sequent has a valid proof?
- ...but first back to the proof rules...

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### **Rules of Double Negation**



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## **Rule for Eliminating Implication**

$$\begin{array}{cc} \phi & \phi \rightarrow \psi \\ \hline \\ \psi \end{array} [ \rightarrow \boldsymbol{e} ]$$

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# **Rule for Eliminating Implication**

$$\begin{array}{cc} \phi & \phi \rightarrow \psi \\ \hline & & \\ \psi \end{array} [\rightarrow e]$$

#### Example

p: It rained.

 $p \rightarrow q$ : If it rained, then the street is wet.

We can conclude from these two that the street is indeed wet.

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Another Rule for Eliminating Implication

The rule

#### is often called "Modus Ponens" (or MP)

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Another Rule for Eliminating Implication

The rule

$$\frac{\phi \quad \phi \to \psi}{\psi} [\to \boldsymbol{e}]$$

is often called "Modus Ponens" (or MP)

#### Origin of term

"Modus ponens" is an abbreviation of the Latin "modus ponendo ponens" which means in English "mode that affirms by affirming". More precisely, we could say "mode that affirms the antecedent of an implication".

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### The Twin Sister of Modus Ponens

The rule

$$\begin{array}{cc} \phi & \phi \rightarrow \psi \\ \hline & \\ \psi \end{array} [\rightarrow e]$$

#### is called "Modus Ponens" (or MP)

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### The Twin Sister of Modus Ponens

### The rule

$$\frac{\phi \quad \phi \to \psi}{\psi} [\to \boldsymbol{e}]$$

### is called "Modus Ponens" (or MP) A similar rule

is called "Modus Tollens" (or MT).

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### The Twin Sister of Modus Ponens

#### The rule

#### is called "Modus Tollens" (or MT).

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### The Twin Sister of Modus Ponens

#### The rule

is called "Modus Tollens" (or MT).

#### Origin of term

"Modus tollens" is an abbreviation of the Latin "modus tollendo tollens" which means in English "mode that denies by denying". More precisely, we could say "mode that denies the consequent of an implication".

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ightarrow r), p, 
eg r dash g \eta$ 

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$$p o (q o r), p, 
eg r dash q$$
  
1  $p o (q o r)$  premise

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$$p 
ightarrow (q 
ightarrow r), p, 
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 $premise$ 
 $premise$ 

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$$p 
ightarrow (q 
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eg r dash \neg r dash \neg r$$
 premise  
 $p 
ightarrow (q 
ightarrow r)$  premise  
 $p 
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$$p 
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ightarrow r), p, 
eg r arrow q$$
  
1  $p 
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ightarrow r)$  premise  
2  $p$  premise  
3  $\neg r$  premise  
4  $q 
ightarrow r$   $ightarrow _e$  1,2

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$$p 
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ightarrow r), p, 
eg r arphi \neg q$$
  
1  $p 
ightarrow (q 
ightarrow r)$  premise  
2  $p$  premise  
3  $\neg r$  premise  
4  $q 
ightarrow r$   $ightarrow _e 1, 2$   
5  $\neg q$  MT 4, 3

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How to introduce implication?

Compare the sequent (MT)

$$p \rightarrow q, \neg q \vdash \neg p$$

with the sequent

$$p 
ightarrow q 
vert \neg q 
ightarrow \neg p$$

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How to introduce implication?

Compare the sequent (MT)

$$p \rightarrow q, \neg q \vdash \neg p$$

with the sequent

$$p 
ightarrow q dash 
eg q 
ightarrow 
eg p$$

The second sequent should be provable, but we don't have a rule to introduce implication yet!

