03—Propositional Logic II

The Importance of Being Formal

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Atoms

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We allow any kind of proposition, for example “The sun is shining today”.

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Atoms

More formally, we fix a set A of propositional atoms.

Meaning of Atoms

Models assign truth values

A *model* assigns truth values (F or T) to each atom.

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More formally

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How do you call them?

Models for propositional logic are called *valuations*.

Syntax of Propositions

Definition

For a given set A of propositional atoms, the set of *well-formed formulas in propositional logic* is the smallest set $\mathcal{F}$ that fulfills the following rules:

- The constant symbols $\bot$ and $\top$ are in $\mathcal{F}$.
- Every element of A is in $\mathcal{F}$.
- If ϕ is in $\mathcal{F}$, then $(\neg\phi)$ is also in $\mathcal{F}$.
- If ϕ and ψ are in $\mathcal{F}$, then $(\phi \wedge \psi)$ is also in $\mathcal{F}$.
- If ϕ and ψ are in $\mathcal{F}$, then $(\phi \vee \psi)$ is also in $\mathcal{F}$.
- If ϕ and ψ are in $\mathcal{F}$, then $(\phi \rightarrow \psi)$ is also in $\mathcal{F}$.

Example

$$(((\neg p) \land q) \to (\top \land (q \lor (\neg r))))$$

is a well-formed formula in propositional logic.

If *the train arrives late* and *there are no taxis at the station*, then *John is late for his meeting.*

can be represented by $(p \land q) \to r$, with the respective abbreviations p, q, r.

Evaluation of Formulas

Definition

The result of *evaluating* a well-formed propositional formula ϕ with respect to a valuation v, denoted $v(\phi)$ is defined as follows:

- If ϕ is the constant $\bot$, then $v(\phi) = F$.
- If ϕ is the constant $\top$, then $v(\phi) = T$.
- If ϕ is an propositional atom p, then $v(\phi) = p^v$.
- If ϕ has the form $(\neg\psi)$, then $v(\phi) = \backslash v(\psi)$.
- If ϕ has the form $(\psi \land \tau)$, then $v(\phi) = v(\psi) \& v(\tau)$.
- If ϕ has the form $(\psi \lor \tau)$, then $v(\phi) = v(\psi) \mid v(\tau)$.
- If ϕ has the form $(\psi \to \tau)$, then $v(\phi) = v(\psi) \Rightarrow v(\tau)$.

Validity and Satisfiability

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A formula is *valid* if it computes $\mathtt{T}$ for all its valuations.

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A formula is *valid* if it computes $\mathtt{T}$ for all its valuations.

Satisfiability

A formula is *satisfiable* if it computes $\mathtt{T}$ for at least one of its valuations.

Example of Truth Table

p	q	$(\neg p)$	$\neg q$	$p \rightarrow \neg q$	$q \vee \neg p$	$(p \rightarrow \neg q) \rightarrow (q \vee \neg p)$
T	T	F	F	F	T	T
T	F	F	T	T	F	F
F	T	T	F	T	T	T
F	F	T	T	T	T	T

The Problem with Truth Tables

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Truth tables get very large.

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Truth tables get very large. How large?

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Consider for example

`prop.txt` from verifying the correctness of a railroad station

Practical Examples

Timetabling and scheduling

Satisfiability of propositional formulas is used for solving scheduling problems (timetabling, sports tournaments etc)

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Software verification using model checking

Satisfiability of propositional formulas is used for verifying the correctness of software (embedded systems etc) (hundreds of thousands of atoms and operators)

Introduction

Objective

We would like to develop a *calculus* for reasoning about propositions, so that we can establish the validity of statements such as “If the train arrives late...”.

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We introduce *proof rules* that allow us to derive a formula ψ from a number of other formulas $\phi_1, \phi_2, \ldots \phi_n$.

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We introduce *proof rules* that allow us to derive a formula ψ from a number of other formulas $\phi_1, \phi_2, \ldots \phi_n$.

