06—From Propositional to Predicate Logic

The Importance of Being Formal

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February 19, 2014

More Declarative Sentences

- Propositional logic can easily handle simple declarative statements such as:

Example

Student Peter Lim enrolled in UIT2206.

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- *But:* How about statements with *"there exists..."* or *"every..."* or *"among..."*?

What is needed?

Example

Every student is younger than *some* instructor.

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These are *properties* of elements of a *set* of objects.

We express them in predicate logic using *predicates*.

Predicates

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- $S(andy)$ could denote that Andy is a student.
- $I(paul)$ could denote that Paul is an instructor.
- $Y(andy, paul)$ could denote that Andy is younger than Paul.

The Need for Variables

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How do we express *"every student"*?

We need *variables* that can stand for constant values, and a *quantifier* symbol that denotes *"every"*.

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Using variables and quantifiers, we can write:

$$\forall x(S(x) \rightarrow (\exists y(I(y) \wedge Y(x, y)))).$$

Literally: For every x, if x is a student, then there is some y such that y is an instructor and x is younger than y.

Another Example

English

Not all birds can fly.

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Predicates

$B(x)$: x is a bird

$F(x)$: x can fly

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The sentence in predicate logic

$$\neg(\forall x(B(x) \to F(x)))$$

A Third Example

English

Every girl is younger than her mother.

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Every girl is younger than her mother.

Predicates

$G(x)$: x is a girl

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$$\forall x \forall y (G(x) \land M(y, x) \rightarrow Y(x, y))$$

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If everyone has exactly one mother, the predicate $M(y, x)$ is a function, when read from right to left.

We introduce a function symbol m that can be applied to variables and constants as in

$$\forall x (G(x) \to Y(x, m(x)))$$

A Drastic Example

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Andy and Paul have the same maternal grandmother.

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The sentence in predicate logic without functions

$$\forall x \forall y \forall u \forall v (M(x, y) \land M(y, andy) \land M(u, v) \land M(v, paul) \rightarrow x = u)$$

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The sentence in predicate logic without functions

$$\forall x \forall y \forall u \forall v (M(x, y) \land M(y, andy) \land M(u, v) \land M(v, paul) \to x = u)$$

The same sentence in predicate logic with functions

$$m(m(andy)) = m(m(paul))$$

Outlook

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Further topics: Soundness/completeness, undecidability, incompleteness results, compactness results

Predicate Vocabulary

At any point in time, we want to describe the features of a particular “world”, using predicates, functions, and constants. Thus, we introduce for this world:

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Arity of Functions and Predicates

Every function symbol in $\mathcal{F}$ and predicate symbol in $\mathcal{P}$ comes with a fixed arity, denoting the number of arguments the symbol can take.

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Predicate symbols with arity 0 denotes predicates that do not depend on any arguments. They correspond to propositional atoms.

Terms

$$t ::= \; x \mid c \mid f(t, \ldots, t)$$

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- c ranges over nullary function symbols in $\mathcal{F}$, and
- f ranges over function symbols in $\mathcal{F}$ with arity $n > 0$.

Examples of Terms

If n is nullary, f is unary, and g is binary, then examples of terms are:

- $g(f(n), n)$
- $f(g(n, f(n)))$

More Examples of Terms

If $0, 1, 2$ are nullary (constants), s is unary, and $+, -$ and $*$ are binary, then

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Occasionally, we allow ourselves to use infix notation for function symbols as in

$$(2 - (s(x) + y)) * x$$

Formulas

$$\begin{aligned}\phi \quad ::= \quad & P(t, \ldots, t) \mid (\neg\phi) \mid (\phi \wedge \phi) \mid (\phi \vee \phi) \mid \\ & (\phi \rightarrow \phi) \mid (\forall x \phi) \mid (\exists x \phi)\end{aligned}$$

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- x are variables in $\mathcal{V}$.

Conventions

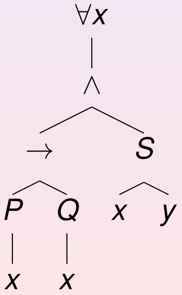
Just like for propositional logic, we introduce convenient conventions to reduce the number of parentheses:

- $\neg, \forall x$ and $\exists x$ bind most tightly;
- then $\land$ and $\lor$;
- then $\rightarrow$, which is right-associative.

