CS 5209: Foundation in Logic and AI

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January 14, 2010
1. Introduction to Foundation in Logic and AI
2. Brief Introduction to CS5209
3. Administrative Matters
4. Propositional Calculus: Declarative Sentences
5. Propositional Calculus: Natural Deduction
1. Introduction to Foundation in Logic and AI
   - Origins of Mathematical Logic
   - Propositional Calculus
   - Predicate Calculus
   - Theorem Proving and Logic Programming
   - Systems of Logic

2. Brief Introduction to CS5209

3. Administrative Matters

4. Propositional Calculus: Declarative Sentences

5. Propositional Calculus: Natural Deduction
What is logic?

1. the branch of philosophy dealing with forms and processes of thinking, especially those of inference and scientific method,

2. a particular system or theory of logic [according to 1].

(from “The World Book Dictionary”)

1

2
Greek origins

The ancient Greek formulated rules of logic as *syllogisms*, which can be seen as precursors of formal logic frameworks.
Example of Syllogism

Premise
All men are mortal.

Premise
Socrates is a man.

Conclusion
Therefore, Socrates is mortal.
Historical Notes

Logic traditions in Ancient Greece

**Stoic logic:** Centers on propositional logic; can be traced back to Euclid of Megara (400 BCE)

**Peripatetic logic:** Precursor of predicate logic; founded by Aristotle (384–322 BCE), focus on syllogisms
Logic Throughout the World

Indian logic: Nyaya school of Hindu philosophy, culminating with Dharmakirti (7th century CE), and Gangea Updhyyya of Mithila (13th century CE), formalized inference

Chinese logic: Gongsun Long (325–250 BCE) wrote on logical arguments and concepts; most famous is the “White Horse Dialogue”; logic typically rejected as trivial by later Chinese philosophers

Islamic logic: Further development of Aristotelian logic, culminating with Algazel (1058–1111 CE)

Medieval logic: Aristotelian; culminating with William of Ockham (1288–1348 CE)

Traditional logic: Port-Royal Logic, influential logic textbook first published in 1665
Remarks on Ockham

Ockham’s razor (in his own words)

For nothing ought to be posited without a reason given, unless it is self-evident or known by experience or proved by the authority of Sacred Scripture.
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English: Entities should not be multiplied without necessity.
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Built-in Skepticism
As a result of this ontological parsimony, Ockham states that human reason cannot prove the immortality of the soul nor the existence, unity, and infinity of God.
**Propositional Calculus**

**Study of atomic propositions**

Propositions are built from sentences whose internal structure is not of concern.

**Building propositions**

Boolean operators are used to construct propositions out of simpler propositions.
Example for Propositional Calculus

Atomic proposition
One plus one equals two.

Atomic proposition
The earth revolves around the sun.

Combined proposition
One plus one equals two and the earth revolves around the sun.
Goals and Main Result

Meaning of formula
Associate meaning to a set of formulas by assigning a value true or false to every formula in the set.

Proofs
Symbol sequence that formally establishes whether a formula is always true.

Soundness and completeness
The set of provable formulas is the same as the set of formulas which are always true.
Uses of Propositional Calculus

**Hardware design**

The production of logic circuits uses propositional calculus at all phases; specification, design, testing.

**Verification**

Verification of hardware and software makes extensive use of propositional calculus.

**Problem solving**

Decision problems (scheduling, timetabling, etc) can be expressed as satisfiability problems in propositional calculus.
Predicate Calculus: Central ideas

Richer language
Instead of dealing with atomic propositions, predicate calculus provides the formulation of statements involving sets, functions and relations on these sets.

Quantifiers
Predicate calculus provides statements that all or some elements of a set have specified properties.

Compositionality
Similar to propositional calculus, formulas can be built from composites using logical connectives.
The meaning of programs such as

\[\text{if } x \geq 0 \text{ then } y := \sqrt{x} \text{ else } y := |x|\]

can be captured with formulas of predicate calculus:

\[\forall x \forall y (x' = x \land (x \geq 0 \rightarrow y' = \sqrt{x}) \land (\neg(x \geq 0) \rightarrow y' = |x|))\]
Other Uses of Predicate Calculus

**Specification:** Formally specify the purpose of a program in order to serve as input for software design,

**Verification:** Prove the correctness of a program with respect to its specification.
Example for Specification

Let \( P \) be a program of the form

```
while a <> b do
    if a > b then a := a - b else a := b - a;
```

The specification of the program is given by the formula

\[
\{a \geq 0 \land b \geq 0\} P \{a = \text{gcd}(a, b)\}
\]
Theorem proving

Formal logic has been used to design programs that can automatically prove mathematical theorems.