Notation

We write a *sequent* $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$
to denote that we can derive ψ from $\phi_1, \phi_2, \ldots, \phi_n$.

Example—Revisited

English

If *the train arrives late* and *there are no taxis at the station*, then *John is late for his meeting.*

John is not late for his meeting. The train did arrive late.

Therefore, *there were taxis at the station.*

Example—Revisited

English

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Sequent

$$p \wedge \neg q \rightarrow r, \neg r, p \vdash q$$

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Sequent

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Remaining task

Find proof rules that allow us to establish such sequents.

Rules for Conjunction

Introduction of Conjunction

$$\frac{\phi \qquad \psi}{\phi \land \psi}[\land i]$$

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$$\frac{\phi \qquad \psi}{\phi \wedge \psi}[\wedge i]$$

Elimination of Conjunction

$$\frac{\phi \wedge \psi}{\phi}[\wedge e_1] \qquad \frac{\phi \wedge \psi}{\psi}[\wedge e_2]$$

Example of Proof

To show

$$p \wedge q, r \vdash q \wedge r$$

How to start?

$$p \wedge q \qquad\qquad r$$

$$q \wedge r$$

Proof Step-by-Step

1. $p \wedge q$ (premise)

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2. r (premise)
3. q (by using Rule $\wedge e_2$ and Item 1)
4. $q \wedge r$ (by using Rule $\wedge i$ and Items 3 and 2)

Graphical Representation of Proof

$$\cfrac{\cfrac{p \land q}{q}\,[\land e_2] \qquad r}{q \land r}\,[\land i]$$

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$$\dfrac{\dfrac{p \wedge q}{q}[\wedge e_2] \qquad r}{q \wedge r}[\wedge i]$$

Find the parts of the corresponding sequent:

$$p \wedge q, r \vdash q \wedge r$$

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Find the parts of the corresponding proof:

1. $p \wedge q$ (premise)
2. r (premise)
3. q (by using Rule $\wedge e_2$ and Item 1)
4. $q \wedge r$ (by using Rule $\wedge i$ and Items 3 and 2)

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- Can we say that sequents with a correct proof are somehow "valid", or "meaningful"?
- What does it mean to be meaningful?
- Can we say that any meaningful sequent has a valid proof?
- ...but first back to the proof rules...

Rules of Double Negation

$$\frac{\neg\neg\phi}{\phi}\,[\neg\neg e] \qquad \frac{\phi}{\neg\neg\phi}\,[\neg\neg i]$$

Rule for Eliminating Implication

$$\frac{\phi \qquad \phi \to \psi}{\psi}\,[\to e]$$

Rule for Eliminating Implication

$$\frac{\phi \qquad \phi \rightarrow \psi}{\psi}[\rightarrow e]$$

Example

p: It rained.

$p \rightarrow q$: If it rained, then the street is wet.

We can conclude from these two that the street is indeed wet.

Another Rule for Eliminating Implication

The rule

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is often called “Modus Ponens” (or MP)

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Origin of term

"Modus ponens" is an abbreviation of the Latin "modus ponendo ponens" which means in English "mode that affirms by affirming". More precisely, we could say "mode that affirms the antecedent of an implication".

The Twin Sister of Modus Ponens

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A similar rule

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}[MT]$$

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Origin of term

“Modus tollens” is an abbreviation of the Latin “modus tollendo tollens” which means in English “mode that denies by denying”. More precisely, we could say “mode that denies the consequent of an implication”.

Example

$$p \to (q \to r), p, \neg r \vdash \neg q$$

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1	$p \to (q \to r)$	premise

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1	$p \to (q \to r)$	premise
2	p	premise

Example

$$p \to (q \to r), p, \neg r \vdash \neg q$$

1	$p \to (q \to r)$	premise
2	p	premise
3	$\neg r$	premise

Example

$$p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q$$

1	$p \rightarrow (q \rightarrow r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \rightarrow r$	$\rightarrow_e$ 1,2

Example

$$p \to (q \to r), p, \neg r \vdash \neg q$$

1	$p \to (q \to r)$	premise
2	p	premise
3	$\neg r$	premise
4	$q \to r$	$\to_e$ 1,2
5	$\neg q$	MT 4,3

How to *introduce* implication?