Parse Trees

$$\forall x((P(x) \rightarrow Q(x)) \land S(x, y))$$

has parse tree



Another Example

Every son of my father is my brother.

Predicates

$S(x, y)$: x is a son of y

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Equality as Predicate

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Example

Instead of the formula

$$=(f(x), g(x))$$

we usually write the formula

$$f(x) = g(x)$$

Free and Bound Variables

Consider the formula

$$\forall x((P(x) \rightarrow Q(x)) \land S(x, y))$$

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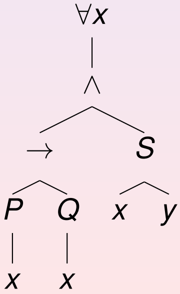

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Which variable *occurrences* are free; which are bound?

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$$(\forall x(P(x) \land Q(x))) \to (\neg P(x) \lor Q(y))$$

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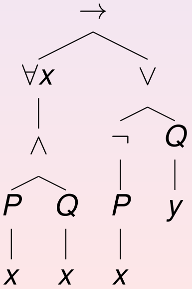

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Given a variable x, a term t and a formula ϕ, we define $[x \Rightarrow t]\phi$ to be the formula obtained by replacing each free occurrence of variable x in ϕ with t.

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Example

$$[x \Rightarrow f(x, y)]((\forall x(P(x) \land Q(x))) \to (\neg P(x) \lor Q(y)))$$
$$= \forall x(P(x) \land Q(x))) \to (\neg P(f(x, y)) \lor Q(y))$$

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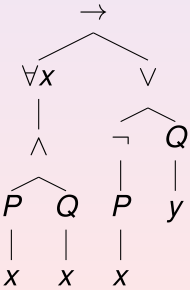

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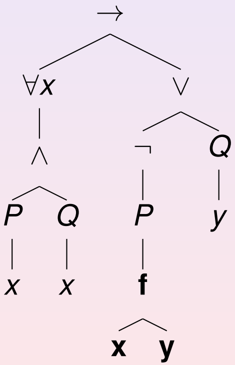

Models

Definition

Let $\mathcal{F}$ contain function symbols and $\mathcal{P}$ contain predicate symbols. A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A non-empty set A, the *universe*;
2. for each nullary function symbol $f \in \mathcal{F}$ a concrete element $f^{\mathcal{M}} \in A$;
3. for each $f \in F$ with arity $n > 0$, a concrete function $f^{\mathcal{M}} : A^n \to A$;
4. for each $P \in \mathcal{P}$ with arity $n > 0$, a function $P^{\mathcal{M}} : U^n \to \{F, T\}$.
5. for each $P \in \mathcal{P}$ with arity $n = 0$, a value from $\{F, T\}$.

Example

Let $\mathcal{F} = \{e, \cdot\}$ and $\mathcal{P} = \{\leq\}$.
Let model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ be defined as follows:

1. Let A be the set of binary strings over the alphabet $\{0, 1\}$;
2. let $e^{\mathcal{M}} = \epsilon$, the empty string;
3. let $\cdot^{\mathcal{M}}$ be defined such that $s_1 \cdot^{\mathcal{M}} s_2$ is the concatenation of the strings s_1 and s_2; and
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Equality Revisited

Interpretation of equality

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Extensionality restriction

This means that allowable models are restricted to those in which $a =^{\mathcal{M}} b$ holds if and only if a and b are the same elements of the model's universe.

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Equality in $\mathcal{M}$

- $000 =^{\mathcal{M}} 000$
- $001 \neq^{\mathcal{M}} 100$

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How To Handle Free Variables?

Idea

We can give meaning to formulas with free variables by providing an environment (lookup table) that assigns variables to elements of our universe:

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Environment extension

We define environment extension such that $I[x \mapsto a]$ is the environment that maps x to a and any other variable y to $I(y)$.

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- in case ϕ is of the form P, if $P^{\mathcal{M}} = T$;
- in case ϕ has the form $\forall x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for all $a \in A$;

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- in case ϕ has the form $\exists x \psi$, if the $\mathcal{M} \models_{l[x \mapsto a]} \psi$ holds for some $a \in A$;

Satisfaction Relation (continued)

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- in case ϕ has the form $\psi_1 \to \psi_2$, if $\mathcal{M} \models_l \psi_2$ holds whenever $\mathcal{M} \models_l \psi_1$ holds.

