Efficient Synergy Computation under MC-Nets

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Abstract. One of the key motivations behind the development of multi-agent systems is the ability of agents to cooperate, and to benefit from the synergies attained through such cooperation. However, despite the considerable research efforts directed at the computational aspects of cooperative game theory over the past decade, the issue of quantifying synergies has attracted relatively little attention. In this paper we consider the problem of computing three domain-independent techniques for measuring synergy: namely the Harsanyi Dividends; the Interaction Index; and the Synergy Index. We prove that each of these measures can be computed in polynomial time under a well-known representation scheme—called marginal-contribution nets (MC-nets)—originally developed to concisely represent coalitional games.

Keywords: synergy, cooperative game theory, Shapley value, MC-nets

1 Introduction

The Oxford English dictionary defines synergy as “the interaction or cooperation of two or more organizations, substances, or other agents to produce a combined effect greater than the sum of their separate effects”\(^1\). Arguably in this definition, the word “greater” does not necessarily mean “better”; there can be greater failings just as there can be greater successes. The phenomenon of synergy is evident and widespread in many diverse settings, from the natural world, through physics and chemistry, to human organisations [10, 21].

Synergy is a recurring theme in the literature on multi-agent systems and cooperative game theory [2, 13]. For instance, by identifying any overlaps in the agents’ tasks, it may be possible for some agents to leave certain tasks to other agents [4]; hence, in this context, the synergy represents the reduced total cost of task execution. In the multi-agent system literature, a specific model of synergy was recently proposed by Liemhetcharat and Veloso [13]. The main idea is based on the notion of a synergy graph—a weighted and undirected graph that defines pairwise compatibility between agents. Given that the compatibility is transitive, the synergy between any two agents is a function of the shortest distance between the nodes representing the agents’ types in the synergy graph. Pairwise synergies between agents were also studied building on the popular notion of Shapley value [18].

\(^1\) http://www.oxforddictionaries.com/definition/english/synergy.
Interestingly, however, while the pursuit of synergies is one of the key motivations for studying cooperative game theory and multi-agent systems [23], the development of computable synergy measures has attracted comparatively little research attention. This could be due to the complexity of measuring synergy when more than two agents are involved. Given a system of two agents—call them $a_1$ and $a_2$—perhaps the most obvious interpretation of the aforementioned dictionary definition of synergy is as follows: denoting by $v(\{a_1\})$ and $v(\{a_2\})$ the values achieved by individual agents, and by $v(\{a_1, a_2\})$ the value achieved by them working together, the synergy between $a_1$ and $a_2$ can be measured as: $v(\{a_1, a_2\}) - v(\{a_1\}) - v(\{a_2\})$. However, as soon as the number of agents exceeds two, a number of combinatorial issues come into play, rendering the measurement more ambiguous and challenging. For instance, when measuring the synergy in $\{a_1, a_2, a_3\}$, it is unclear whether, and how, we should exclude the synergies in the subgroups $\{a_1, a_2\}$, $\{a_1, a_3\}$, and $\{a_2, a_3\}$.

The most general, domain-independent measures of synergy include:

- **Synergy Coalition Groups** [3]—according to this measure, synergy in a given group, $C$, is understood as the value added over the value of the best partition of $C$ into smaller subgroups;

- **Harsanyi Dividends** [11]—when measuring the synergy in a group, $C$, this measure excludes the synergies that are present in all strict subsets of $C$; this is done in a way that resembles the inclusion-exclusion principle;

- **Interaction Index** [8]—originally developed in the field of fuzzy systems, this measure focuses on quantifying the average impact that the group makes when joining other groups. The basic idea resembles that of prominent solution concepts from coalitional game theory such as the Shapley value [20], the probabilistic Banzhaf index [6], and, more generally, Semivalues [22];

- **Synergy Index** [19]—this measure quantifies the degree by which a group’s performance deviates from expectation. More specifically, the value of any given group, $C$, is compared against the average impact that each member of $C$ makes when joining an arbitrary group.

In general, all of these measures are computationally challenging; to compute the synergy of a group of size $m$ in a system of $n$ agents, one needs to consider all $2^m - 1$ subgroups of this group (according to the Harsanyi Dividends or the Synergy Coalition Groups), or even all $2^n - 1$ possible groups in the entire system (according to the Interaction Index or the Synergy Index). One way around this combinatorial explosion is to consider alternative representations whereby the values of different groups are expressed in a more concise manner. We focus on one such representation, called **Marginal Contribution-nets (MC-nets)** [12]. This representation scheme, based on logical rules, is arguably the most influential of its kind in the literature on the computational aspects of coalition formation [2]. Out of all the above domain-independent measures of synergy, the efficient computation of only one measure has been considered in the literature to date, which is Synergy Coalition Groups; this measure was shown to be efficiently computable under the MC-net representation [16].
Against this background, we present in this paper the first positive computational results for the Harsanyi Dividends, the Interaction Index, and the Synergy Index. In particular, we prove that given a set of MC-net rules, \( R \), the synergy attained by an arbitrary group, \( C \subseteq A \), can be computed in \( O(|R|) \), \( O(|R| \times |A \setminus C|) \), and \( O(|R| \times |A|^2) \) time according to the Harsanyi Dividends, the Interaction Index, and the Synergy Index, respectively, where \( A \) is the set of all agents in the system. Importantly, while all of these synergy measures are related to the Shapley value—a prominent solution concept for cooperative games—they are nevertheless different, in the sense that the Shapley value does not attempt to measure the synergies obtained by a group, but rather the expected contribution made by a single agent. Also note that, although our exposition draws upon concepts originally developed for coalitional game theory, our results are applicable to any setting in which synergies may emerge in groups.

