Introduction

We are interested in knowing “how many” elements there are in a set. This leads to the concept of two sets having “equally many” elements. Infinite sets with “equally many” elements have some surprising properties. Furthermore, not all infinite sets have “equally many” elements.
Cardinality

Let $A$ and $B$ be any sets. Set $A$ has the same cardinality as set $B$ if and only if there is a 1-1 correspondence from $A$ to $B$. 
Finite Sets

For any positive integer $n$, let the set consisting of the first $n$ positive integers be

$$C_n = \{1, 2, \ldots, n\}.$$  

A set is finite if and only if either it is empty or there is a positive integer $n$ such that there is a 1-1 correspondence from the set to $C_n$.

That is, a set is finite if and only if either it is empty or it has the same cardinality as $C_n$ for some positive $n$.

The cardinality of $\emptyset$ is 0 and the cardinality of $C_n$ is $n$. Thus the cardinality of a finite set is a non-negative integer.
Relexivity, Symmetry, and Transitivity of Cardinality

Theorem 7.5.1.

1. (Reflexive property of cardinality) A set has the same cardinality as itself.

2. (Symmetric property of cardinality) If a set has the same cardinality as another set, then the latter also has the same cardinality as the former.

3. (Transitive property of cardinality) If set $A$ has the same cardinality as set $B$ and set $B$ has the same cardinality as set $C$, then set $A$ has the same cardinality as set $C$. 
Reflexivity, Symmetry, and Transitivity of Cardinality

Proof:

The following proofs are about exhibiting a 1-1 correspondence from a set to another with respect to which it is said to have the same cardinality.
Reflexivity of Cardinality

Take the identity function which is a 1-1 correspondence from any non-empty set to itself.
Symmetry of Cardinality

If $f : A \rightarrow B$ is a 1-1 correspondence, then $f^{-1} : B \rightarrow A$ is also a 1-1 correspondence.

Because of the symmetric property, instead of saying a set has the same cardinality as another set, we can simply say the sets have the same cardinality.
Transitivity of Cardinality

If $f : A \to B$, $g : B \to C$, are 1-1 correspondences, the $gf : A \to C$ is also a 1-1 correspondence.
Distinguishing Finite and Infinite Sets: An Infinite Set and a Proper Subset Can Have the Same Cardinality

- Let $2\mathbb{Z}$ be the set of even integers.

- Clearly, $2\mathbb{Z} \subset \mathbb{Z}$. That is, $2\mathbb{Z} \subseteq \mathbb{Z}$ and $2\mathbb{Z} \neq \mathbb{Z}$.

- To show that $2\mathbb{Z}$ and $\mathbb{Z}$ have the same cardinality, we claim

$$f : 2\mathbb{Z} \to \mathbb{Z}, \ f(n) = \frac{n}{2}$$

is a 1-1 correspondence.
2\(\mathbb{Z}\) and \(\mathbb{Z}\) Have the Same Cardinality

The function \(f\) can be seen to be:

\[
\begin{array}{c|c}
2\mathbb{Z} & \mathbb{Z} \\
\hline
\vdots & \vdots & \vdots \\
-6 & \mapsto & -3 \\
-4 & \mapsto & -2 \\
-2 & \mapsto & -1 \\
0 & \mapsto & 0 \\
2 & \mapsto & 1 \\
4 & \mapsto & 2 \\
6 & \mapsto & 3 \\
\vdots & \vdots & \vdots \\
\end{array}
\]
2\(\mathbb{Z}\) and \(\mathbb{Z}\) Have the Same Cardinality

- Claim: \(f\) is well-defined. This is because for every \(n \in 2\mathbb{Z}\), we have (1) \(f(n) \in \mathbb{Z}\) and (2) \(f(n)\) is uniquely defined.

- Claim: \(f\) is 1-1. From 
  
  \[ f(m) = f(n) \]

  indeed we have

  \[ m = n. \]

- Claim: \(f\) is onto. For any \(n \in \mathbb{Z}\), we have \(2n \in 2\mathbb{Z}\) and \(f(2n) = n\).