Satisfaction of Closed Formulas

If a formula ϕ has no free variables, we call ϕ a *sentence*.

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If a formula ϕ has no free variables, we call ϕ a *sentence*.
$\mathcal{M} \models_I \phi$ holds or does not hold regardless of the choice of I.
Thus we write $\mathcal{M} \models \phi$ or $\mathcal{M} \not\models \phi$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

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$\Gamma \models \psi$ iff for all models $\mathcal{M}$ and environments l, whenever $\mathcal{M} \models_l \phi$ holds for all $\phi \in \Gamma$, then $\mathcal{M} \models_l \psi$.

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Satisfiability of Formula Sets

Γ is satisfiable iff there is some model $\mathcal{M}$ and some environment l such that $\mathcal{M} \models_l \phi$, for all $\phi \in \Gamma$.

Semantic Entailment and Satisfiability

Let Γ be a possibly infinite set of formulas in predicate logic and ψ a formula.

Validity

ψ is valid iff for all models $\mathcal{M}$ and environments l, we have $\mathcal{M} \models_l \psi$.

The Problem with Predicate Logic

Entailment ranges over models

Semantic entailment between sentences: $\phi_1, \phi_2, \ldots, \phi_n \models \psi$ requires that in *all* models that satisfy $\phi_1, \phi_2, \ldots, \phi_n$, the sentence ψ is satisfied.

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How to effectively argue about all possible models?

Usually the number of models is infinite; it is very hard to argue on the semantic level in predicate logic.

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Idea from propositional logic

Can we use natural deduction for showing entailment?

Natural Deduction for Predicate Logic

Relationship between propositional and predicate logic

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If we consider propositions as nullary predicates, propositional logic is a sub-language of predicate logic.

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Example

$$\frac{\phi \qquad \psi}{\phi \wedge \psi}\,[\wedge i]$$

Built-in Rules for Equality

$$\frac{}{t = t}[= i] \qquad \frac{t_1 = t_2 \qquad [x \Rightarrow t_1]\phi}{[x \Rightarrow t_2]\phi}[= e]$$

Properties of Equality

We show:

$$f(x) = g(x) \vdash h(g(x)) = h(f(x))$$

using

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1	$f(x) = g(x)$	premise
2	$h(f(x)) = h(f(x))$	$= i$
3	$h(g(x)) = h(f(x))$	$= e$ 1,2

Elimination of Universal Quantification

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}\,[\forall x\ e]$$

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Once you have proven $\forall x \phi$, you can replace x by any term t in ϕ, provided that t is free for x in ϕ.

Example

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}[\forall x\ e]$$

We prove: $S(g(john)), \forall x(S(x) \rightarrow \neg L(x)) \vdash \neg L(g(john))$

Example

$$\frac{\forall x \phi}{[x \Rightarrow t]\phi}\,[\forall x\ e]$$

We prove: $S(g(john)), \forall x(S(x) \to \neg L(x)) \vdash \neg L(g(john))$

1	$S(g(john))$	premise
2	$\forall x(S(x) \to \neg L(x))$	premise
3	$S(g(john)) \to \neg L(g(john))$	$\forall x\ e$ 2
4	$\neg L(g(john))$	$\to e$ 3,1

Introduction of Universal Quantification

$$\frac{\boxed{\begin{array}{c}\vdots\\ [x \Rightarrow x_0]\phi\end{array}}^{\,x_0}}{\forall x \phi}\;[\forall x\ i]$$

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If we manage to establish a formula ϕ about a fresh variable x_0, we can assume $\forall x \phi$.
The variable x_0 must be *fresh*; we cannot introduce the same variable twice in nested boxes.

Example

$$\forall x(P(x) \to Q(x)), \forall x P(x) \vdash \forall x Q(x) \text{ via } \frac{\boxed{\begin{array}{c} \vdots \\ [x \Rightarrow x_0]\phi \end{array}}^{\,x_0}}{\forall x \phi}$$

Example

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1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\forall x P(x)$	premise	
3	$P(x_0) \rightarrow Q(x_0)$	$\forall x\ e$ 1	x_0
4	$P(x_0)$	$\forall x\ e$ 2	
5	$Q(x_0)$	$\rightarrow e$ 3,4	
6	$\forall x Q(x)$	$\forall x\ i$ 3–5	

Introduction of Existential Quantification

$$\frac{[x \Rightarrow t]\phi}{\exists x \phi}\,[\exists x\ i]$$

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In order to prove $\exists x \phi$, it suffices to find a term t as "witness", provided that t is free for x in ϕ.

