09—Modal Logic

UIT2206: The Importance of Being Formal

Martin Henz

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    - Maybe the cook did it before dinner?
    - Maybe the maid did it after dinner?
- But: “The victim Ms Smith made a phone call *before* she was killed.” is *necessarily* true.
- “Necessarily” means in all possible scenarios (worlds) under consideration.

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- We need to consider *modalities* if truth, such as:
  - necessity (“in all possible scenarios”)
  - morality/law (“in acceptable/legal scenarios”)
- Modal logic constructs a framework using which modalities can be formalized and reasoning methods can be established.

1. Motivation
2. Basic Modal Logic
   - Syntax
   - Semantics
3. Logic Engineering

Syntax of Basic Modal Logic

$$
\begin{aligned}
\phi \quad ::= \quad & \top \mid \bot \mid p \mid (\neg\phi) \mid (\phi \wedge \phi) \\
& \mid (\phi \vee \phi) \mid (\phi \rightarrow \phi) \\
& \mid (\Box\phi) \mid (\Diamond\phi)
\end{aligned}
$$

Pronunciation and Examples

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$$(p \wedge \Diamond(p \rightarrow \Box\neg r))$$

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$$(p \wedge \Diamond(p \rightarrow \Box\neg r))$$

$$\Box((\Diamond q \wedge \neg r) \rightarrow \Box p)$$

Kripke Models

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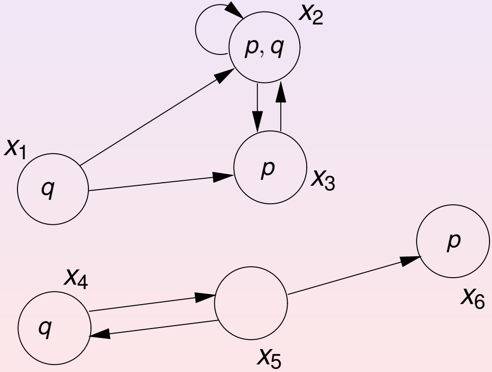
Definition

A model $\mathcal{M}$ of propositional modal logic over a set of propositional atoms A is specified by three things:

1. A W of *worlds*;
2. a relation R on W, meaning $R \subseteq W \times W$, called the *accessibility relation*;
3. a function $L : W \rightarrow A \rightarrow \{T, F\}$, called *labeling function*.

Example

$$
\begin{aligned}
W &= \{x_1, x_2, x_3, x_4, x_5, x_6\} \\
R &= \{(x_1, x_2), (x_1, x_3), (x_2, x_2), (x_2, x_3), (x_3, x_2), (x_4, x_5), (x_5, x_4), (x_5, x_6)\} \\
L &= \{(x_1, \{q\}), (x_2, \{p, q\}), (x_3, \{p\}), (x_4, \{q\}), (x_5, \{\}), (x_6, \{p\})\}
\end{aligned}
$$



When is a formula true in a possible world?

Definition

Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ as the smallest relation satisfying:

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- $x \Vdash p$ iff $L(x)(p) = T$

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- $x \Vdash \neg\phi$ iff $x \nVdash \phi$

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- $x \Vdash \phi \wedge \psi$ iff $x \Vdash \phi$ and $x \Vdash \psi$
- $x \Vdash \phi \vee \psi$ iff $x \Vdash \phi$ or $x \Vdash \psi$
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Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ as the smallest relation satisfying:

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Let $\mathcal{M} = (W, R, L)$, $x \in W$, and ϕ a formula in basic modal logic. We define $x \Vdash \phi$ as the smallest relation satisfying:

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- $x \Vdash \phi \to \psi$ iff $x \Vdash \psi$, whenever $x \Vdash \phi$
- $x \Vdash \Box\phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \Vdash \phi$

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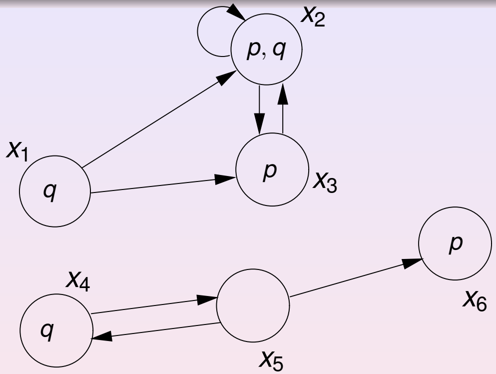
- $x \Vdash \Box\phi$ iff for each $y \in W$ with $R(x, y)$, we have $y \Vdash \phi$
- $x \Vdash \Diamond\phi$ iff there is a $y \in W$ such that $R(x, y)$ and $y \Vdash \phi$.

Example



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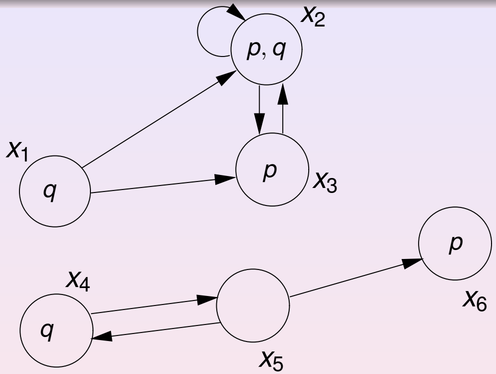

- $x_1 \Vdash q$

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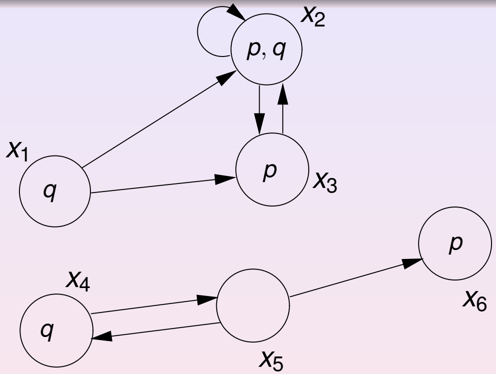