2 Preliminaries

Section 2.1 presents the main notation, whereas Section 2.2 presents the necessary background on Marginal Contribution-nets.

2.1 Basic Notation and Relevant Solution Concepts

Let \( A = \{a_1, \ldots, a_{|A|}\} \) denote the set of agents (or players), and let \( 2^A \) denote the set of all subsets of \( A \). An element of \( 2^A \), i.e., a subset (or a group) of agents, is called a coalition, and an arbitrary such coalition is often denoted by \( C, S, \) or \( T \). The coalition that includes all the agents in the game is called the grand coalition. A characteristic function, \( v \), assigns to every coalition \( C \subseteq A \) a real number representing the coalition’s value, i.e., its payoff. Formally, \( v : 2^A \rightarrow \mathbb{R} \). As common in the literature, we assume that \( v(\emptyset) = 0 \). A coalitional game in the characteristic function form is a pair \((A,v)\). The marginal contribution of an agent, \( a_i \), to a coalition, \( C \), is the difference in payoff that \( a_i \) creates when joining \( C \), which is simply: \( v(C \cup \{a_i\}) - v(C) \).

Assuming that the agents have joined the grand coalition one by one, there would be a total of \( |A|! \) possible such joining orders. For example, given \( A = \{a_1, a_2, a_3\} \), a possible joining order would be \((a_3, a_1, a_2)\), whereby \( a_3 \) is the first to “join”, followed by \( a_1 \) and then finally \( a_2 \). Under this assumption, if we wanted to divide the payoff of the grand coalition among all agents, a seemingly reasonable thing to do would be to compute the shares as follows: each agent, \( a_i \in A \), receives the difference in payoff created when \( a_i \) joins those who arrived before it (i.e., \( a_i \) receives its marginal contribution to the coalition of agents who arrived before it). For instance, given the joining order \((a_3, a_1, a_2)\), the share of \( a_3 \) would be \( v(\{a_3\}) \), the share of \( a_1 \) would be \( v(\{a_3, a_1\}) - v(\{a_3\}) \), and the share of \( a_2 \) would be \( v(\{a_3, a_1, a_2\}) - v(\{a_3, a_1\}) \).

The above payoff-division scheme seems reasonable when the agents join the grand coalition one by one, but what if this was not the case? A typical game in characteristic function form does not specify any order in which the agents have
joined the game; it is as if all the agents have joined the game together at once. To handle such a case, Shapley [20] proposed a payoff-division scheme—now known as the Shapley value—whereby each agent receives its average marginal contribution taken over all possible joining orders. More formally, let \( \Pi(A) \) denote the set of all possible permutations of \( A \). Furthermore, let \( C_{\pi_i} \) denote the coalition consisting of all the agents that precede \( a_i \) in the permutation \( \pi \in \Pi(A) \). Then, the Shapley value of agent \( a_i \) is defined as follows:

\[
SV_i(A, v) = \frac{1}{|A|!} \sum_{\pi \in \Pi(A)} [v(C_{\pi_i} \cup \{a_i\}) - v(C_{\pi_i})].
\]

The above equation can be rewritten as follows:

\[
SV_i(A, v) = \frac{1}{2^{|A|-1}} \sum_{C \subseteq A \setminus \{a_i\}} \frac{v(C \cup \{a_i\}) - v(C)}{\binom{|A| - 1}{|C|}}.
\]

Viewed from a different perspective, the Shapley value of \( a_i \) is the expected marginal contribution of \( a_i \) if the agents join the coalition one by one in a random joining order.

Another prominent payoff-division scheme which is similar to the Shapley value is the probabilistic Banzhaf index [6]. Specifically, the probabilistic Banzhaf index of an agent, \( a_i \in A \), is defined as the expected marginal contribution of \( a_i \) if each agent decides whether to join the coalition independently with probability \( \frac{1}{2} \). Formally, the probabilistic Banzhaf index of \( a_i \) is denoted by \( BI_i \), and is computed as follows:

\[
BI_i(A, v) = \frac{1}{2^{|A|-1}} \sum_{C \subseteq A \setminus \{a_i\}} [v(C \cup \{a_i\}) - v(C)].
\]

Whenever \( A \) and \( v \) are obvious from the context, we may sometimes write \( SV_i \) and \( BI_i \) instead of \( SV_i(A, v) \) and \( BI_i(A, v) \).