Example

$$\forall x \phi \vdash \exists x \phi$$

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Recall: Definition of Models

A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

1. A *non-empty* set U, the *universe*;
2. ...

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Remark

Compare this with Traditional Logic.

Example

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A model $\mathcal{M}$ for $(\mathcal{F}, \mathcal{P})$ consists of:

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Remark

Compare this with Traditional Logic.

Because U must not be empty, we should be able to prove the sequent above.

Example (continued)

$$\forall x \phi \vdash \exists x \phi$$

Example (continued)

$$\forall x \phi \vdash \exists x \phi$$

1	$\forall x \phi$	premise
2	$[x \Rightarrow x]\phi$	$\forall x$ *e* 1
3	$\exists x \phi$	$\exists x$ *i* 2

Elimination of Existential Quantification

$$\dfrac{\exists x \phi \qquad \boxed{\begin{array}{c} [x \Rightarrow x_0]\phi \\ \vdots \\ \chi \end{array}}^{\,x_0}_{\,[x\Rightarrow x_0]\phi}}{\chi}\,[\exists e]$$

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Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds.

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Elimination of Existential Quantification

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$$

Making use of $\exists$

If we know $\exists x \phi$, we know that there exist at least one object x for which ϕ holds. We call that element x_0, and assume

Example

$$\forall x(P(x) \rightarrow Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

1	$\forall x(P(x) \rightarrow Q(x))$	premise	
2	$\exists x P(x)$	premise	
3	$P(x_0)$	assumption	x_0
4	$P(x_0) \rightarrow Q(x_0)$	$\forall x$ *e* 1	
5	$Q(x_0)$	$\rightarrow$ *e* 4,3	
6	$\exists x Q(x)$	$\exists x$ *i* 5	
7	$\exists x Q(x)$	$\exists x$ *e* 2,3–6	

Example

$$\forall x(P(x) \to Q(x)), \exists x P(x) \vdash \exists x Q(x)$$

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5	$Q(x_0)$	$\to$ *e* 4,3	
6	$\exists x Q(x)$	$\exists x$ *i* 5	
7	$\exists x Q(x)$	$\exists x$ *e* 2,3–6	

Note that $\exists x Q(x)$ within the box does not contain x_0, and therefore can be "exported" from the box.

Another Example

1	$\forall x(Q(x) \to R(x))$	premise	
2	$\exists x(P(x) \wedge Q(x))$	premise	
3	$P(x_0) \wedge Q(x_0)$	assumption	x_0
4	$Q(x_0) \to R(x_0)$	$\forall x\ e$ 1	
5	$Q(x_0)$	$\wedge e_2$ 3	
6	$R(x_0)$	$\to e$ 4,5	
7	$P(x_0)$	$\wedge e_1$ 3	
8	$P(x_0) \wedge R(x_0)$	$\wedge i$ 7, 6	
9	$\exists x(P(x) \wedge R(x)$	$\exists x\ i$ 8	
10	$\exists x(P(x) \wedge R(x))$	$\exists x\ e$ 2,3–9	

Variables must be fresh! This is not a proof!

1	$\exists x P(x)$	premise	
2	$\forall x(P(x) \rightarrow Q(x))$	premise	
3			x_0
4	$P(x_0)$	assumption	x_0
5	$P(x_0) \rightarrow Q(x_0)$	$\forall x\ e$ 2	
6	$Q(x_0)$	$\rightarrow e$ 5,4	
7	$Q(x_0)$	$\exists x\ e$ 1, 4–6	
8	$\forall y Q(y)$	$\forall y\ i$ 3–7	

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

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Some simple equivalences

$$\neg\forall x\phi \quad \dashv\vdash \quad \exists x\neg\phi$$

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$$\neg\forall x\phi \quad \dashv\vdash \quad \exists x\neg\phi$$
$$\neg\exists x\phi \quad \dashv\vdash \quad \forall x\neg\phi$$