- Remark. Intuitively we would think \(2\mathbb{Z}\) has “about half as many elements as” \(\mathbb{Z}\). But this intuition is wrong!
Another Example Concerning Infinite Cardinalities—Geometry
A short line segment and a long line segment have the same number of points. There is a 1-1 correspondence from the shorter line segment $DE$ to the longer line segment $BC$ by projecting a point $X \in DE$ to a point $Y \in BC$ from the point $A$. 
Another Example Concerning Infinite Cardinalities

- If you have $100, can you buy 100 XYZ shares, each costs $1 and also 100 ABC shares, each costs $1?
- Of course not, you need to have $200.
- Let $\mathbb{Z}^+$ be the set of positive integers and $|\mathbb{Z}^+| = \aleph_0$.
- If you have $\aleph_0$ dollars, can you buy $\aleph_0$ XYZ shares, each costs $1 and also $\aleph_0$ ABC shares, each costs $1?
- The answer is yes, if you know how to pay.
A Payment Plan

- Number your $\aleph_0$ dollars as
  
  $1, 2, 3, \cdots$

- Number the $\aleph_0$ XYZ shares as
  
  $X_1, X_2, X_3, \cdots$

- Number the $\aleph_0$ ABC shares as
  
  $A_1, A_2, A_3, \cdots$
• Pay as follows:

<table>
<thead>
<tr>
<th>Cash</th>
<th>XYZ</th>
<th>ABC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
</tr>
<tr>
<td>$2k-1$</td>
<td>$k$</td>
<td></td>
</tr>
<tr>
<td>$2k$</td>
<td>$k$</td>
<td></td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>

• Indeed, all the XYZ and ABC shares are paid!
Countably Infinite Sets

A set is **countably infinite** if and only if it has the same cardinality as \( \mathbb{Z}^+ \).

Other examples of countably infinite sets are \( \mathbb{Z} \), the set of even integers, the set of positive even integers, the set of odd integers, the set of positive odd integers, the set of rational numbers, the cartesian product of two countably infinite sets, etc.
Z is Countably Infinite

- List Z as follows:
  \[0, 1, -1, 2, -2, \cdots\]

- A 1-1 correspondence from \(Z^+\) is then obtained by numbering the list:
  \[0_1, 1_2, -1_3, 2_4, -2_5, \cdots, n_{2n}, -n_{2n+1}, \cdots\]

- That is,
  \[
  f(1) = 0, \quad f(2) = 1, \quad f(3) = -1, \quad f(4) = 2, \quad f(5) = -2, \cdots, \\
  f(2n) = n, \quad f(2n + 1) = -n, \cdots
  \]
Numbering a Grid — 1

We can number an infinite grid as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>6</td>
<td>10</td>
<td>15</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>14</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>4</td>
<td>8</td>
<td>13</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>12</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>5</td>
<td>11</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

The observation is: there is a 1-1 correspondence from \( \mathbb{Z}^+ \) to the set of grid positions.
Numbering a Grid — 2

We can also number an infinite grid as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>⋯</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>7</td>
<td>11</td>
<td>⋯</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>⋯</td>
<td>⋯</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>13</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
<td>14</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
</tr>
<tr>
<td>⋮</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
<td>⋯</td>
</tr>
</tbody>
</table>

The observation is: there is a 1-1 correspondence from $\mathbb{Z}^+$ to the set of grid positions.
Numbering a Grid — 3

We can also number an infinite grid as follows:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>10</td>
<td>11</td>
<td>...</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td>9</td>
<td>12</td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>8</td>
<td>13</td>
<td></td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>4</td>
<td>7</td>
<td>14</td>
<td></td>
<td></td>
<td></td>
<td>...</td>
</tr>
<tr>
<td>5</td>
<td>15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>...</td>
</tr>
</tbody>
</table>