Both the Shapley value and the probabilistic Banzhaf index belong to a class of solution concepts called Semivalues [5]. To define this class, let \( \beta \) be a discrete probability distribution over coalition sizes, excluding the size of the grand coalition. To put it differently, let \( \beta : \{0, 1, \ldots, |A| - 1\} \rightarrow [0, 1] \) such that \( \sum_{k=0}^{|A|-1} \beta(k) = 1 \). Clearly, one can define \( \beta \) in infinitely many ways. Every such \( \beta \) defines a unique Semivalue, \( \phi^\beta_i(A, v) \), based on which the payoff of an agent \( a_i \) in a cooperative game \( (A, v) \) is computed as follows:

\[
\phi^\beta_i(A, v) = \sum_{0 \leq k \leq |A|-1} \beta(k) E_{C_k} [v(C_k \cup \{a_i\}) - v(C_k)],
\]

where \( C_k \) is a random variable over the coalitions of size \( k \) chosen from the set \( A \setminus \{a_i\} \) with uniform probability, and \( E_{C_k} \) is the expected value operator for this variable. Intuitively, \( \beta(k) \) can be interpreted as the probability that any agent makes a marginal contribution to a coalition of size \( k \). Alternatively, \( \phi^\beta_i(A, v) \)
can be computed as follows:
\[
\phi^\beta_i(A, v) = \sum_{0 \leq k \leq |A| - 1} \beta(k) \frac{1}{|A| - 1} \sum_{C \subseteq A \setminus \{a_i\}, |C| = k} [v(C \cup \{a_i\}) - v(C)].
\]

The Shapley value and the probabilistic Banzhaf index are in fact the two Semivalues, defined by the following two probability distributions:
\[
\beta_{\text{Shapley}}(k) = \frac{1}{|A|}, \quad \beta_{\text{Banzhaf}}(k) = \frac{(|A| - 1)}{2(|A| - 1)}.
\]

As we will see in Section 3, Semivalues are strongly related to an important measure of synergy, namely the Interaction Index, whereas the Shapley value is strongly related to a more recent measure of synergy, namely the Synergy Index.

2.2 Marginal Contribution-nets (MC-nets)

The Shapley value, like many other solution concepts, is hard to compute when the coalitional game is represented in characteristic function form. This is due to the number of possible coalitions to be considered, which grows exponentially with the number of agents involved. To handle this computational explosion, a number of alternative representation schemes have been proposed in the literature (see [2] for an overview). Perhaps the most influential among them is the Marginal Contribution-nets (MC-nets) scheme, introduced by Ieong and Shoham [12]. With this scheme, a game is represented by a set of agents, \( A \), and a set of rules, \( \mathcal{R} \). Every rule \( r \in \mathcal{R} \) is of the form \( \mathcal{F}_r \rightarrow \vartheta_r \), where \( \mathcal{F}_r \) is a propositional formula over a set of variable, \( \{b_1, \ldots, b_{|A|}\} \), and \( \vartheta_r \) is a real number. Here, the variable \( b_i \) corresponds to the agent \( a_i \). We say that a rule \( r = \mathcal{F}_r \rightarrow \vartheta_r \) is applicable to a given coalition, \( C \), if \( \mathcal{F}_r \) is satisfied by the following truth assignment:
\[
b_i = \text{true} \text{ if } a_i \in C \text{ and } b_i = \text{false} \text{ if } a_i \notin C.
\]
The value of a coalition is then determined by the rules that are applicable to it. More precisely, if we denote by \( \mathcal{R}^C \subseteq \mathcal{R} \) the set of rules that are applicable to \( C \), then the game defined by \( \mathcal{R} \) would have the following characteristic function:
\[
\forall C \subseteq A, \quad v(C) = \sum_{r \in \mathcal{R}^C} \vartheta_r.
\]
For example, the MC-net where \( \mathcal{R} = \{a_2 \rightarrow 3, a_1 \land a_2 \rightarrow 5\} \) corresponds to the characteristic function game \( (\{a_1, a_2\}, v) \) where \( v(\{a_1\}) = 0, v(\{a_2\}) = 3, \)
\( v(\{a_1, a_2\}) = 8 \). Intuitively, in this example, the rules state that whenever \( a_2 \) is present in a coalition, the value of that coalition increases by 3, and whenever \( a_1 \) and \( a_2 \) are present together in a coalition, its value increases by 5.

Ieong and Shoham [12] focused on a restricted version of their representation, known as basic MC-nets, where \( \mathcal{F}_r \) is a conjunction of literals, i.e., variables and their negations. Any such basic rule will be written as \( (P_r, N_r) \rightarrow \vartheta_r \), where \( P_r \) and \( N_r \) are the sets of agents that correspond to the positive and negative literals in \( \mathcal{F}_r \), respectively. For instance, given the basic rule \( r \) of which \( \mathcal{F}_r = \{b_1 \land b_4 \land b_5 \land \neg b_2 \land \neg b_8\} \), we have: \( P_r = \{a_1, a_4, a_5\} \) and \( N_r = \{a_2, a_8\} \), and
the rule is written as: \((P,N) \rightarrow \varnothing_r\). Whenever we deal with a game consisting of just a single basic rule, we will omit \(r\) from the subscript, and simply write the rule as: \((P,N) \rightarrow \varnothing\). Furthermore, we will denote by \(U\) the agents that are not in \(P\) nor in \(N\). That is, \(U = A \setminus (P \cup N)\). We will refer to the agents in \(P\), \(N\), and \(U\) as positive, negative, and neutral agents, respectively.

