This examination question booklet has 9 pages, including this cover page, and contains 16 questions. Answer all questions. All questions have equal weight.

You have 50 minutes to complete the examination. Use a B2 pencil to fill up the provided MCQ form. Leave Section A blank. Fill up Sections B and C.

After finishing, place the MCQ sheet on top of the question sheet and leave both on the table, when you exit the room.
**Question 1:** Consider the following two formulas in propositional logic:

\[
\phi = p \land q, \quad \psi = r \rightarrow (p \land q)
\]

Which of the following statement is true?

1. [A] If a truth assignment makes \(\phi\) true, it also makes \(\psi\) true.
2. [B] If a truth assignment makes \(\psi\) true, it also makes \(\phi\) true.
3. [C] If a truth assignment makes \(\phi\) false, it also makes \(\psi\) false.
4. [D] None of the above.

**Answer 1:**

1. [A] If the conclusion of an implication is true, the implication is true, regardless of the premise. The other alternatives are false by similar simple arguments.

**Question 2:** Which formula captures the following statement most accurately?

When the next large bank gets into trouble (\(t\)), the financial system collapses (\(c\)) unless the Fed buys the bank (\(b\)).

1. [A] \((-c \rightarrow b) \rightarrow t\)
2. [B] \((c \land \neg b) \rightarrow \neg t\)
3. [C] \((c \land \neg b) \rightarrow t\)
4. [D] \(t \rightarrow (\neg c \rightarrow b)\)
5. [E] \(t \rightarrow (\neg b \rightarrow c)\)

**Answer 2:**

2. [D] “\(b\) unless \(c\)” in English means “if not \(c\) then \(b\)”. Of course, \(t\) is the premise of an implication, where the unless-clause is the conclusion. Note that \(t \rightarrow (\neg c \rightarrow b)\) and \(t \rightarrow (\neg b \rightarrow c)\) are equivalent. Thus both answers D and E are valid.
Question 3: Let us say you managed to prove that
\[(p \land q) \land r \rightarrow (r \land q) \land p\]
is a tautology. Which known fact about propositional logic allows you to conclude that there is a natural deduction proof for
\[\vdash (p \land q) \land r \rightarrow (r \land q) \land p?\]

3 [A] Law of Excluded Middle (LEM)
3 [B] Modus Tollens (MT)
3 [C] soundness of natural deduction
3 [D] completeness of natural deduction
3 [E] compactness of natural deduction

Answer 3:
3 [D] The fact that a tautology is provable using natural deduction follows from completeness of natural deduction, see Section 1.4.4.

Question 4: There is no polynomial algorithm known that can test the satisfiability of arbitrary formulas in propositional logic. However, for clauses of a formula in conjunctive normal form (CNF), Lemma 1.43 (page 56) claims:

A clause \(L_1 \lor L_2 \lor \cdots \lor L_m\) is valid iff there are indices \(i, j\) with \(1 \leq i, j \leq m\) such that \(L_i \equiv \neg L_j\).

This suggests that we can test the validity of a formula by translating it to CNF, and then check each clause for validity. Which of the following statements is most accurate?

4 [A] There is no polynomial algorithm that can check any clause for validity.
4 [B] There is no polynomial algorithm that transforms any formula in propositional logic into CNF.
4 [C] To show that a conjunction of clauses is valid, it is not enough to show that each clause is valid.
4 [D] There is no polynomial algorithm known that test the satisfiability of arbitrary formulas, but there are polynomial algorithms that test the validity of arbitrary formulas.
4 [E] There is an error in the book; Lemma 1.43 does not hold.
4 [B] For example, translating the formula

\[(p_1 \land q_1) \lor (p_2 \land q_2) \lor \cdots \lor (p_n \land q_n)\]

to CNF leads to

\[(p_1 \lor \cdots \lor p_{n-1} \lor p_n) \land (p_1 \lor \cdots \lor q_{n-1} \lor q_n) \land (p_1 \lor \cdots \lor q_{n-1} \lor q_n) \land \cdots \land (p_1 \lor \cdots \lor q_{n-1} \lor q_n)\]

which has \(2^n\) clauses. Clearly, in order to generate an exponentially-sized formula, an exponential time is required. Other facts:

- Validity testing may be “harder” than satisfiability testing. It is in a class called co-NP, which is known to be not easier than NP, and is assumed to be harder than NP.
- If every clause of a CNF formula is valid, it holds for any truth assignment. The conjunction of all clauses of course then also holds for any truth assignment.
- One way of testing whether there is a pair of complementary literals in the clause is to keep an array \(a\) of \(n\) 3-valued variables, where \(n\) is the overall number of propositional variables. Initially, the value is “not seen”. You go through the clause literal by literal, and for each literal \(L\) you do:
  * if \(L = p\) and \(a[p] = \text{“not seen”}\), set \(a[p] = \text{“positive”}\),
  * if \(L = \neg p\) and \(a[p] = \text{“not seen”}\), set \(a[p] = \text{“negative”}\),
  * if \(L = p\) and \(a[p] = \text{“negative”}\), return “valid”,
  * if \(L = \neg p\) and \(a[p] = \text{“positive”}\), return “valid”.