The observation is: there is a 1-1 correspondence from $\mathbb{Z}^+$ to the set of grid positions.
Numbering a Grid — 4

We can also number an infinite grid as follows:

```
  1  2  3  4  5  ...
1  1  2  6  7 15  ...
2  3  5  8 14  ...
3  4  9 13  ...
4 10 12  ...
5 11  ...
:  ...
```

The observation is: there is a 1-1 correspondence from $\mathbb{Z}^+$ to the set of grid positions.
Numbering a Grid

Clearly there are other ways of setting up a 1-1 correspondence from $\mathbb{Z}^+$ to the set of grid positions.
Z \times Z is Countably Infinite

List Z \times Z as follows:

\[
\begin{array}{cccccccc}
\text{ } & 0 & 1 & -1 & 2 & -2 & \ldots \\
0 & (0,0)_1 & (0,1)_3 & (0,-1)_6 & (0,2)_10 & (0,-2)_15 & \ldots \\
1 & (1,0)_2 & (1,1)_5 & (1,-1)_9 & (1,2)_14 & \ldots & \ldots \\
-1 & (-1,0)_4 & (-1,1)_8 & (-1,-1)_{13} & \ldots & \ldots & \ldots \\
2 & (2,0)_7 & (2,1)_{12} & \ldots & \ldots & \ldots & \ldots \\
-2 & (-2,0)_{11} & \ldots & \ldots & \ldots & \ldots & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]

Observation: there is a 1-1 correspondence from the set of grid positions to Z \times Z.
That is, the 1-1 correspondence $f : \mathbb{Z}^+ \rightarrow \mathbb{Z} \times \mathbb{Z}$ is

$$f(1) = (0,0), f(2) = (1,0), f(3) = (0,1), \cdots$$
\( \mathbb{Q}^+ \) is Countably Infinite

Simply exhibit a 1-1 correspondence from \( \mathbb{Z}^+ \) to \( \mathbb{Q}^+ \).

1. Populate a grid with members of \( \mathbb{Q}^+ \).

\[
\begin{array}{c|cccccc}
 & 1 & 2 & 3 & 4 & 5 & \ldots \\
1 & 1/1 & 1/2 & 1/3 & 1/4 & 1/5 & \ldots \\
2 & 2/1 & 2/2 & 2/3 & 2/4 & 2/5 & \ldots \\
3 & 3/1 & 3/2 & 3/3 & 3/4 & 3/5 & \ldots \\
4 & 4/1 & 4/2 & 4/3 & 4/4 & 4/5 & \ldots \\
5 & 5/1 & 5/2 & 5/3 & 5/4 & 5/5 & \ldots \\
\vdots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\end{array}
\]
2. Unfortunately, this does not set up a 1-1 correspondence between grid positions and $\mathbb{Q}^+$: one rational number may occupy many grid positions. For example, the number 1 occupies all the diagonal positions.

3. The non-one-to-one problem can be overcome with an appropriate numbering strategy.

4. At each grid position, if the rational number has appeared in a previously numbered grid position, skip the grid position and move to the next grid position; otherwise, number the grid position.

5. For the from-south-west/to-north-east numbering strategy, the resulting 1-1 correspondence is
|   | 1   | 2   | 3   | 4   | 5   |   ...
|---|-----|-----|-----|-----|-----|------|
| 1 | \(
\frac{1}{11}
\) | \(
\frac{1}{23}
\) | \(
\frac{1}{35}
\) | \(
\frac{1}{49}
\) | \(
\frac{1}{511}
\) |   ...
| 2 | \(
\frac{2}{12}
\) | \(
\frac{2}{22}
\) | \(
\frac{2}{38}
\) | \(
\frac{2}{44}
\) | \(
\frac{2}{516}
\) |   ...
| 3 | \(
\frac{3}{14}
\) | \(
\frac{3}{27}
\) | \(
\frac{3}{33}
\) | \(
\frac{3}{415}
\) | \(
\frac{3}{520}
\) |   ...
| 4 | \(
\frac{4}{16}
\) | \(
\frac{4}{22}
\) | \(
\frac{4}{314}
\) | \(
\frac{4}{44}
\) | \(
\frac{4}{525}
\) |   ...
| 5 | \(
\frac{5}{110}
\) | \(
\frac{5}{213}
\) | \(
\frac{5}{319}
\) | \(
\frac{5}{424}
\) | \(
\frac{5}{55}
\) |   ...
|   |   ... |   ... |   ... |   ... |   ... |   ... |
That is,