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### A Proof We Would Like To Have

1	$oldsymbol{ ho}  ightarrow oldsymbol{q}$	premise
2	eg q	assumption
3	eg p	MT 1,2
4	eg q  ightarrow  eg p	<i>→i</i> 2–3

 $p \rightarrow q \vdash \neg q \rightarrow \neg p$ 

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## A Proof We Would Like To Have

1	$oldsymbol{ ho}  o oldsymbol{q}$	premise
2	eg q	assumption
3	eg p	MT 1,2
4	eg q  ightarrow  eg p	$\rightarrow_i 2-3$

We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

 $p \rightarrow q \vdash \neg q \rightarrow \neg p$ 

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## A Proof We Would Like To Have

		, , , ,
1	$oldsymbol{ ho}  o oldsymbol{q}$	premise
2	eg q	assumption
3	eg p	MT 1,2
4	$\neg a \rightarrow \neg b$	$\rightarrow_i 2-3$

We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

 $p \rightarrow q \vdash \neg q \rightarrow \neg p$ 

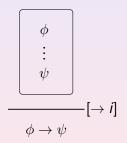
We cannot use assumptions from inside the box in rules outside the box.

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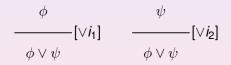
## Rule for Introduction of Implication



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## **Rules for Introduction of Disjunction**

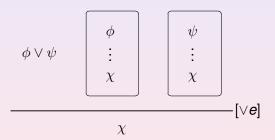


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## Rule for Elimination of Disjunction



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### Example

1	$m{p} \wedge (m{q} ee m{r})$	premise
2	р	∧ <i>e</i> 1 1
3	$q \lor r$	∧ <i>e</i> ₂ 1
4	q	assumption
5	$oldsymbol{p} \wedge oldsymbol{q}$	∧ <i>i</i> 2,4
6	$(p \wedge q) \lor (p \wedge r)$	∨ <i>i</i> 1 5
7	r	assumption
8	$p \wedge r$	<i>∧i</i> 2,7
9	$(p \wedge q) \lor (p \wedge r)$	∨ <i>i</i> ₂ 8
10	$(p \land q) \lor (p \land r)$	∨ <i>e</i> 3, 4–6, 7–9

The Problem with Truth Tables Sequents Basic Rules Basic and Derived Rules

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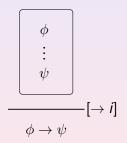
# **Rules for Eliminating Implication**

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# Rule for Introduction of Implication

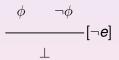


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# Elimination of Negation

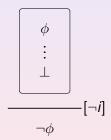


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# Introduction of Negation



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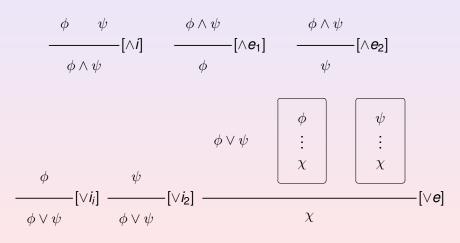
# Elimination of $\perp$



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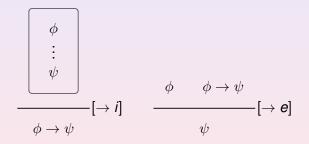
# Basic Rules (conjunction and disjunction)



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## **Basic Rules (implication)**

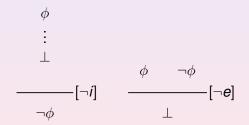


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# **Basic Rules (negation)**



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# Basic Rules ( $\perp$ and double negation)



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# Some Derived Rules: Introduction of Double Negation



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# Example: Deriving $[\neg \neg i]$ from $[\neg i]$ and $[\neg e]$

1	$\phi$	premise
2	$\neg \phi$	assumption
3	$\perp$	¬e 1,2
4	$\neg \neg \phi$	i 2–3

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# Some Derived Rules: Modus Tollens

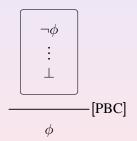
$$\begin{array}{c} \phi \to \psi & \neg \psi \\ \hline & \hline & \\ \neg \phi \end{array} [MT]$$

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# Some Derived Rules: Proof By Contradiction



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# Some Derived Rules: Law of Excluded Middle