This leads to a linear algorithm for clause validity checking. It requires space proportional to the number of propositional variables that might occur in the clause.
**Question 5:** Which of the sequents below are valid, and which ones are not valid?

1. \( \neg p, p \lor q \vdash q \)
2. \( p \lor q, \neg q \lor r \vdash p \lor r \)
3. \( \neg p \land \neg q \vdash \neg (p \lor q) \)
4. \( p \land \neg p \vdash \neg (r \rightarrow q) \land (r \rightarrow q) \)
5. \( \neg (q \lor q) \vdash p \)

5 A 1, 2, 3, 5 are valid, but 4 is not valid.
5 B 1, 2, 3 are valid, but 4 and 5 are not valid.
5 C All sequents are valid.
5 D 1, 2, 4, 5 are valid, but 3 is not valid.
5 E 3, 4, 5 are valid, but 1 and 2 are not valid.

**Answer 5:**

5 C It turns out that all these sequents are valid. If in doubt, the easiest way to check may be to construct a truth table, and use completeness.

**Question 6:** Let us say a formula \( D \) is in disjunctive normal form (DNF) if it is a disjunction of conjunctive clauses, where each conjunctive clause \( C \) is a conjunction of literals:

\[
L ::= p \mid \neg p \\
C ::= L \mid L \land C \\
D ::= C \mid C \lor D
\]

Which one of the following formulas is not in DNF?

6 A \( p \)
6 B \( (\neg p \land q) \lor r \)
6 C \( \neg (p \lor q) \)
6 D \( p \lor \neg r \lor (p \land \neg p) \)
6 E \( (q \land \neg q \land r) \lor (\neg p \land \neg r) \)
Answer 6:

6. C $\neg(p \lor q)$ has a negation over a disjunction. In DNF, only negation of literals is allowed. The other formulas are all in DNF.

Question 7: Recall that a formula $D$ is in disjunctive normal form (DNF) if it is a disjunction of conjunctive clauses, where each conjunctive clause $C$ is a conjunction of literals:

$$L ::= p \mid \neg p$$
$$C ::= L \mid L \land C$$
$$D ::= C \mid C \lor D$$

Which one of the following statements about conjunctive clauses of the form $L_1 \land L_2 \land \cdots \land L_m$ is correct?

7. A A conjunctive clause of the form $L_1 \land L_2 \land \cdots \land L_m$ is unsatisfiable iff there are indices $i, j$ with $1 \leq i, j \leq m$ such that $L_i$ is $\neg L_j$.

7. B A conjunctive clause of the form $L_1 \land L_2 \land \cdots \land L_m$ is valid iff there are indices $i, j$ with $1 \leq i, j \leq m$ such that $L_i$ is $\neg L_j$.

7. C A conjunctive clause of the form $L_1 \land L_2 \land \cdots \land L_m$ is satisfiable iff there are indices $i, j$ with $1 \leq i, j \leq m$ such that $L_i$ is $\neg L_j$.

7. D A conjunctive clause of the form $L_1 \land L_2 \land \cdots \land L_m$ is unsatisfiable iff for every index $i$ with $1 \leq i \leq m$ such that $L_i$ is an atom, there is an index $j$ with $1 \leq j \leq m$ such that $L_j$ is $\neg L_i$.

7. E None of the above.

Answer 7:

7. A If two of the conjuncts in a conjunctive clause are complementary literals of the atom $p$, then the conjunction is false for both of the possible assignments of $p$, and thus unsatisfiable. All other statements are not correct (considering that “iff” means “if and only if”).
Question 8: Which one of the following statements on propositional formulas does not hold?

8 A If \( \phi \) is valid, then \( \neg \phi \) is unsatisfiable.
8 B If \( \phi \) is satisfiable, then \( \neg \phi \) is unsatisfiable.
8 C If \( \phi \) is unsatisfiable, then \( \neg \phi \) is valid.
8 D If \( \phi \) is satisfiable and not valid, then \( \neg \phi \) is satisfiable and not valid.
8 E Exactly one of the above does not hold.