\[ f(1) = 1, f(2) = 2, f(3) = \frac{1}{2}, f(4) = 3, f(5) = \frac{1}{3}, \ldots \]
Q is Countably Infinity

Given a 1-1 correspondence

\[ f : \mathbb{Z}^+ \rightarrow \mathbb{Q}^+ \]

and

\[ \mathbb{Q} = (\mathbb{Q}^+) \cup \{0\} \cup \mathbb{Q}^+, \]

we can construct a 1-1 correspondence \( g : \mathbb{Z}^+ \rightarrow \mathbb{Q} \) as follows:

\[
\begin{align*}
    g(1) &= 0 \\
    g(2k) &= f(k) \\
    g(2k + 1) &= -f(k) 
\end{align*}
\]
Q is Countably Infinity

Given that $f : \mathbb{Z}^+ \to \mathbb{Q}^+$ is a 1-1 correspondence:

$r_1, r_2, r_3, \cdots$

we can construct a 1-1 correspondence $g : \mathbb{Z}^+ \to \mathbb{Q}$ as follows:

$0, r_1, -r_1, r_2, -r_2, r_3, -r_3, \cdots$
Lemma. If $B$ is countably infinite and $f : B \to A$ is a 1-1 correspondence, then $A$ is countably infinite.

Proof:

There is a 1-1 correspondence $g : \mathbb{Z}^+ \to B$ since $B$ is countably infinite. Then $fg : \mathbb{Z}^+ \to A$ is a 1-1 correspondence thus $A$ is countably infinite.
Lemma. If $B$ is countably infinite and $f : A \to B$ is a 1-1 correspondence, then $A$ is countably infinite.

Proof:

There is a 1-1 correspondence $g : \mathbb{Z}^+ \to B$ since $B$ is countably infinite. Then $f^{-1}g : \mathbb{Z}^+ \to A$ is a 1-1 correspondence thus $A$ is countably infinite.
Countable Sets

A set is **countable** if and only if it is either finite or countably infinite.
A set is **uncountable** if and only if it is not countable.
Example: Countable Sets

- The empty set $\emptyset = \{\}$.  
- Singletons: $\{1\}$, $\{1.23\}$, $\{\sqrt{2}\}$, $\{\emptyset\}$, $\{\mathbb{R}\}$, $\{(0, 1)\}$, $\{(1, 2, 3)\}$, etc.
- Sets with 2 elements: $\{1, 2\}$, etc.
- Finite sets with 3 or more elements.
- Countably infinite sets: $\mathbb{Z}^+$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^+$, $\mathbb{Z} \times \mathbb{Z}$, etc.
Larger Infinities

- Let $|\mathbb{Z}^+| = \aleph_0$.

- We have seen that

$$|\mathbb{Q}| = |\mathbb{Q}^+| = |\mathbb{Z} \times \mathbb{Z}| = |X| = \aleph_0$$

where $X$ can be the set of integers, the set of even numbers, the set of positive even numbers, the set of odd numbers, the set of positive odd numbers, etc.

- It can be proved that the countably union of disjoint countable sets is countable.
• It can be proved that the cartesian product of finitely many countable sets is countable.

• Are there larger infinities?
Larger Infinities

Theorem 7.5.2 (Cantor) The set of all real numbers between 0 and 1 is uncountable.