Compare the sequent (MT)

$$p \rightarrow q, \neg q \vdash \neg p$$

with the sequent

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

How to *introduce* implication?

Compare the sequent (MT)

$$p \rightarrow q, \neg q \vdash \neg p$$

with the sequent

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

The second sequent should be provable, but we don't have a rule to introduce implication yet!

A Proof We Would Like To Have

$$p \rightarrow q \vdash \neg q \rightarrow \neg p$$

1	$p \rightarrow q$	premise
2	$\neg q$	assumption
3	$\neg p$	MT 1,2
4	$\neg q \rightarrow \neg p$	$\rightarrow_i$ 2–3

A Proof We Would Like To Have

$$p \to q \vdash \neg q \to \neg p$$

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We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

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We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.
We cannot use assumptions from inside the box in rules outside the box.

Rule for Introduction of Implication

$$\frac{\boxed{\begin{array}{c}\phi \\ \vdots \\ \psi\end{array}}}{\phi \rightarrow \psi} [\rightarrow i]$$

Rules for Introduction of Disjunction

$$\frac{\phi}{\phi \lor \psi}\,[\lor i_1] \qquad \frac{\psi}{\phi \lor \psi}\,[\lor i_2]$$

Rule for Elimination of Disjunction

$$\dfrac{\phi \lor \psi \qquad \boxed{\begin{array}{c}\phi\\ \vdots\\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi\\ \vdots\\ \chi\end{array}}}{\chi}\;[\lor e]$$

Example

1	$p \wedge (q \vee r)$	premise
2	p	$\wedge e_1$ 1
3	$q \vee r$	$\wedge e_2$ 1
4	q	assumption
5	$p \wedge q$	$\wedge i$ 2,4
6	$(p \wedge q) \vee (p \wedge r)$	$\vee i_1$ 5
7	r	assumption
8	$p \wedge r$	$\wedge i$ 2,7
9	$(p \wedge q) \vee (p \wedge r)$	$\vee i_2$ 8
10	$(p \wedge q) \vee (p \wedge r)$	$\vee e$ 3, 4–6, 7–9

Rules for Eliminating Implication

$$\frac{\phi \qquad \phi \to \psi}{\psi}\,[\to e]$$

$$\frac{\phi \to \psi \qquad \neg\psi}{\neg\phi}\,[MT]$$

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Elimination of Negation

$$\frac{\phi \qquad \neg\phi}{\bot}\,[\neg e]$$

Introduction of Negation

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}}{\neg\phi}\ [\neg i]$$

Elimination of $\perp$

$$\frac{\perp}{\phi}\,[\perp e]$$

Basic Rules (conjunction and disjunction)

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\,[\wedge i] \qquad \frac{\phi \wedge \psi}{\phi}\,[\wedge e_1] \qquad \frac{\phi \wedge \psi}{\psi}\,[\wedge e_2]$$

$$\frac{\phi}{\phi \vee \psi}\,[\vee i_i] \quad \frac{\psi}{\phi \vee \psi}\,[\vee i_2] \quad \frac{\phi \vee \psi \qquad \boxed{\begin{array}{c}\phi \\ \vdots \\ \chi\end{array}} \qquad \boxed{\begin{array}{c}\psi \\ \vdots \\ \chi\end{array}}}{\chi}\,[\vee e]$$

Basic Rules (implication)

$$\frac{\boxed{\begin{array}{c}\phi\\ \vdots\\ \psi\end{array}}}{\phi \to \psi}\,[\to i] \qquad \frac{\phi \qquad \phi \to \psi}{\psi}\,[\to e]$$

Basic Rules (negation)

$$\frac{\begin{array}{c}\phi\\ \vdots\\ \bot\end{array}}{\neg\phi}\,[\neg i] \qquad \frac{\phi \qquad \neg\phi}{\bot}\,[\neg e]$$