Logic programming

Research in theorem proving has led to an efficient way of proving formulas in predicate calculus, called resolution, which forms the basis for logic programming.
### Other Systems of Logic

<table>
<thead>
<tr>
<th>System</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Three-valued logic</strong></td>
<td>A third truth value (denoting “don’t know” or “undetermined”) is often useful.</td>
</tr>
<tr>
<td><strong>Intuitionistic logic</strong></td>
<td>A mathematical object is accepted only if a finite construction can be given for it.</td>
</tr>
<tr>
<td><strong>Temporal logic</strong></td>
<td>Integrates time-dependent constructs such as (“always” and “eventually”) explicitly into a logic framework; useful for reasoning about real-time systems.</td>
</tr>
</tbody>
</table>
Introduction to Foundation in Logic and AI

Brief Introduction to CS5209
- Style: Broad, elementary, rigorous
- Method: From Theory to Practice
- Overview of Module Content

Administrative Matters

Propositional Calculus: Declarative Sentences

Propositional Calculus: Natural Deduction
Style: Broad, elementary, rigorous

**Broad:** Cover a good number of logical frameworks

**Elementary:** Focus on a minimal subset of each framework

**Rigorous:** Cover topics formally, preparing students for advanced studies in logic in computer science
Method: From Theory to Practice

Cover theory and back it up with practical exercises that apply the theory and give new insights.
Overview of Module Content

1. Propositional calculus (3 lectures, including today)
2. Predicate calculus (3 lectures)
3. Verification by Model Checking (1 lectures)
4. Program Verification (2 lectures)
5. Modal Logics (2 lectures; to be confirmed)
Use www.comp.nus.edu.sg/~cs5209 and IVLE

Textbook

Assignments (one per week, starting next week; marked)

Self-assessments (occasional; not marked)

Discussion forums (IVLE)

Announcements (IVLE)

Webcast (IVLE)

Blog (IVLE, just for fun)

Tutorials (one per week); register!
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4. Propositional Calculus: Declarative Sentences
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Declarative Sentences

The language of propositional logic is based on *propositions* or *declarative sentences*.

Declarative Sentences

Sentences which one can—in principle—argue as being true or false.
Examples

1. The sum of the numbers 3 and 5 equals 8.
2. Jane reacted violently to Jack’s accusations.
3. Every natural number > 2 is the sum of two prime numbers.
4. All Martians like pepperoni on their pizza.
Not Examples

- Could you please pass me the salt?
- Ready, steady, go!
- May fortune come your way.
Example 1.1

If the train arrives late and there are no taxis at the station then John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, there were taxis at the station.
Example 1.2

If *it is raining* and *Jane does not have her umbrella with her* then *she will get wet.*

*Jane is not wet.*

*It is raining.*

Therefore, *Jane has her umbrella with her.*
Focus on Structure

We are primarily concerned about the structure of arguments in this class, not the validity of statements in a particular domain.
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We therefore simply abbreviate sentences by letters such as $p$, $q$, $r$, $p_1$, $p_2$ etc.
From Concrete Propositions to Letters

Example 1.1

If the train arrives late and there are no taxis at the station then John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, there were taxis at the station.

becomes
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becomes

Letter version

If \(p\) and not \(q\), then \(r\). Not \(r\). \(p\). Therefore, \(q\).
Example 1.2

If *it is raining* and *Jane does not have her umbrella with her* then *she will get wet.*

*Jane is not wet.*

*It is raining.*

Therefore, *Jane has her umbrella with her.*

has
Example 1.2

If it is raining and Jane does not have her umbrella with her then she will get wet.

Jane is not wet.

It is raining.

Therefore, Jane has her umbrella with her.

has

the same letter version

If p and not q, then r. Not r. p. Therefore, q.
Sentences like “If $p$ and not $q$, then $r$.” occur frequently. Instead of English words such as “if...then”, “and”, “not”, it is more convenient to use symbols such as $\rightarrow$, $\land$, $\neg$. 
Logical Connectives

¬: negation of $p$ is denoted by $\neg p$

∨: disjunction of $p$ and $r$ is denoted by $p \lor r$, meaning at least one of the two statements is true.

∧: conjunction of $p$ and $r$ is denoted by $p \land r$, meaning both are true.

→: implication between $p$ and $r$ is denoted by $p \rightarrow r$, meaning that $r$ is a logical consequence of $p$. $p$ is called the antecedent, and $r$ the consequent.
Example 1.1 Revisited

From Example 1.1

If *the train arrives late* and *there are no taxis at the station* then *John is late for his meeting.*
Example 1.1 Revisited

From Example 1.1

If the train arrives late and there are no taxis at the station then John is late for his meeting.

