Relations (Partial Orders), Functions. (Oct 16–20)

1. Let \( A = \{ x \in \mathbb{Z} : -3 \leq x \leq 3 \} \). Consider the following relations on \( A \): \( R_\prec \) (the less than relation), \( R_\leq \) (the less than or equal relation), \( R_\equiv \) (the equal relation), \( R_\geq \) (the greater than or equal relation), \( R_\succ \) (the greater than relation), \( R_\divides \) (the “divides” relation).

   - Which of these relations are partial orders?
   - For relations that are partial orders, determine (if any) their least elements, greatest elements, minimal elements, maximal elements.

2. Let \( R \subseteq A \times B \) be a relation from \( A \) to \( B \). Show that \( R \) induces naturally a function \( f : P(A) \to P(B) \) and a function \( g : P(B) \to P(A) \).

   If \( A = \{0, 1, 2, 3\} \), \( B = \{a, b, c, d, e, f\} \), and \( R = \{0, 2\} \times \{a, c, e\} \), what are \( f \) and \( g \)?

3. Let \( f : A \to B \). That is, \( f \) is a function from \( A \) to \( B \). Prove the following claims.

   - If \( X \subseteq Y \subseteq A \), then \( f(X) \subseteq f(Y) \).
   - If \( X \subseteq Y \subseteq B \), then \( f^{-1}(X) \subseteq f^{-1}(Y) \).
   - For any \( X, Y \subseteq A \), \( f(X \cap Y) \subseteq f(X) \cap f(Y) \). Show that the inclusion can be proper.
   - For any \( X, Y \subseteq A \), \( f(X \cup Y) = f(X) \cup f(Y) \).
   - For any \( X, Y \subseteq B \), \( f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y) \).
   - For any \( X, Y \subseteq B \), \( f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y) \).

4. Consider the following functions \( f : \mathbb{R} \to \mathbb{R} \) where \( f \) is respectively the floor function \( (f(x) = \lfloor x \rfloor) \), the ceiling function \( (f(x) = \lceil x \rceil) \), the absolute value function \( (f(x) = |x|) \), the exponential function \( (f(x) = e^x) \), the identity function \( (f(x) = x) \), the quadratic function \( (f(x) = x^2) \), the cubic function \( (f(x) = x^3) \), and the sine function \( (f(x) = \sin(x)) \). For each function, determine if it is 1-1, onto, its range, and \( f^{-1}(\{x\}) \) for each \( x \in \mathbb{R} \).