Proof:

1. Consider the set $X = \{x \in \mathbb{R} \mid 0 < x < 1\}$.

2. Each $x \in X$ has a unique decimal representation if we disallow infinitely repeating 9’s.

3. That $X$ is uncountable can be proved by contradiction with the Cantor diagonalization argument.
4. Suppose $X$ is countable, then all the members of $X$ can be listed as follows:

\[
\begin{align*}
x_1 &= 0 \cdot a_{11}a_{12}a_{13}a_{14} \cdots \\
x_2 &= 0 \cdot a_{21}a_{22}a_{23}a_{24} \cdots \\
x_3 &= 0 \cdot a_{31}a_{32}a_{33}a_{34} \cdots \\
x_4 &= 0 \cdot a_{41}a_{42}a_{43}a_{44} \cdots \\
\vdots &= \vdots
\end{align*}
\]

5. Construct a number as follows

\[
d = 0 \cdot d_1d_2d_3d_4 \cdots
\]
such that

\[ d_i = \begin{cases} 
1 & a_{ii} \neq 1, \\
2 & a_{ii} = 1. 
\end{cases} \]

6. Note that for all \( i \in \mathbb{Z}^+ \), \( d_i \neq a_{ii} \).

7. Clearly \( d \in X \) but \( d \neq x_n \) for all \( n \in \mathbb{Z}^+ \).

8. This is a contradiction so \( X \) cannot be countable.
Any Subset of a Countable Set is Countable

Theorem 7.5.3: Any subset of a countable set is countable.

Proof:

• Let $B \subseteq A$ and $A$ is countable.

• Case 1. $B$ is finite. Then $B$ is countable.

• Case 2. $B$ is infinite. Then $A$ is countably infinite so we can write

$$A = \{a_1, a_2, a_3, \cdots \}.$$
• We then go through the elements of $A$ and pick out elements of $B$:

$$B = \{a_{k_1}, a_{k_2}, a_{k_3}, \cdots\}$$

where $1 \leq a_{k_1} < a_{k_2} < a_{k_3} < \cdots$.

• The function $f : \mathbb{Z}^+ \rightarrow B$ given as

$$f(n) = a_{kn}$$

is then a 1-1 correspondence from $\mathbb{Z}^+$ to $B$.

• That is, $B$ is countably infinite and so countable.
Illustration

Marked those elements of $B$ with "*":

$$A = \{a_1, a_2, a_3^*, a_4^*, a_5, a_6, a_7^*, a_8, a_9^*, \ldots\}.$$ 

That is,

$$B = \{a_3, a_4, a_7, a_9, \ldots\}.$$ 

Then

$$k_1 = 3, k_2 = 4, k_3 = 7, k_4 = 9, \ldots$$
The Cardinality of the Set of (All) Real Numbers

Lemma. $|(-1, 1)| = |\mathbb{R}|$.

- Recall

  $$\ (-1, 1) = \{x \in \mathbb{R} \mid -1 < x < 1\}.$$ 

- Note the notational “polymorphism”: $(-1, 1)$ here denotes an open interval and not an ordered pair.

- Claim. $f: \mathbb{R} \rightarrow (-1, 1)$ given below is a 1-1 correspondence:

  $$f(x) = \frac{x}{1 + |x|}.$$
Proof: \(|(-1, 1)| = |\mathbb{R}|\) 

1. \(f\) is well-defined as a function.

This amounts to checking that for all \(x \in \mathbb{R}\),

\[-1 < \frac{x}{1 + |x|} < 1\]

which is obviously true.

2. \(f\) is 1-1.

Let

\[f(x) = \frac{x}{1 + |x|} = \frac{y}{1 + |y|} = f(y).\]
Consider 3 cases: $x = 0$, $x < 0$, $x > 0$.

When $x = 0$, then $y = 0$ so $x = y$.

When $x > 0$, then $y > 0$. The equality becomes

$$\frac{x}{1+x} = \frac{y}{1+y}$$

and so $x = y$.