Symbolic Propositions

We replaced “the train arrives late” by $p$ etc

The statement becomes: If $p$ and not $q$, then $r$. 
Example 1.1 Revisited

From Example 1.1
If the train arrives late and there are no taxis at the station then John is late for his meeting.

Symbolic Propositions
We replaced “the train arrives late” by p etc
The statement becomes: If p and not q, then r.

Symbolic Connectives
With symbolic connectives, the statement becomes:

\[ p \land \neg q \rightarrow r \]
Introduction to Foundation in Logic and AI

Brief Introduction to CS5209

Administrative Matters

Propositional Calculus: Declarative Sentences

Propositional Calculus: Natural Deduction

- Sequents
- Rules for Conjunction
- Rules for Double Negation and Implication
- Rules for Disjunction
Objective

We would like to develop a calculus for reasoning about propositions, so that we can establish the validity of statements such as Example 1.1.
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Idea

We introduce proof rules that allow us to derive a formula $\psi$ from a number of other formulas $\phi_1, \phi_2, \ldots, \phi_n$. 


Objective

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Idea

We introduce proof rules that allow us to derive a formula $\psi$ from a number of other formulas $\phi_1, \phi_2, \ldots, \phi_n$.

Notation

We write a sequent $\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$ to denote that we can derive $\psi$ from $\phi_1, \phi_2, \ldots, \phi_n$. 
Example 1.1 Revisited

English

If the train arrives late and there are no taxis at the station then John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, there were taxis at the station.
Example 1.1 Revisited

**English**

If the train arrives late and there are no taxis at the station then John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, there were taxis at the station.

**Sequent**

\[ p \land \neg q \rightarrow r, \neg r, p \vdash q \]
Example 1.1 Revisited

English

If the train arrives late and there are no taxis at the station then John is late for his meeting.

John is not late for his meeting.

The train did arrive late.

Therefore, there were taxis at the station.

Sequent

\[ p \land \neg q \rightarrow r, \neg r, p \vdash q \]
What Next?

Sequent

\[ p \land \neg q \rightarrow r, \neg r, p \vdash q \]
What Next?

Sequent

\[ p \land \neg q \rightarrow r, \neg r, p \vdash q \]

Remaining task

Develop a set of proof rules that allows us to establish such sequents.
**Rules for Conjunction**

**Introduction of Conjunction**

\[
\phi \quad \psi \\
\hline
[\wedge i] \\
\phi \wedge \psi
\]
**Rules for Conjunction**

**Introduction of Conjunction**

\[
\begin{align*}
\phi & \quad \psi \\
\hline
\phi \land \psi
\end{align*}
\]

**Elimination of Conjunction**

\[
\begin{align*}
\phi \land \psi & \quad \phi \land \psi \\
\hline
\phi & \quad \psi
\end{align*}
\]
Example of Proof

To show

\[ p \land q, r \vdash q \land r \]

How to start?

\[ p \land q \quad r \\
q \land r \]
Proof Step-by-Step

1. \( p \land q \) (premise)
Proof Step-by-Step

1. \( p \land q \) (premise)
2. \( r \) (premise)
Proof Step-by-Step

1. $p \land q$ (premise)
2. $r$ (premise)
3. $q$ (by using Rule $\land e_2$ and Item 1)
Proof Step-by-Step

1. $p \land q$ (premise)
2. $r$ (premise)
3. $q$ (by using Rule $\land e_2$ and Item 1)
4. $q \land r$ (by using Rule $\land i$ and Items 3 and 2)
Graphical Representation of Proof

\[
p \land q \\
\hline 
q \\
\hline 
q \land r
\]

\[
\begin{array}{c}
p \land q \\
\hline 
q \\
\hline 
q \land r
\end{array}
\]

\[
\frac{p \land q}{\[\land e_2]} r \\
\frac{q}{\[\land i]} q \land r
\]
Graphical Representation of Proof

\[
p \land q \quad [\land e_2] \quad r
\]
\[
q \quad [\land i]
\]
\[
q \land r
\]

Find the parts of the corresponding sequent:

\[p \land q, r \vdash q \land r\]
Graphical Representation of Proof

\[ p \land q \]

\[ \frac{}{q} \quad \land e_2 \]

\[ r \]

\[ q \land r \]

\[ \frac{}{q \land r} \quad \land i \]
Graphical Representation of Proof

\[ p \land q \]
\[ \quad [\land e_2] \quad r \]
\[ \quad q \]
\[ \quad [\land i] \quad q \land r \]

Find the parts of the corresponding proof:

1. \( p \land q \) (premise)
2. \( r \) (premise)
3. \( q \) (by using Rule \( \land e_2 \) and Item 1)
4. \( q \land r \) (by using Rule \( \land i \) and Items 3 and 2)
Where are we heading with this?