\_\_\_\_[LEM]  $\phi \vee \neg \phi$ 

The Importance of Being Formal 03—Propositional Logic II

## Review: Syntax and Semantics of Propositional Logic

- 2 Natural Deduction
- Soundness and Completeness

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# Provability

## Definition

If there is a natural deduction proof of  $\psi$  using the premises  $\phi_1, \phi_2, \ldots, \phi_n$ , we say that  $\psi$  is *provable* from  $\phi_1, \phi_2, \ldots, \phi_n$  and write

 $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ 

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# Semantic Entailment

#### Definition

If, for all valuations in which all  $\phi_1, \phi_2, \ldots, \phi_n$  evaluate to T, the formula  $\psi$  evaluates to T as well, we say that  $\phi_1, \phi_2, \ldots, \phi_n$  semantically entail  $\psi$ , written:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

## Some More Definitions

## Semantic equivalence

Let  $\phi$  and  $\psi$  be formulas of propositional logic. We say that  $\phi$  and  $\psi$  are semantically equivalent iff  $\phi \models \psi$  and  $\psi \models \phi$  hold. We write  $\psi \equiv \phi$ .

## Some More Definitions

## Semantic equivalence

Let  $\phi$  and  $\psi$  be formulas of propositional logic. We say that  $\phi$  and  $\psi$  are semantically equivalent iff  $\phi \models \psi$  and  $\psi \models \phi$  hold. We write  $\psi \equiv \phi$ .

#### Validity

If  $\models \phi$  holds, we call  $\phi$  *valid*.

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## Soundness of Natural Deduction

## Soundness

Let  $\phi_1, \phi_2, \dots, \phi_n$  and  $\psi$  be propositional formulas. If  $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ , then  $\phi_1, \phi_2, \dots, \phi_n \models \psi$ .

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# **Completeness of Propositional Logic**

## Completeness

Let  $\phi_1, \phi_2, \dots, \phi_n$  and  $\psi$  be propositional formulas. If  $\phi_1, \phi_2, \dots, \phi_n \models \psi$ , then  $\phi_1, \phi_2, \dots, \phi_n \vdash \psi$ .

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# Questions about Propositional Formula

- Is a given formula valid?
- Is a given formula satisfiable?
- Is a given formula invalid?
- Is a given formula unsatisfiable?
- Are two formulas equivalent?

## **Decision Problems**

## Definition

A *decision problem* is a question in some formal system with a yes-or-no answer.

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# **Decision Problems**

## Definition

A *decision problem* is a question in some formal system with a yes-or-no answer.

## Examples

The question whether a given propositional formula is satisifiable (unsatisfiable, valid, invalid) is a decision problem.

The question whether two given propositional formulas are equivalent is also a decision problem.

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## How to Solve the Decision Problem?

Question

How do you decide whether a given propositional formula is satisfiable/valid?

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## How to Solve the Decision Problem?

#### Question

How do you decide whether a given propositional formula is satisfiable/valid?

## The good news

We can construct a truth table for the formula and check if some/all rows have T in the last column.

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# Satisifiability is Decidable

## An algorithm for satisifiability

Using a truth table, we can implement an *algorithm* (systematic method) that returns *"yes"* if the formula is satisifiable, and that returns *"no"* if the formula is unsatisfiable.

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## Decidability

Decision problems for which there is an algorithm computing "yes" whenever the answer is "yes", and "no" whenever the answer is "no", are called *decidable*.

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Using a truth table, we can implement an *algorithm* (systematic method) that returns *"yes"* if the formula is satisifiable, and that returns *"no"* if the formula is unsatisfiable.

## Decidability

Decision problems for which there is an algorithm computing "yes" whenever the answer is "yes", and "no" whenever the answer is "no", are called *decidable*.

## Decidability of satisfiability

The question, whether a given propositional formula is satisifiable, is decidable.



- Next homework: Homework 3: Due Wednesday morning before class
- Wednesday material: Predicate Logic
- After that: Homework 4: Due Friday morning before class

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