Answer 8:

8 B The negation of the satisfiable formula \( p \) is \( \neg p \), which is also satisfiable. Other facts:
  - If \( \phi \) is valid, then \( \neg \phi \) is unsatisfiable: Correct. Of course, if any assignment makes \( \phi \) true, no assignment makes \( \phi \) false, and thus no assignment makes \( \neg \phi \) true.
  - If \( \phi \) is unsatisfiable, then \( \neg \phi \) is valid: Correct. Similar argument.
  - If \( \phi \) is satisfiable and not valid, then \( \neg \phi \) is satisfiable and not valid: Correct. If \( \phi \) is satisfiable and not valid, there are assignments that make it true and there are assignments that make it false. The same holds then for \( \neg \phi \).
8 C Exactly one of the above does not hold: Correct, namely Alternative 2.

Question 9: Consider the following statements on formulas in propositional logic:

1. If \( \phi \) and \( \psi \) are both valid, then \( \phi \rightarrow \psi \) is valid.
2. If \( \phi \) and \( \psi \) are both satisfiable, then \( \phi \rightarrow \psi \) is satisfiable.
3. If \( \phi \) and \( \psi \) are both unsatisfiable, then \( \phi \rightarrow \psi \) is unsatisfiable.

Which of these statements are correct?

9 A none of Statement 1 through 3
9 B Statement 1 and Statement 3 only
9 C Statement 1 only
9 D Statement 1 and Statement 2 only
Answer 9:

1. If $\phi$ and $\psi$ are both valid, then $\phi \rightarrow \psi$ is valid: correct. $\phi \rightarrow \psi$ is true whenever $\psi$ is true, and $\psi$ is true for all assignments. Thus $\phi \rightarrow \psi$ is also true for all assignments.

2. If $\phi$ and $\psi$ are both satisfiable, then $\phi \rightarrow \psi$ is satisfiable: correct. $\phi \rightarrow \psi$ is true whenever $\psi$ is true, and $\psi$ is true for some truth assignment. $\phi \rightarrow \psi$ is also true for that assignment, and thus $\phi \rightarrow \psi$ is satisfiable. Assignment that makes it true.

3. If $\phi$ and $\psi$ are both unsatisfiable, then $\phi \rightarrow \psi$ is unsatisfiable: Not correct. If $\phi$ is false, then $\phi \rightarrow \psi$ is true. $\phi$ is false for all truth assignments. Thus $\phi \rightarrow \psi$ is true for all truth assignments. It is therefore valid, not unsatisfiable.

9 [D]

Question 10: Which variables occur free in the following formula?

$$P(x) \land \exists y \ Q(y, z) \lor \forall x \ P(x, w)$$

10 [A] z, w

10 [B] w

10 [C] x, z, w

10 [D] y, x

10 [E] z, w, x, y

Answer 10:

10 [C]
Question 11: Which one of the following statements is false?

11 A A term t is free for x in a formula in predicate logic φ, if x does not occur free in φ.

11 B A term t is free for x in a formula in predicate logic φ, if x is the only variable in t.

11 C A term t is free for x in a formula in predicate logic φ, if t has no variables.

11 D A term t is free for x in a formula in predicate logic φ, if there are no subformulas ∀y(⋯) or ∃y(⋯) in φ, such that y occurs in t.

11 E A term t is free for x in a formula in predicate logic φ, if there are no subformulas ∀x(⋯) or ∃x(⋯) in φ.

Answer 11:

11 E Counter-example: f(y) is not free for x in ∀yP(x), but there are no subformulas ∀x(⋯) or ∃x(⋯) in ∀yP(x). Other facts (consider Definition 2.8, page 106):

- A term t is free for x in a formula in predicate logic φ, if x does not occur free in φ: correct. There is no free x leaf in φ.
- A term t is free for x in a formula in predicate logic φ, if x is the only variable in t: correct. No free x leaf occurs in the scope of ∀x or ∃x; they are all bound, of course.
- A term t is free for x in a formula in predicate logic φ, if t has no variables: correct. “...for any variable y” of course holds then, vacuously.
- A term t is free for x in a formula in predicate logic φ, if there are no subformulas ∀y(⋯) or ∃y(⋯) in φ, such that y occurs in t: correct. This is more general than Definition 2.8.

Question 12: What is the result of applying the following substitution?