- We would like to prove sequents of the form

$$\phi_1, \phi_2, \ldots, \phi_n \vdash \psi$$
Where are we heading with this?

- We would like to prove sequents of the form
  \[ \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \]
- We introduce rules that allow us to form “legal” proofs
Where are we heading with this?

- We would like to prove sequents of the form
  \[ \phi_1, \phi_2, \ldots, \phi_n \vdash \psi \]
- We introduce rules that allow us to form “legal” proofs
- Then any proof of any formula \( \psi \) using the premises \( \phi_1, \phi_2, \ldots, \phi_n \) is considered “correct”.
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- Can we say that sequents with a correct proof are somehow “valid”, or “meaningful”?
- What does it mean to be meaningful?
- Can we say that any meaningful sequent has a valid proof?
- ...but first back to the proof rules...
Rules of Double Negation

\[\neg\neg\phi \Rightarrow \phi \quad \text{[\neg\neg e]} \]

\[\phi \Rightarrow \neg\neg\phi \quad \text{[\neg\neg i]} \]
Rule for Eliminating Implication

\[
\begin{align*}
\phi & \quad \phi \rightarrow \psi \\
\frac{}{\psi} & \quad \text{[} \rightarrow \text{]}
\end{align*}
\]
**Rule for Eliminating Implication**

\[
\phi \quad \phi \rightarrow \psi \\
\hline
[\rightarrow e] \\
\psi
\]

**Example**

\( p: \) It rained.  
\( p \rightarrow q: \) If it rained, then the street is wet.

We can conclude from these two that the street is indeed wet.
Another Rule for Eliminating Implication

The rule

\[
\phi \quad \phi \rightarrow \psi \\
\hline
\psi
\]

\[
[\rightarrow e]
\]

is often called “Modus Ponens” (or MP)
Another Rule for Eliminating Implication

The rule

\[
\phi \quad \phi \to \psi \\
\hline
\psi
\]

is often called “Modus Ponens” (or MP)

Origin of term

“Modus ponens” is an abbreviation of the Latin “modus ponendo ponens” which means in English “mode that affirms by affirming”. More precisely, we could say “mode that affirms the antecedent of an implication”.

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The Twin Sister of Modus Ponens

The rule

\[
\phi \quad \phi \rightarrow \psi \\
\hline
\psi
\]

is called “Modus Ponens” (or MP)
The Twin Sister of Modus Ponens

The rule

\[
\phi \quad \phi \rightarrow \psi \\
\hline
[\rightarrow e] \\
\psi
\]

is called “Modus Ponens” (or MP)

A similar rule

\[
\phi \rightarrow \psi \quad \neg \psi \\
\hline
[MT] \\
\neg \phi
\]

is called “Modus Tollens” (or MT).
The Twin Sister of Modus Ponens

The rule

\[
\phi \rightarrow \psi \quad \neg \psi \\
\hline
\neg \phi
\]

is called “Modus Tollens” (or MT).
The Twin Sister of Modus Ponens

The rule

\[ \phi \rightarrow \psi \quad \neg \psi \]

\[ \underline{\neg \phi} \]

[MT]

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Origin of term

“Modus tollens” is an abbreviation of the Latin “modus tollendo tollens” which means in English “mode that denies by denying”. More precisely, we could say “mode that denies the consequent of an implication”.

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The rule

\[ \phi \rightarrow \psi \quad \neg \psi \]

\[ \underline{\neg \phi} \]

[MT]
Example

\[ p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q \]
Example

\[ p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q \]

1. \[ p \rightarrow (q \rightarrow r) \] premise
Example

\[ p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q \]

1 \hspace{1em} p \rightarrow (q \rightarrow r) \quad \text{premise}
2 \hspace{1em} p \quad \text{premise}
Example

\[ p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q \]

1. \( p \rightarrow (q \rightarrow r) \)  
   premise
2. \( p \)  
   premise
3. \( \neg r \)  
   premise
Example

\[ p \to (q \to r), p, \neg r \vdash \neg q \]

1. \[ p \to (q \to r) \]  premise
2. \[ p \]  premise
3. \[ \neg r \]  premise
4. \[ q \to r \]  \( \to_e 1,2 \)
Example

\[ p \rightarrow (q \rightarrow r), p, \neg r \vdash \neg q \]

1. \( p \rightarrow (q \rightarrow r) \)  
   premise
2. \( p \)  
   premise
3. \( \neg r \)  
   premise
4. \( q \rightarrow r \)  
   \( \rightarrow_e 1,2 \)
5. \( \neg q \)  
   MT 4,3
How to *introduce* implication?