Ieong and Shoham showed that if the coalitional game is represented by a set of basic rules, \(R\), then the Shapley value can be computed in \(O(|A||R|)\) time. To see why this is the case, recall that one of the well-known properties of the Shapley value (and any Semivalue) is additivity. Similarly, the MC-net rules are also additive (in the sense that the value of a coalition is computed by adding up the values of all the rules that are applicable to it). Consequently, it can be shown that, when computing the Shapley value of a game represented by a set of basic rules, it is possible to think of each such rule as an independent game on its own—a game represented by just a single rule. Once the Shapley value has been computed for every such game, all that remains is to add up the results, and this would yield the Shapley value of the original game. We note that for any game, \((A,v)\), that is represented by a single basic rule, \((P,N) \rightarrow \varnothing\), we have:

\[
\text{if } P \neq \emptyset, \text{ then: } SV_i(A,v) = \begin{cases} 
\varnothing \times \frac{(|P|-1)!|N|!}{(|P|+|N|)!} & \text{if } i \in P, \\
-\varnothing \times \frac{P|(|N|-1)!}{(|P|+|N|)!} & \text{if } i \in N, \\
0 & \text{if } i \in U.
\end{cases}
\]

\[
\text{if } P = \emptyset, \text{ then: } SV_i(A,v) = \begin{cases} 
-\varnothing \times \frac{|U|}{|N|(|N|+|U|)} & \text{if } i \in N, \\
\varnothing \times \frac{1}{|N|(|N|+|U|)} & \text{if } i \in U.
\end{cases}
\]

We also note that, when the game is represented by a single basic rule, all positive agents are symmetric (i.e., they play identical roles in the game), and so the Shapley value of all these agents is identical (since the Shapley value is known to reward symmetric agents equally). The same argument applies for negative agents, and for neutral agents. Throughout the remainder of this paper, we restrict our attention to basic MC-net rules, and hence the word “basic” may sometimes be omitted.

### 3 Synergy Measures

In this section, we formally define the Harsanyi Dividends, the Interaction Index and the Synergy Index.

#### 3.1 The Harsanyi Dividends

One possible way to compute the synergy in a group of two agents, \(\{a_1, a_2\}\) would be as follows: \(v(\{a_1, a_2\}) - v(\{a_1\}) - v(\{a_2\})\). The basic idea behind the Harsanyi Dividends [11] is to extend this calculation to a group of arbitrary size,
using the inclusion-exclusion principle. We will denote the Harsanyi Dividends of an arbitrary coalition \( C \in 2^A \) by \( \psi^{Hd}(C) \), which is formally defined as:

\[
\psi^{Hd}(C) = \sum_{S \subseteq C} (-1)^{|C|-|S|} v(S).
\]  

(2)

Figure 1 illustrates the similarity between the inclusion-exclusion principle and the Harsanyi Dividends. On one hand, Figure 1(A) depicts an example of inclusion-exclusion; given 3 sets, \( X, Y, Z \), the size of \( X \cup Y \cup Z \) is computed as:

\[
|X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z|.
\]

On the other hand, Figure 1(B) depicts an example of Harsanyi Dividends; given three agents, \( a_1, a_2, a_3 \), the synergy in \( \{a_1, a_2, a_3\} \) is computed as:

\[
\psi^{Hd}(\{a_1, a_2, a_3\}) = v(\{a_1\}) + v(\{a_2\}) + v(\{a_3\}) - v(\{a_1, a_2\}) - v(\{a_1, a_3\}) - v(\{a_2, a_3\}) + v(\{a_1, a_2, a_3\}).
\]

3.2 The Interaction Index

Roughly speaking, the Interaction Index quantifies the added value that a group of agents brings to the entire game, by analyzing the impact that this group has when cooperating with other agents in various other coalitions. It was first proposed by Owen [17], and then rediscovered by Murofushi and Soneda [15] in the field of fuzzy control systems. The original definition of the Interaction Index is only for two agents, \( a_i, a_j \in N \) (here “ii” stands for “Interaction Index”):

\[
\psi^{ii}(\{a_i, a_j\}) = \sum_{T \subseteq N \setminus \{a_i, a_j\}} \frac{(|A| - |T| - 2)! |T|!}{(|A| - 1)!} \left[ v(T \cup \{a_i, a_j\}) - v(T \cup \{a_i\}) - v(T \cup \{a_j\}) + v(T) \right]
\]

Here, the Harsanyi Dividends, \( v(\{a_1, a_2\}) - v(\{a_1\}) - v(\{a_2\}) + v(\emptyset) \), is extended as follows: \( v(T \cup \{a_i, a_j\}) - v(T \cup \{a_i\}) - v(T \cup \{a_j\}) + v(T) \), thereby computing the synergy between \( a_1 \) and \( a_2 \) when they both join coalition \( T \).
The Interaction Index is then a weighted average of the aforementioned synergy, taken over all possible coalitions that both $a_1$ and $a_2$ can join. Grabisch and Roubens [8, 9] generalized the Interaction Index to a coalition $C$ of arbitrary size as:

$$\psi^{ii}(C) = \sum_{k=0}^{\lvert A \rvert - \lvert C \rvert} \frac{\beta(k)}{\binom{\lvert A \rvert - \lvert C \rvert}{k}} \sum_{S \subseteq A \setminus C : \lvert S \rvert = k} Q_S,$$

where $Q_S = \sum_{T \subseteq C} (-1)^{\lvert C \rvert - \lvert T \rvert} v(S \cup T)$, and $\beta$ is a discrete probability distribution over coalition sizes. In addition to game theory and fuzzy control systems, the above generalization has been studied in other fields, including: multi-criteria decision making, aggregation function theory, statistics, and data analysis [14].

