Methods of Proof. (Sep 18–22)

1. Prove Bernoulli’s inequality. (This is a very useful inequality in analysis.)
\[ \forall n \geq 1, \forall x \in (-1, \infty), (1 + x)^n \geq 1 + nx. \]

Note that here \((-1, \infty)\) does not denote an ordered pair, but denotes the set \(\{ x \in \mathbb{R} : x > -1 \}\).

2. Let \(F_0 = 0, F_1 = 1\), for \(n \geq 2\), \(F_n = F_{n-2} + F_{n-1}\). Prove that for \(n \geq 0\),
\[ F_n = \frac{A^n - B^n}{\sqrt{5}} \]

where \(A = \frac{1+\sqrt{5}}{2}, B = \frac{1-\sqrt{5}}{2}\).

Note. The sequence is known as the Fibonacci sequence and the value \((1+\sqrt{5})/2\) is known as the golden ratio.

3. Prove that the principle of strong mathematical induction implies the principle of mathematical induction. That is, if we have
\[ P(1) \land (\forall n \geq 1, P(1) \land \cdots \land P(n) \rightarrow P(n+1)) \rightarrow \forall n \geq 1, P(n); \]
then we have
\[ P(1) \land (\forall n \geq 1, P(n) \rightarrow P(n+1)) \rightarrow \forall n \geq 1, P(n). \]

4. Prove the converse of the previous question. That is, if we have
\[ P(1) \land (\forall n \geq 1, P(n) \rightarrow P(n+1)) \rightarrow \forall n \geq 1, P(n); \]
then we have
\[ P(1) \land (\forall n \geq 1, P(1) \land \cdots \land P(n) \rightarrow P(n+1)) \rightarrow \forall n \geq 1, P(n). \]

5. Do you think the following proves that any \(n \geq 1\) horses have the same color?

When \(n = 1\), then all the \(n\) horses have the same color.
Assume when there are \(n \geq 1\) horses, they all have the same color.

Consider \(n + 1\) horses. Number the horses from 1 to \(n + 1\). Horses number 1 to number \(n\) are \(n\) horses, so have the same color by the inductive hypothesis. Similarly, horses number 2 to number \(n + 1\) are \(n\) horses, so have the same color. This shows all horses have the same color as horse number 2. So \(n + 1\) horses have the same color.

By the principle of mathematical induction, all \(n \geq 1\) horses have the same color.