∀x(P(x) → Q(y))[f(y)/x]

12 A ∀x(P(f(y)) → Q(y))

12 B ∀w(P(w) → Q(y))

12 C ∀w(P(f(y)) → Q(y))

12 D ∀x(P(x) → Q(y))

12 E ∀w(P(w) → Q(f(y))))
Answer 12:
12  D  There are no free occurrences of $x$, and thus the formula remains unchanged.

Question 13: Consider the following attempt to prove the validity of the sequent

$\neg \exists x P(x) \vdash \forall x \neg P(x)$

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</tr>
<tr>
<td>2</td>
<td>$x_0$</td>
</tr>
<tr>
<td>3</td>
<td>$P(x_0)$</td>
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<tr>
<td>4</td>
<td>$\exists x P(x)$</td>
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<tr>
<td>5</td>
<td>$\bot$</td>
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<tr>
<td>6</td>
<td>$\neg P(x_0)$</td>
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<tr>
<td>7</td>
<td>$\forall x \neg P(x)$</td>
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13  A  The proof is valid.

13  B  Step 5 is wrong.

13  C  Step 6 is wrong.

13  D  Step 7 is wrong.

13  E  Step 4 is wrong.

Answer 13:
**Question 14:** Predicate logic is undecidable. Which of the statements below captures the undecidability result that was discussed in the lectures?

14 A There are sentences in predicate logic that are both valid and invalid.
14 B There is no algorithm that decides for every formula, whether the formula is valid or not.
14 C There are sentences in predicate logic that are both satisfiable and unsatisfiable.
14 D There is a valid formula such that there is no algorithm that proves its validity.
14 E There is no algorithm that decides if a given model satisfies a given formula.

**Answer 14:**
14 B This is Theorem 2.22, page 133. The other statements are not connected to this undecidability result.

**Question 15:** Consider the following formula in predicate logic, where we assume that equality is always modelled extensionally (page 131).

\[ \forall x \forall y \forall z \forall w (\neg(x = y \lor x = z \lor y = z) \rightarrow (w = x \lor w = y \lor w = x)) \]

Which of the following statements holds?

15 A Every model has at least 3 elements.
15 B Every model has at most 3 elements.
15 C Every model has exactly 3 elements.
15 D Every model has at least 4 elements.
15 E Every model has exactly 4 elements.

**Answer 15:**
15 C The correct answer would have been:
B’: Every model has at most 2 elements.
This is due to the fact that \(w\) must be equal to \(x\) or \(y\), regardless of the choices for \(x\) and \(y\) (and \(z\), which does not appear on the right hand side).
Answer B is the closest to B’:

- 1 element: models meet formula: covered by both B and B’
- 2 elements: models meet formula: covered by both B and B’
- 3 elements: the given answer B includes such models, but the models do not meet the formula.
- 4 and more elements: models do not meet formula: covered by both B and B’

That means alternative B is wrong for only one case, whereas all other alternatives are wrong for an infinite number of cases.

This situation came about because of a typo. I intended to write $w = x \lor w = y \lor w = z$, which would have made B the correct answer.

**Question 16**: Consider a predicate logic over $(\mathcal{F}, \mathcal{P})$, where $\mathcal{F} = \emptyset$ and $\mathcal{P} = \{P\}$, where $P$ is ternary. Furthermore, consider the following formula $\phi$:

$$\forall x \forall y \exists z \ P(x, y, z)$$

and a model $\mathcal{M}$ such that $A^\mathcal{M} = \{a, b\}$ and $P^\mathcal{M} = \{(a, a, b), (a, b, a), (a, b, b), (b, b, a), (b, b, b)\}$.

Which of the following statements is correct?

16 [A] $\mathcal{M}$ is a model for $(\mathcal{F}, \mathcal{P})$ and $\phi$ is a formula over $(\mathcal{F}, \mathcal{P})$, but $\mathcal{M}$ does not satisfy $\phi$.

16 [B] $\phi$ is not a formula over $(\mathcal{F}, \mathcal{P})$.

16 [C] $\mathcal{M}$ is not a model for $(\mathcal{F}, \mathcal{P})$.

16 [D] $\mathcal{M}$ is a model for $(\mathcal{F}, \mathcal{P})$ and $\phi$ is a formula over $(\mathcal{F}, \mathcal{P})$, and $\mathcal{M}$ satisfies $\phi$.

**Answer 16:**

16 [A] Nothing wrong with $\mathcal{M}$ and $\phi$, but $\mathcal{M}$ does not satisfy $\phi$. $P(x, y, z)$ does not hold for any $z$ if $x$ is $b$ and $y$ is $a$. 

12