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We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$\begin{aligned} \neg\forall x\phi &\dashv\vdash \exists x\neg\phi \\ \neg\exists x\phi &\dashv\vdash \forall x\neg\phi \\ \forall x\forall y\phi &\dashv\vdash \forall y\forall x\phi \end{aligned}$$

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$\begin{array}{rcl} \neg\forall x\phi & \dashv\vdash & \exists x\neg\phi \\ \neg\exists x\phi & \dashv\vdash & \forall x\neg\phi \\ \forall x\forall y\phi & \dashv\vdash & \forall y\forall x\phi \\ \exists x\exists y\phi & \dashv\vdash & \exists y\exists x\phi \end{array}$$

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Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$
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\neg\exists x\phi &\dashv\vdash \forall x\neg\phi \\
\forall x\forall y\phi &\dashv\vdash \forall y\forall x\phi \\
\exists x\exists y\phi &\dashv\vdash \exists y\exists x\phi \\
\forall x\phi \wedge \forall x\psi &\dashv\vdash \forall x(\phi \wedge \psi)
\end{aligned}
$$

Equivalences

Two-way-provable

We write $\phi \dashv\vdash \psi$ iff $\phi \vdash \psi$ and also $\psi \vdash \phi$.

Some simple equivalences

$$
\begin{array}{rcl}
\neg\forall x \phi & \dashv\vdash & \exists x \neg\phi \\
\neg\exists x \phi & \dashv\vdash & \forall x \neg\phi \\
\forall x \forall y \phi & \dashv\vdash & \forall y \forall x \phi \\
\exists x \exists y \phi & \dashv\vdash & \exists y \exists x \phi \\
\forall x \phi \land \forall x \psi & \dashv\vdash & \forall x (\phi \land \psi) \\
\exists x \phi \lor \exists x \psi & \dashv\vdash & \exists x (\phi \lor \psi)
\end{array}
$$

$\neg\forall x\phi \vdash \exists x\neg\phi$

1	$\neg\forall x\phi$	premise
2	$\neg\exists x\neg\phi$	assumption
3		x_0
4	$\neg[x \Rightarrow x_0]\phi$	assumption
5	$\exists x\neg\phi$	$\exists x\ i$ 4
6	$\bot$	$\neg e$ 5, 2
7	$[x \Rightarrow x_0]\phi$	PBC 4–6
8	$\forall x\phi$	$\forall x\ i$ 3–7
9	$\bot$	$\neg e$ 8, 1
10	$\exists x\neg\phi$	PBC 2–9

$\exists x \exists y \phi \vdash \exists y \exists x \phi$

Assume that x and y are different variables.

$\exists x \exists y \phi \vdash \exists y \exists x \phi$

Assume that x and y are different variables.

1	$\exists x \exists y \phi$	premise	
2	$[x \Rightarrow x_0](\exists y \phi)$	assumption	x_0
3	$\exists y([x \Rightarrow x_0]\phi$	def of subst (x, y different)	
4	$[y \Rightarrow y_0][x \Rightarrow x_0]\phi$	assumption	y_0
5	$[x \Rightarrow x_0][y \Rightarrow y_0]\phi$	def of subst (x, y, x_0, y_0 different)	
6	$\exists x[y \rightarrow y_0]\phi$	$\exists x\ i$ 5	
7	$\exists y \exists x \phi$	$\exists y\ i$ 6	
8	$\exists y \exists x \phi$	$\exists y\ e$ 3, 4–7	
9	$\exists y \exists x \phi$	$\exists x\ e$ 1, 2–8	

More Equivalences

Assume that x is not free in ψ

$$\begin{array}{rcl}
\forall x\phi \wedge \psi & \dashv\vdash & \forall x(\phi \wedge \psi) \\
\forall x\phi \vee \psi & \dashv\vdash & \forall x(\phi \vee \psi) \\
\exists x\phi \wedge \psi & \dashv\vdash & \exists x(\phi \wedge \psi) \\
\exists x\phi \vee \psi & \dashv\vdash & \exists x(\phi \vee \psi)
\end{array}$$

Central Result of Natural Deduction

$$\phi_1, \ldots, \phi_n \models \psi$$

iff

$$\phi_1, \ldots, \phi_n \vdash \psi$$

proven by Kurt Gödel, in 1929 in his doctoral dissertation

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proven by Kurt Gödel, in 1929 in his doctoral dissertation (just one year before his most famous result, the incompleteness results of mathematical logic)