09—Modal Logic

UIT2206: The Importance of Being Formal

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Necessity

- You are crime investigator and consider different suspects. You know that the victim Ms Smith had called the police.
  - Maybe the cook did it before dinner?
  - Maybe the maid did it after dinner?
- But: “The victim Ms Smith made a phone call *before* she was killed.” is *necessarily* true.
- “Necessarily” means in all possible scenarios (worlds) under consideration.

Notions of Truth

- Often, it is not enough to distinguish between “true” and “false”.
- We need to consider *modalities* if truth, such as:
  - necessity (“in all possible scenarios”)
  - morality/law (“in acceptable/legal scenarios”)
- Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.

Syntax of Basic Modal Logic

$$
\begin{aligned}
\phi \quad ::= \quad & \top \mid \bot \mid p \mid (\neg\phi) \mid (\phi \wedge \phi) \\
& \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \\
& \mid (\Box\phi) \mid (\Diamond\phi)
\end{aligned}
$$

Pronunciation and Examples

Pronunciation

If we want to keep the meaning open, we simply say “box” and “diamond”.
If we want to appeal to our intuition, we may say “necessarily” and “possibly”

Examples

$$(p \wedge \Diamond(p \rightarrow \Box\neg r))$$

$$\Box((\Diamond q \wedge \neg r) \rightarrow \Box p)$$

Kripke Models

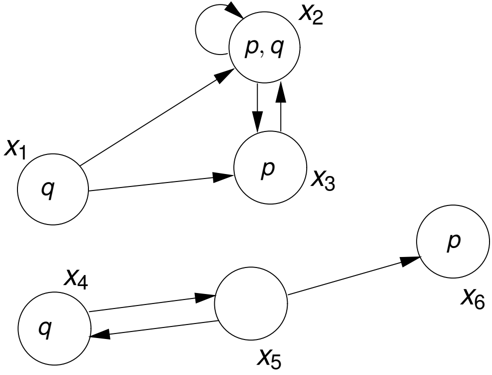
Definition

A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms A is specified by three things:

1. A W of *worlds*;
2. a relation R on W, meaning $R \subseteq W \times W$, called the *accessibility relation*;
3. a function $L : W \to A \to \{T, F\}$, called *labeling function*.

Example

$$W = \{x_1, x_2, x_3, x_4, x_5, x_6\}$$
$$R = \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6)\}$$
$$L = \{(x_1, \{q\}), (x_2, \{p, q\}), (x_3, \{p\}), (x_4, \{q\}), (x_5, \{\}), (x_6, \{p\})\}$$



When is a formula true in a possible world?

Definition

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ as the smallest relation satisfying:

- $x \Vdash \top$
- $x \nVdash \bot$
- $x \Vdash p$ iff $L(x)(p) = T$
- $x \Vdash \neg\phi$ iff $x \nVdash \phi$
- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
- ...

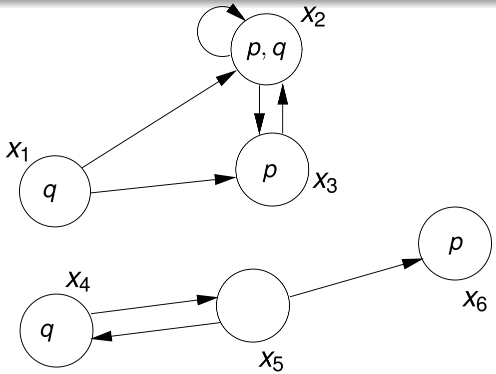
When is a formula true in a possible world?

Definition (continued)

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ as the smallest relation satisfying:

- ...
- $x \Vdash \phi \rightarrow \psi$ iff $x \Vdash \psi$, whenever $x \Vdash \phi$
- $x \Vdash \Box\phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \Vdash \phi$
- $x \Vdash \Diamond\phi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \phi$.