### 3.3 The Synergy Index

Rahwan et al. [19] proposed a synergy measure called the Synergy Index, defined for every coalition $C \in 2^A$ as:

$$\psi^{si}(C) = v(C) - \sum_{a_i \in C} SV_i(A, v),$$

where $SV_i(A, v)$ is the average Shapley value of agent $a_i$ taken over all sub-games $(C, v) : C \in 2^A$. That is,

$$SV_i(A, v) = 2^{1-\lvert A \rvert} \sum_{S \subseteq 2^A \setminus \{S : a_i \in S\}} SV_i(S, v).$$

Intuitively, the Synergy Index compares the outcome of $C$ against the outcomes that the members of $C$ usually produce when working in an arbitrary group.

### 4 Algorithms

In this section, we present algorithms for computing the Harsanyi Dividends, the Interaction Index, and the Synergy Index. Importantly, it can easily be shown that all three measures satisfy Shapley’s Additivity axiom [20], e.g., if we know the Synergy Index of coalition $C$ in game $(A, v_1)$ and in game $(A, v_2)$, then we can easily compute the Synergy Index of $C$ in the game $(A, v_3) : v_3(S) = v_1(S) + v_2(S), \forall S \subseteq A$, by simply adding up the aforementioned two indices. To put it differently, the synergy measures are additive over games involving the same set of agents. Thus, when computing the synergy in a coalition, $C$, given a game of multiple MC-net rules, $R$, one can handle each rule independently (by creating a game represented by just that rule, and computing the synergy of $C$ in that game) and then add up the resulting synergies over all such single-rule games. Based on this, without loss of generality, we restrict our attention to games consisting of just a single, basic MC-net rule.
4.1 Computing the Harsanyi Dividends

One of the celebrated results of Harsanyi [11] was to prove that the Shapley value coincides with the payoff that results from the equal division of Harsanyi Dividends within each coalition. This relationship between Harsanyi Dividends and the Shapley value, as well as the fact that under the MC-net representation the Shapley value can be computed in polynomial time, suggest that Harsanyi Dividends also can be computed in polynomial time. This is what we show in our first theorem.

**Theorem 1.** Let \((A, v)\) be a coalitional game represented with a single MC-net rule \((P, N) \rightarrow \varnothing\). Then, for every \(C \subseteq A\):

\[
\psi_{Hd}(C) = \begin{cases} 
(−1)^{|C|−|P|} \varnothing & \text{if } C \cap U = \emptyset \land P \neq \emptyset \land P \subseteq C, \\
(−1)^{|C|−1} \varnothing & \text{if } C \cap U \neq \emptyset \land P = \emptyset, \\
0 & \text{otherwise}.
\end{cases}
\]

The time-complexity of this formula is \(O(|P| + |N|)\).

**Proof.** Let \(C \subseteq A\) be an arbitrary coalition. From Formula (2) we know that the Harsanyi Dividends of coalition \(C\) depends solely on the values of its subsets, \(S \subseteq C\). Recall that in a game represented with a single MC-net rule \((P, N) \rightarrow \varnothing\) it holds that:

\[
v(S) \neq 0 \iff (P \subseteq S \land N \cap S = \emptyset \land S \neq \emptyset).
\]

Moreover, all such coalitions have value \(\varnothing\). First, we will calculate the Harsanyi Dividends assuming that there are no neutral agents in \(C\), i.e., \(C \cap U = \emptyset\). Then, we will consider the case when there is at least one neutral agent in \(C\), i.e., \(C \cap U \neq \emptyset\).

First, assume that \(C \cap U = \emptyset\), i.e., \(C\) contains only positive and negative agents. If \(C\) does not contain all agents from \(P\), then we know from Formula (5) that every subset of \(C\) has a value of zero. Consequently, the Harsanyi Dividends also equal zero. On the other hand, if \(C\) does contain all agents from \(P\), i.e., \(P \subseteq C\), then \(P\) is the only subset that contains all positive agents and does not contain any negative ones. Thus, we know from the formula of the Harsanyi Dividends that:

\[
\psi_{Hd}(C) = (−1)^{|C|−|P|} v(P).
\]

We know from Formula (5) that \(v(P) = \varnothing\) if \(P \neq \emptyset\), and \(v(P) = 0\), otherwise. To conclude, we proved that if \(C\) does not contain any neutral agents, then \(\psi_{Hd}(C) = (−1)^{|C|−|P|} \varnothing\) if \(C\) contains all positive agents and the set of all positive agents is not empty.