Compare the sequent (MT)

\[ p \rightarrow q, \neg q \vdash \neg p \]

with the sequent

\[ p \rightarrow q \vdash \neg q \rightarrow \neg p \]
How to *introduce* implication?

Compare the sequent (MT)

\[ p \rightarrow q, \neg q \vdash \neg p \]

with the sequent

\[ p \rightarrow q \vdash \neg q \rightarrow \neg p \]

The second sequent should be provable, but we don’t have a rule to introduce implication yet!
A Proof We Would Like To Have

\[
p \rightarrow q \vdash \neg q \rightarrow \neg p
\]

1. \( p \rightarrow q \)  \hspace{1cm} \text{premise}
2. \( \neg q \)  \hspace{1cm} \text{assumption}
3. \( \neg p \)  \hspace{1cm} \text{MT 1,2}
4. \( \neg q \rightarrow \neg p \)  \hspace{1cm} \rightarrow_i 2–3
A Proof We Would Like To Have

\[ p \rightarrow q \vdash \neg q \rightarrow \neg p \]

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
<th>Justification</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( p \rightarrow q )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( \neg q )</td>
<td>assumption</td>
</tr>
<tr>
<td>3</td>
<td>( \neg p )</td>
<td>MT 1,2</td>
</tr>
<tr>
<td>4</td>
<td>( \neg q \rightarrow \neg p )</td>
<td>( \rightarrow i ) 2–3</td>
</tr>
</tbody>
</table>

We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.
A Proof We Would Like To Have

\[ p \rightarrow q \vdash \neg q \rightarrow \neg p \]

1. \( p \rightarrow q \)  
   \( p \rightarrow q \) \hspace{1cm} \text{premise}

2. \( \neg q \)  
   \( \neg q \) \hspace{1cm} \text{assumption}

3. \( \neg p \)  
   \( \neg p \) \hspace{1cm} \text{MT 1,2}

4. \( \neg q \rightarrow \neg p \)  
   \( \neg q \rightarrow \neg p \) \hspace{1cm} \rightarrow_i 2–3

We can start a box with an *assumption*, and use previously proven propositions (including premises) from the outside in the box.

We cannot use assumptions from inside the box in rules outside the box.
Rule for Introduction of Implication

\[ \phi \rightarrow \psi \]
Rules for Introduction of Disjunction

\[
\frac{\phi}{\phi \lor \psi} \quad \frac{\psi}{\phi \lor \psi}
\]

\[
[\lor i_1] \\
\phi \lor \psi
\]

\[
[\lor i_2] \\
\phi \lor \psi
\]
Rule for Elimination of Disjunction

\[ \frac{\phi \lor \psi}{\chi} \quad [\lor e] \]

\[ \vdots \]

\[ \phi \]
\[ \psi \]

\[ \chi \]
Example

<table>
<thead>
<tr>
<th></th>
<th>Proposition</th>
<th>Rule</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( p \land (q \lor r) )</td>
<td>premise</td>
</tr>
<tr>
<td>2</td>
<td>( p )</td>
<td>( \land e_1 \ 1 )</td>
</tr>
<tr>
<td>3</td>
<td>( q \lor r )</td>
<td>( \land e_2 \ 1 )</td>
</tr>
<tr>
<td>4</td>
<td>( q )</td>
<td>assumption</td>
</tr>
<tr>
<td>5</td>
<td>( p \land q )</td>
<td>( \land i \ 2,4 )</td>
</tr>
<tr>
<td>6</td>
<td>( (p \land q) \lor (p \land r) )</td>
<td>( \lor i_1 \ 5 )</td>
</tr>
<tr>
<td>7</td>
<td>( r )</td>
<td>assumption</td>
</tr>
<tr>
<td>8</td>
<td>( p \land r )</td>
<td>( \land i \ 2,7 )</td>
</tr>
<tr>
<td>9</td>
<td>( (p \land q) \lor (p \land r) )</td>
<td>( \lor i_2 \ 8 )</td>
</tr>
<tr>
<td>10</td>
<td>( (p \land q) \lor (p \land r) )</td>
<td>( \lor e \ 3, 4–6, 7–9 )</td>
</tr>
</tbody>
</table>
Summary

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Remaining tasks:
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Next Week

- More rules for negation
- Excursion: Intuitionistic logic
- Propositional logic as a formal language
- Semantics of propositional logic