When $x < 0$, then $y < 0$. The equality becomes

$$\frac{x}{1-x} = \frac{y}{1-y}$$

and so $x = y$.

Since $f(x) = f(y)$ leads to $x = y$, $f$ is 1-1.
3. $f$ is onto.

If $-1 < y < 0$, $f\left(\frac{y}{1+y}\right) = y$.

If $y = 0$, $f(0) = y$.

If $0 < y < 1$, $f\left(\frac{y}{1-y}\right) = y$. 
The Cardinality of the Set of (All) Real Numbers

Lemma. There is a 1-1 correspondence from $\mathbb{R}$ to the open interval $(0, 1)$.

A 1-1 correspondence $g : (-1, 1) \rightarrow (0, 1)$ is easily obtained:

$$g(x) = \frac{x + 1}{2}.$$ 

The previously constructed $f : \mathbb{R} \rightarrow (-1, 1)$ is a 1-1 correspondence.

Thus there is a 1-1 correspondence $gf : \mathbb{R} \rightarrow (0, 1)$.

That is, $\mathbb{R}$ and $(0, 1)$ have the same cardinality.
A Common Confusion

- A useful concept for both finite and infinite sets:

  \[
  \bullet \leftrightarrow \bullet \\
  \bullet \leftrightarrow \bullet \\
  \bullet \leftrightarrow \bullet \\
  \vdots \leftrightarrow \vdots \\
  \bullet \leftrightarrow \bullet 
  \]

If there is a 1-1 correspondence from one set to another, then the sets have equally many elements.
• A useful concept for finite set but a useless concept for infinite sets:

If there is a 1-1 function $f$ from $A$ to $B$ and $f(A)$ is a proper subset of $B$, then $A$ has fewer elements than $B$ does if $A$ is finite.

$A : B :$

\[
\begin{array}{c}
\bullet & \rightarrow & \bullet \\
\bullet & \rightarrow & \bullet \\
\bullet & \rightarrow & \bullet \\
\vdots & \rightarrow & \vdots \\
\bullet & \rightarrow & \bullet \\
\bullet & \rightarrow & \bullet \\
\vdots & & \\
\bullet & & \\
\end{array}
\]

But if $A$ is infinite it is useless to say $A$ has fewer elements than $B$ does.
This is because it can also be true that there is a 1-1 function \( g \) from \( B \) to \( A \) such that \( g(B) \) is a proper subset of \( A \). What use it is to say \( A \) has fewer elements than \( B \) and also \( B \) has fewer elements than \( A \)?

\[
\begin{align*}
A &: \quad B : \\
\bullet &: \quad \bullet \\
\bullet &: \quad \bullet \\
\bullet &: \quad \bullet \\
\vdots &: \quad \vdots \\
\bullet &: \quad \bullet \\
\bullet &: \quad \bullet
\end{align*}
\]
Illustration: A Common Confusion

- Consider $\mathbb{Z}^+$ and $\mathbb{Z}$.

- Let $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}$ be $f(x) = x$.

- Clearly $f$ is a 1-1 function and $f(\mathbb{Z}^+) = \mathbb{Z}^+ \subseteq \mathbb{Z}$.

- Let $g : \mathbb{Z} \rightarrow \mathbb{Z}^+$ be

$$
g(x) = \begin{cases} 
1, & x = 0; \\
\overbrace{2\ldots2}^{x}, & x > 0; \\
\overbrace{-3\ldots3}^{x}, & x < 0.
\end{cases}
$$
• That is,

\[
\begin{array}{c|c}
Z : & Z^+ : \\
\hline
\vdots & \vdots \\
-3 & 333 \\
-2 & 33 \\
-1 & 3 \\
0 & 1 \\
1 & 2 \\
2 & 22 \\
3 & 222 \\
\vdots & \vdots \\
\end{array}
\]

• Clearly \( g \) is a 1-1 function and \( f(Z) \subset Z^+ \).