- $x_1 \Vdash q$
- $x_1 \Vdash \Diamond q$, $x_1 \nVdash \Box q$

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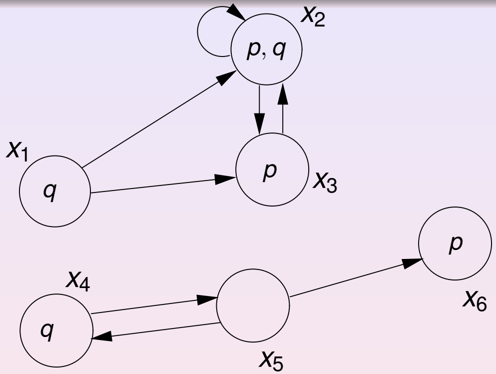

- $x_1 \Vdash q$
- $x_1 \Vdash \Diamond q$, $x_1 \nVdash \Box q$
- $x_5 \nVdash \Box p$, $x_5 \nVdash \Box q$, $x_5 \nVdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$

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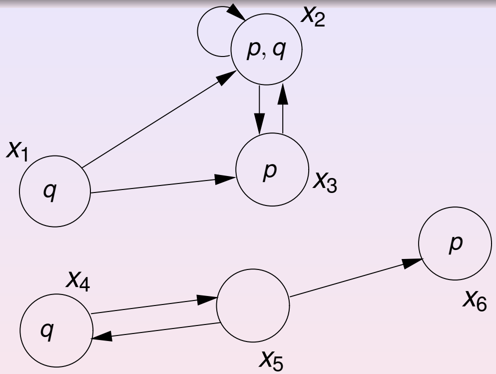

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- $x_5 \nVdash \Box p$, $x_5 \nVdash \Box q$, $x_5 \nVdash \Box p \vee \Box q$, $x_5 \Vdash \Box(p \vee q)$
- $x_6 \Vdash \Box\phi$ holds for all ϕ, but $x_6 \nVdash \Diamond\phi$ regardless of ϕ

Some Equivalences

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- $\Box\top \equiv \top$, $\Diamond\bot \equiv \bot$

Validity

Definition

A formula ϕ is valid if it is true in every world of every model, i.e. iff $\models \phi$ holds.

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- $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$
- $\Diamond(\phi \vee \psi) \rightarrow \Diamond\phi \vee \Diamond\psi$
- Formula K: $\Box(\phi \rightarrow \psi) \rightarrow \Box\phi \rightarrow \Box\psi$.

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Since $\Diamond\phi \equiv \neg\Box\neg\phi$, we can infer the meaning of $\Diamond$ in each context.

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From the meaning of $\Box\phi$, we can conclude the meaning of $\Diamond\phi$, since $\Diamond\phi \equiv \neg\Box\neg\phi$:

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It ought to be that ϕ	It is permitted to be that ϕ

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It ought to be that ϕ	It is permitted to be that ϕ
Agent Q believes that ϕ	ϕ is consistent with Q's beliefs
Agent Q knows that ϕ	For all Q knows, ϕ

Formula Schemes that hold wrt some Modalities

$\Box\phi$	$\Box\phi \to \phi$	$\Box\phi \to \Box\Box\phi$	$\Diamond\phi \to \Box\Diamond\phi$	$\Diamond\top$	$\Box\phi \to \Diamond\phi$	$\Box\phi \vee \Box\neg\phi$	$\Box(\phi \to \psi) \wedge \Box\phi \to \Box\psi$	$\Diamond\phi \wedge \Diamond\psi \to \Diamond(\phi \wedge \psi)$
It is necessary that ϕ	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
It ought to be that ϕ	$\times$	$\times$	$\times$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Agent Q believes that ϕ	$\times$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$
Agent Q knows that ϕ	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\times$	$\surd$	$\times$

Modalities lead to Interpretations of R

$\Box\phi$	$R(x, y)$
It is necessarily true that ϕ	y is possible world according to info at x
It ought to be that ϕ	y is an acceptable world according to the information at x
Agent Q believes that ϕ	y could be the actual world according to Q's beliefs at x
Agent Q knows that ϕ	y could be the actual world according to Q's knowledge at x

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- transitive: for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$.

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- transitive: for every $x, y, z \in W$, we have $R(x, y)$ and $R(y, z)$ imply $R(x, z)$.
- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.

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- Euclidean: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$.
- functional: for each x there is a unique y such that $R(x, y)$.

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- functional: for each x there is a unique y such that $R(x, y)$.
- linear: for every $x, y, z \in W$ with $R(x, y)$ and $R(x, z)$, we have $R(y, z)$ or $y = z$ or $R(z, y)$.

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- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$.

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- total: for every $x, y \in W$, we have $R(x, y)$ and $R(y, x)$.
- equivalence: reflexive, symmetric and transitive.

Example

Consider the modality in which $\Box\phi$ means "Agent Q knows ϕ".

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- Should R be reflexive?

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Consider the modality in which $\Box\phi$ means "Agent Q *believes* ϕ".

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Necessarily true and Reflexivity

Guess

R is reflexive if and only if $\Box\phi \to \phi$ is valid.

Correspondence Theory

Theorem 1

The following statements are equivalent:

- R is reflexive;
- All models with R as accessibility relation satisfy $\Box\phi \rightarrow \phi$ for all formulas ϕ.

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Theorem 2

The following statements are equivalent:

- R is transitive;
- All models with R as accessibility relation satisfy $\Box\phi \to \Box\Box\phi$ for all formulas ϕ.