Second, assume \(C \cap U \neq \emptyset\), i.e., \(C\) contains at least one neutral agent, \(u \in U\). Splitting Formula (2) into two sums—the sum over coalitions with agent \(u\) and coalitions without agent \(u\)—we obtain:

\[
\psi_{Hd}(C) = \sum_{S \subseteq C \setminus \{u\}} (−1)^{|C|−|S|−1} (v(S \cup \{u\}) − v(S)).
\]
Since \( u \) is a neutral agent, we know that its marginal contribution to every non-empty coalition equals zero:
\[
v(S \cup \{u\}) - v(S) = 0 \quad \text{for every } S \subseteq C \setminus \{u\}, S \neq \emptyset.
\] (7)

Taken together, formulas (6) and (7) imply that:
\[
\psi^{Hd}(C) = (-1)^{|C|-1}v(\{u\}).
\]

Now, we know from Formula (5) that \( v(\{u\}) = \vartheta \) if there are no positive literals in the formula, i.e., \( P = \emptyset \). On the other hand, if \( P \neq \emptyset \), then \( v(\{u\}) = 0 \) and consequently: \( \psi^{Hd}(C) = 0 \). This concludes the proof of Theorem 1. \( \square \)

### 4.2 Computing the Interaction Index

The Interaction Index is based on the Harsanyi Dividends combined with Semi-values. We have just derived the Harsanyi Dividends from MC-nets. Now, as the first step of our analysis we extend the formulas for the Shapley value from MC-nets to Semi-values.

**Theorem 2.** Let \((A,v)\) be a coalitional game represented with a single MC-net rule \((P,N) \rightarrow \vartheta\). Given a probability distribution \(\beta\), the Semivalue of \( a_i \in A \) is:

\[
\phi^\beta_i(A,v) = \begin{cases} 
\vartheta \sum_{j=|P|-1}^{|A|-1} \beta(j) \binom{|A|-|P|-1}{j} & \text{if } a_i \in P, \\
-\vartheta \sum_{j=|N|}^{A-|P|-1} \beta(j) \binom{|A|-|P|-|N|}{j} & \text{if } a_i \in N, \\
\vartheta \beta(0) & \text{if } a_i \in U \land P = \emptyset, \\
0 & \text{otherwise.}
\end{cases}
\]

The time-complexity of this formula is \(O(|A|)\).

**Proof.** Let us denote by \(SUM_i(k)\) the sum of marginal contributions of agent \( a_i \) to coalitions of size \( k \). More formally:
\[
SUM_i(k) = \sum_{C \subseteq A \setminus \{a_i\}, |C|=k} (v(C \cup \{a_i\}) - v(C)).
\]

Following Formula (1), we find that:
\[
\phi^\beta_i(A,v) = \sum_{0 \leq k \leq |A|-1} \beta(k) \frac{SUM_i(k)}{\binom{|A|-1}{k}}.
\] (8)

Let us fix a coalition size \( k \in \{0, \ldots, |A|-1\} \). We will calculate \( SUM_i(k) \) for all types of agents—positive, negative and neutral—separately.

Let \( p \in P \) be a positive agent. Agent \( p \) has a non-zero marginal contribution to \( C \subseteq A \setminus \{p\} \), if and only if:

- \( C \) contains all other positive agents, i.e., \( P \setminus \{p\} \subseteq C \); and
- \( C \) does not contain any negative agent, i.e., \( N \cap C = \emptyset \).
If both conditions are satisfied, then: \( v(C \cup \{p\}) - v(C) = \vartheta \). Let us calculate the number of coalitions that satisfy both conditions. If \( k < |P| - 1 \) or \( k > (|A| - 1) - |N| \), then there exists no coalition that satisfies both conditions (since a coalition of size \( k \) cannot contain all positive agents, or must contain at least one negative agent). Otherwise, coalition \( C \) is the union of the set \( P \setminus \{p\} \) and an arbitrary subset of neutral nodes of size \( j - (|P| - 1) \). Thus, there are \( \binom{|A| - |P| - |N|}{j - |P| + 1} \) such subsets, implying that:

\[
SUM_p(k) = \begin{cases} 
\vartheta \binom{|A| - |P| - |N|}{j - |P| + 1} & \text{if } |P| - 1 \leq k \leq |A| - 1 - |N|, \\
0 & \text{otherwise.}
\end{cases}
\]

This formula, together with (8), concludes the proof of correctness for the Semi-value of positive agents.

Analogously, consider a negative agent \( n \in N \). Agent \( n \) has non-zero marginal contribution only to coalitions that contain all positive agents and none of the negative ones. Since agent \( n \) changes the value of the coalition from \( \vartheta \) to \( 0 \), the marginal contribution equals \( -\vartheta \). If \( k < |P| \), then the coalition cannot contain all positive agents. On the other hand, if \( k > |A| - |N| \), then it must contain at least one negative agent. If \( k \geq |P| \) and \( k \leq |A| - |N| \), then there are \( \binom{|A| - |P| - |N|}{k} \) possible coalitions obtained as a union of \( P \) and a subset of neutral nodes. Consequently, we have:

\[
SUM_n(k) = \begin{cases} 
-\vartheta \binom{|A| - |P| - |N|}{|P|} & \text{if } |P| \leq k \leq |A| - |N|, \\
0 & \text{otherwise.}
\end{cases}
\]

This formula taken together with (8) is equivalent to the formula for the Semi-value for negative agents.