Basic Rules (⊥ and double negation)

$$\frac{\bot}{\phi}\;[\bot e]$$

$$\frac{\neg\neg\phi}{\phi}\;[\neg\neg e]$$

Some Derived Rules: Introduction of Double Negation

$$\frac{\phi}{\neg\neg\phi}\,[\neg\neg i]$$

Example: Deriving [¬¬ i] from [¬ i] and [¬ e]

1	ϕ	premise
2	$\neg\phi$	assumption
3	$\bot$	¬e 1,2
4	$\neg\neg\phi$	¬i 2–3

Some Derived Rules: Modus Tollens

$$\frac{\phi \rightarrow \psi \qquad \neg\psi}{\neg\phi}\,[MT]$$

Some Derived Rules: Proof By Contradiction

$$\frac{\boxed{\begin{array}{c}\neg\phi \\ \vdots \\ \bot\end{array}}}{\phi}\,[\mathrm{PBC}]$$

Some Derived Rules: Law of Excluded Middle

$$\frac{}{\phi \vee \neg\phi}\,[\mathrm{LEM}]$$

Provability

Definition

If there is a natural deduction proof of ψ using the premises $\phi_1, \phi_2, \ldots, \phi_n$, we say that ψ is *provable* from $\phi_1, \phi_2, \ldots, \phi_n$ and write

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$

Semantic Entailment

Definition

If, for all valuations in which all $\phi_1, \phi_2, \ldots, \phi_n$ evaluate to $\mathtt{T}$, the formula ψ evaluates to $\mathtt{T}$ as well, we say that $\phi_1, \phi_2, \ldots, \phi_n$ semantically entail ψ, written:

$$\phi_1, \phi_2, \ldots, \phi_n \models \psi$$

Some More Definitions

Semantic equivalence

Let ϕ and ψ be formulas of propositional logic. We say that ϕ and ψ are semantically equivalent iff $\phi \models \psi$ and $\psi \models \phi$ hold. We write $\psi \equiv \phi$.

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Semantic equivalence

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Validity

If $\models \phi$ holds, we call ϕ *valid*.

Soundness of Natural Deduction

Soundness

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional formulas. If $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$, then $\phi_1, \phi_2, \ldots, \phi_n \models \psi$.

Completeness of Propositional Logic

Completeness

Let $\phi_1, \phi_2, \ldots, \phi_n$ and ψ be propositional formulas. If $\phi_1, \phi_2, \ldots, \phi_n \models \psi$, then $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$.

Questions about Propositional Formula

- Is a given formula valid?
- Is a given formula satisfiable?
- Is a given formula invalid?
- Is a given formula unsatisfiable?
- Are two formulas equivalent?

Decision Problems

Definition

A *decision problem* is a question in some formal system with a yes-or-no answer.

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Examples

The question whether a given propositional formula is satisifiable (unsatisfiable, valid, invalid) is a decision problem.

The question whether two given propositional formulas are equivalent is also a decision problem.

How to Solve the Decision Problem?

Question

How do you decide whether a given propositional formula is satisfiable/valid?

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How do you decide whether a given propositional formula is satisfiable/valid?

The good news

We can construct a truth table for the formula and check if some/all rows have T in the last column.

Satisifiability is Decidable

An algorithm for satisifiability

Using a truth table, we can implement an *algorithm* (systematic method) that returns *"yes"* if the formula is satisifiable, and that returns *"no"* if the formula is unsatisfiable.

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An algorithm for satisifiability

Using a truth table, we can implement an *algorithm* (systematic method) that returns *"yes"* if the formula is satisifiable, and that returns *"no"* if the formula is unsatisfiable.

Decidability

Decision problems for which there is an algorithm computing "yes" whenever the answer is "yes", and "no" whenever the answer is "no", are called *decidable*.

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Decidability of satisfiability

The question, whether a given propositional formula is satisifiable, is decidable.

Admin

- Next homework: Homework 3: Due Wednesday morning before class
- Wednesday material: Predicate Logic
- After that: Homework 4: Due Friday morning before class