Example



- $x_1 \Vdash q$
- $x_1 \Vdash \Diamond q$, $x_1 \nVdash \Box q$
- $x_5 \nVdash \Box p$, $x_5 \nVdash \Box q$, $x_5 \nVdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$
- $x_6 \Vdash \Box\phi$ holds for all ϕ, but $x_6 \nVdash \Diamond\phi$ regardless of ϕ

Some Equivalences

- De Morgan rules: $\neg\Box\phi \equiv \Diamond\neg\phi$, $\neg\Diamond\phi \equiv \Box\neg\phi$.
- Distributivity of $\Box$ over $\wedge$:

$$\Box(\phi \wedge \psi) \equiv \Box\phi \wedge \Box\psi$$

- Distributivity of $\Diamond$ over $\vee$:

$$\Diamond(\phi \vee \psi) \equiv \Diamond\phi \vee \Diamond\psi$$

- $\Box\top \equiv \top$, $\Diamond\bot \equiv \bot$

Validity

Definition

A formula ϕ is valid if it is true in every world of every model, i.e. iff $\models \phi$ holds.

Examples of Valid Formulas

- All valid formulas of propositional logic
- $\neg\Box\phi \to \Diamond\neg\phi$
- $\Box(\phi \land \psi) \to \Box\phi \land \Box\psi$
- $\Diamond(\phi \lor \psi) \to \Diamond\phi \lor \Diamond\psi$
- Formula *K*: $\Box(\phi \to \psi) \to \Box\phi \to \Box\psi$.

1. Motivation
2. Basic Modal Logic
3. Logic Engineering
   - Valid Formulas wrt Modalities
   - Correspondence Theory

A Range of Modalities

In a particular context $\Box\phi$ could mean:

- It is necessarily true that ϕ
- It ought to be that ϕ
- Agent Q believes that ϕ
- Agent Q knows that ϕ

Since $\Diamond\phi \equiv \neg\Box\neg\phi$, we can infer the meaning of $\Diamond$ in each context.

A Range of Modalities

From the meaning of $\Box\phi$, we can conclude the meaning of $\Diamond\phi$, since $\Diamond\phi \equiv \neg\Box\neg\phi$:

$\Box\phi$	$\Diamond\phi$
It is necessarily true that ϕ	It is possibly true that ϕ
It ought to be that ϕ	It is permitted to be that ϕ
Agent Q believes that ϕ	ϕ is consistent with Q's beliefs
Agent Q knows that ϕ	For all Q knows, ϕ

Formula Schemes that hold wrt some Modalities

$\Box\phi$	$\Box\phi \to \phi$	$\Box\phi \to \Box\Box\phi$	$\Diamond\phi \to \Box\Diamond\phi$	$\Diamond\top$	$\Box\phi \to \Diamond\phi$	$\Box\phi \vee \Box\neg\phi$	$\Box(\phi \to \psi) \wedge \Box\phi \to \Box\psi$	$\Diamond\phi \wedge \Diamond\psi \to \Diamond(\phi \wedge \psi)$
It is necessary that ϕ	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
It ought to be that ϕ	$\times$	$\times$	$\times$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Agent Q believes that ϕ	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Agent Q knows that ϕ	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$

Modalities lead to Interpretations of R

$\Box\phi$	$R(x, y)$
It is necessarily true that ϕ	y is possible world according to info at x
It ought to be that ϕ	y is an acceptable world according to the information at x
Agent Q believes that ϕ	y could be the actual world according to Q's beliefs at x
Agent Q knows that ϕ	y could be the actual world according to Q's knowledge at x

Possible Properties of R

- reflexive: for every $w \in W$, we have $R(x, x)$.
- symmetric: for every $x, y \in W$, we have $R(x, y)$ implies $R(y, x)$.
- serial: for every x there is a y such that $R(x, y)$.
- transitive: for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$.
- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
- functional: for each x there is a unique y such that $R(x, y)$.
- linear: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$ or $y = z$ or $R(z, y)$.
- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$.
- equivalence: reflexive, symmetric and transitive.

Example

Consider the modality in which $\Box\phi$ means "Agent Q knows ϕ".

- Should R be reflexive?

Example

Consider the modality in which $\Box\phi$ means "Agent Q *believes* ϕ".

- Should R be reflexive?

Necessarily true and Reflexivity

Guess

R is reflexive if and only if $\Box\phi \to \phi$ is valid.

Correspondence Theory

Theorem 1

The following statements are equivalent:

- R is reflexive;
- All models with R as accessibility relation satisfy $\Box\phi \to \phi$ for all formulas ϕ.

Theorem 2

The following statements are equivalent:

- R is transitive;
- All models with R as accessibility relation satisfy $\Box\phi \to \Box\Box\phi$ for all formulas ϕ.