Finally, we will prove that for every neutral agent \( u \in U \), we have \( \phi_u^\beta(A,v) = 0 \) if \( P \neq \emptyset \), and \( \phi_u^\beta(A,v) = v \beta(0) \) otherwise. To this end, observe that the marginal contribution of a neutral agent \( u \) equals zero if \( P \neq \emptyset \), because the value of the coalition would depend solely on the positive and negative agents therein. On the other hand, if \( P = \emptyset \), then agent \( u \) has a marginal contribution of \( \vartheta \) to \( \emptyset \). Consequently:

\[ v(C \cup \{u\}) - v(C) \neq 0 \text{ iff } P = \emptyset \land C = \emptyset. \]

This concludes the proof of Theorem 2. \( \square \)

It is interesting to see how the formulas for the Interaction Index from MC-nets can be viewed as a combination of the formulas for Harsanyi Dividends and Semi values. In particular:

**Theorem 3.** Let \((A,v)\) be a coalitional game represented with a single MC-net rule \((P,N) \rightarrow \vartheta \). Then, for every \( C \subseteq A \):

\[
psi^u(C) = \begin{cases} 
\vartheta(-1)^{|C\cap U|} \sum_{k=0}^{|A| - |C|} \binom{|A| - |C|}{k} \beta(k) \binom{|U| - |P \setminus C|}{k} & \text{if } C \cap U = \emptyset, \\
\vartheta \beta(0)(-1)^{|C\cap U| - 1} & \text{if } C \cap U \neq \emptyset \land P = \emptyset, \\
0 & \text{otherwise.}
\end{cases}
\]
The time-complexity of this formula is $O(|A \setminus C|)$.

Proof. Let $C \subseteq A$ be an arbitrary coalition and let us fix a coalition size $k \in \{0, \ldots, |A| - 1\}$. Following Formula (3) we will calculate the following sum:

$$SUM_C(k) = \sum_{S \subseteq A \setminus C: |S| = k} Q_S = \sum_{S \subseteq A \setminus C, T \subseteq C} (-1)^{|C| - |T|} v(S \cup T).$$

(Note that $Q_S$ is equal to the Harsanyi Dividends of coalition $C$ in the game where values $v(T)$ are replaced with $v(S \cup T)$ for every $T \subseteq C$. Thus, we will conduct a similar analysis to the one from the proof of Theorem 1. We begin by considering the value $v(S \cup T)$ for the arbitrary coalitions $T \subseteq C$ and $S \subseteq A \setminus C$. Note that $S$ and $T$ are subsets of complementary sets. From the definition of MC-net rules, we know that $v(S \cup T)$ has non-zero value if and only if it contains all positive agents, and does not contain any negative agents. Thus, $P \subseteq (S \cup T)$ and $N \cap (S \cup T) = \emptyset$. This happens only if:

- $T$ contains all positive agents from $C$; and
- $S$ contains all positive agents from $N \setminus C$; and
- $T$ does not contain any negative agents from $C$; and
- $S$ does not contain any negative agents from $N \setminus C$.

We consider separately cases when $C$ does or does not contain neutral agents.

First, assume that $C \cap U = \emptyset$, i.e., that $C$ contains only positive and negative agents. The above conditions imply that $v(S \cup T)$ is non-zero only if $T = (C \cap P)$. Thus, we know that:

$$SUM_C(k) = (-1)^{|C| - |C \cap P|} \sum_{S \subseteq A \setminus C: |S| = k} v(S \cup (C \cap P)).$$

Now consider the conditions on $S$. We know that $S$ must contain all positive agents from $A \setminus C$, cannot contain any negative agents, and may contain neutral agents from $A \setminus C$. Thus, $S = (P \cap (A \setminus C)) \cup R$ where $R \subseteq U$ is an arbitrary subset of neutral agents; there are exactly $\binom{|U|}{|k - |P \cap C|}$ such coalitions $S$. Taken together, formulas (9) and (3) imply that:

$$\psi^u_i(C) = \partial(-1)^{|C \cap N|} \sum_{k=0}^{|A| - |C|} \beta(k) \binom{|U|}{k - |P \cap C|} \binom{|A| - |C|}{k}.$$

Second, assume that $C \cap U \neq \emptyset$, i.e., $C$ contains at least one neutral agent, $u \in U$. By splitting the $Q_S$ in Formula (9) into two sums—the sum over coalitions with agent $u$ and coalitions without agent $u$—we find that:

$$SUM_C(k) = \sum_{S \subseteq A \setminus C: |S| = k} \sum_{T \subseteq C \setminus \{u\}} (-1)^{|C| - |T|} (v(S \cup T \cup \{u\}) - v(S \cup T)).$$
Since \( u \) is a neutral agent, we know that its marginal contribution to every non-empty coalition \( S \cup T \) equals zero. Thus,

\[
SUM_C(k) = (-1)^{|C|-1}v(\{u\}).
\]

We also know from Formula (5) that \( v(\{u\}) = \emptyset \) if there are no positive literals, i.e., \( P = \emptyset \). In such a case we have:

\[
\psi^{ti}(C) = \beta(0)(-1)^{|C|-1}\emptyset
\]

On the other hand, if \( P \neq \emptyset \), then \( v(\{u\}) = 0 \) and consequently: \( \psi^{ti}(C) = 0 \). This concludes the proof of Theorem 3. \( \square \)

4.3 Computing the Synergy Index

For the Synergy Index, the following holds:

**Theorem 4.** Let \((A, v)\) be a coalitional game represented with a single MC-net rule \((P, N) \rightarrow \emptyset \). Then, \( \forall C \subseteq A \):

\[
\psi^{si}(C) = \left\{ \begin{array}{ll}
\frac{v(C)}{2^{|A|-|C|-1}} \sum_{k=0}^{N} \binom{|N|}{k} \binom{|N|-1}{k+1} & \text{if } P \neq \emptyset, \\
\frac{v(C)}{2^{|A|-1}} \left( \sum_{k=1}^{A} \binom{|A|-1}{k} \sum_{h=1}^{k} \frac{\binom{|N|-1}{h-1} \binom{|U|}{k-h}}{k} \right) & \text{otherwise.}
\end{array} \right.
\]

The time-complexity of this formula is \( O(|A|^2) \).

**Proof.** Assume that \( P \neq \emptyset \). For every \( S \subseteq A : P \not\subseteq S \), the value of every coalition in the subgame \((S, v)\) equals 0. Thus:

\[
SV_i = 2^{1-|A|} \sum_{S \subseteq A: a_i \in S \cap P \subseteq S} SV(S, v).
\]

Consider a positive agent, \( p \in P \). We know that for any subgame \((S, v)\) in which there are exactly \( k \) negative agents, all positive agents, and any number of neutral agents, the Shapley value of agent \( p \) equals:

\[
SV_p(S, v) = \frac{\partial(|P|-1)!k!}{(|P|+k)!} \quad \text{if } P \subseteq S, |N \cap S| = k, p \in P.
\]

There are precisely \( 2^{||S||} \binom{|N|}{k} \) different subsets of \( A \) that consist of \( k \) negative agents, all positive agents, and any number of neutral agents. Thus, for \( p \in P \):

\[
SV_p = 2^{1-|A|} \sum_{k=0}^{N} \frac{\partial(|P|-1)!k!2^{||U||}}{(|P|+k)!} \binom{|N|}{k} = \sum_{k=0}^{N} \frac{2^{||S||} \binom{|N|}{k} \partial}{(|P|+k)! \binom{|U|}{k} \binom{|P|}{k}}. \]

Analogously, consider a negative agent, \( n \in N \). We have:

\[
SV_n(S, v) = \frac{-\partial |P|!(k-1)!}{(|P|+k)!} \quad \text{if } P \subseteq S, |N \cap S| = k, n \in N.
\]
Thus, in this subgame, the Shapley value of agent $n$ equals

$$SV_n = 2^{1-|A|} \sum_{k=1}^{|N|} -\vartheta |P|! (k-1)! \frac{1}{(|P|+k)!} 2^{|U|} \binom{|N|-1}{k-1}. $$

Note that a neutral agent has no impact in any subgame $(S,v)$ since $P \neq \emptyset$. This implies that $SV_u = 0$ for every $u \in U$. Finally, from Formula (4) we know that:

$$\psi^u(C) = v(C) - (P|SV_p + |N||SV_n)$$

$$= v(C) - \frac{\vartheta}{2^{|A|-|U|-1}} \sum_{k=0}^{|N|} \binom{|N|}{k} \sum_{k=1}^{|N|} \binom{|N|-1}{k-1} \frac{\vartheta}{k}. $$

Having dealt with the case where $P \neq \emptyset$, next we deal with the case where $P = \emptyset$.

Consider a neutral agent, $u \in U$. The Shapley value of agent $u$ in a subgame, $(S,v)$, such that $|S| = k$ and $u \in S$, equals $\frac{\vartheta}{k}$. Therefore, we have:

$$SV_u = 2^{1-|A|} \sum_{k=1}^{|A|} \binom{|A|-1}{k-1} \frac{\vartheta}{k} \text{ if } u \in U.$$

Consider a negative agent, $n \in N$, and a subgame $(S,v)$ in which there are exactly $k$ agents (including $n$), $h$ of which are negative. Here, in a permutation $\pi$ where $n$ is the first negative agent, and there is at least one neutral agent before $n$, the marginal contribution of $n$ to the agents that precede it is $-\vartheta$.

Thus, in this subgame, the Shapley value of agent $n$ equals $SV_n(S,v) = \frac{(k-h)\vartheta}{kh}$. Consequently, we have:

$$SV_n = 2^{1-|A|} \sum_{k=1}^{|N|} \sum_{h=1}^{k-1} \binom{|N|-1}{h-1} \binom{|U|}{k} \frac{-(k-h)\vartheta}{kh} \text{ if } n \in N.$$ 

We conclude the proof by showing that from Formula (4):

$$\psi^u(C) = v(C) - (|U||SV_u + |N||SV_n)$$

$$= v(C) - \frac{\vartheta}{2^{|A|-1}} \binom{|A|-1}{k} - \sum_{k=1}^{|N|} \sum_{h=1}^{k-1} \binom{|N|-1}{h-1} \binom{|U|}{k} \frac{k-h}{kh}. $$

This concludes the proof of Theorem 4. 

5 Discussion & Future Work

In this paper, we showed that Harsanyi Dividends, the Interaction Index, and the Synergy Index can be computed in polynomial time if the values of groups are represented with MC-nets [12]. As a future work, one can explore the possibility of extending the above results to read-once MC-nets [7] and Algebraic Decision Diagrams [1]. The positive results obtained in this paper suggest that such an extension may yield polynomial-time algorithms for all three measures.
References