Problem Formulation

CS6240 Multimedia Analysis

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Before attempting to solve a problem, we need to first formulate or define the problem.

It is important to precisely define the problem you intend to solve.

*The more difficult it is to define the problem, the harder you have to try.*

Why?
A Practical Example: Image Mosaicking

How to register images (a) and (b) to produce image (c)?
Maybe, we can write the problem as:

*Find the transformation between image (a) and image (b), then transform one of them and blend them together.*

This description is not precise.
How to write a program according to the English description?
What makes a good problem definition?

- It is precise.
- It states the objectives (what is required):
  - the inputs and their characteristics, including initial conditions
  - the outputs and their desired characteristics, including goal conditions
  - the relationships between the inputs and the desired outputs

How to state a good problem definition?
Answer: Use mathematical notations.

One way to learn how to write good problem definitions is to start with generic problems.
Abstract Mapping Problem

Let $S$ denote a set $\{x_i\}$ of $n$ points $x_i$ called the source.

Let $S'$ denote a set $\{x'_i\}$ of $n$ points $x'_i$ called the target.

Suppose we know that there is a mapping from each $x_i \in S$ to $x'_i \in S'$.

We want to determine the mapping function.

How to formulate this problem?
A possible problem formulation:

*Given a set $S$ of $n$ points $x_i$ and a set $S'$ of $n$ points $x'_i$, determine the function $f : S \rightarrow S'$ such that $x'_i = f(x_i)$ for $i = 1, \ldots, n$."

“Determine the function” means “determine the form and parameters of the function”.

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Example: Linear Case

In the linear case, \( x'_i = f(x_i) \) can be written in matrix form:

\[
x'_i = F x_i.
\]  

(1)

In this case, the form is a linear equation and the parameters are the values of the matrix elements in \( F \).

In practice, we usually cannot obtain the exact \( F \). There is an error \( e_i = \|x'_i - Fx_i\| \).
So, we can re-formulate the problem as:

*Given a set $S$ of $n$ points $x_i$ and a set $S'$ of $n$ points $x'_i$, determine the matrix $F$ that minimizes the sum-squared error $E$:

$$E = \sum_{i=1}^{n} \|x'_i - Fx_i\|^2.$$  \hfill (2)

- Now, the problem becomes an optimization problem.
- Eq. 2 is called the objective function.
In the previous example, the mapping or correspondence is known.

What if the mapping is unknown?

- Let $S$ denote a set $\{x_i\}$ of $m$ points $x_i$.
- Let $S'$ denote a set $\{x'_i\}$ of $n$ points $x'_j$, $m$ may or may not be equal to $n$. 
We know that a point in $S$ can map to some points in $S'$, but don’t know which one.

So, must use different subscripts for points in $S$ and $S'$. Same subscript implicitly means “known correspondence”.

A possible problem formulation:

Given a set $S$ of $m$ points $x_i$ and a set $S'$ of $n$ points $x'_j$, determine the function $f : S \rightarrow S'$ such that for each $x_i \in S$, there is a $x'_j \in S'$ such that $x'_j = f(x_i)$.

This formulation is ambiguous.

There are many possible $f$. Which one is it talking about?
Suppose we know how to measure the difference $d$ between any $x_i \in S$ and $x'_j \in S'$.

We can formulate the problem as one of finding the best mapping:

> Given a set $S$ of $m$ points $x_i$, a set $S'$ of $n$ points $x'_j$, and a difference measure $d(x_i, x'_j)$, determine the function $f : S \rightarrow S'$ that minimizes the sum-squared error $E$:

$$E = \sum_{i=1}^{m} d^2(x_i, f(x_i)).$$

Questions:

- In the above formulation, why use $d(x_i, f(x_i))$ instead of $d(x_i, x'_j)$?
- Can we use $d(x'_j, f(x_i))$ as in Eq. 2?
- What are the differences between this problem definition and the one in Eq. 2?
A Practical Example: Image Mosaicking

This problem can be divided into three sub-problems:

1. Identify corresponding points between the two images.
2. Compute transformation between the two images.
3. Transform and blend images.

Let’s consider Sub-Problem 2.

Suppose the transformation $T$ is linear.

Then, ideally

$$x'_i = T x_i, \text{ for each point } i.$$  \hfill (4)

In reality, there is an error $e_i = \|x'_i - T x_i\|$. 

So, we can formulate the problem as follows:

Given a set $S$ of $n$ points $x_i$ and a set $S'$ of $n$ points $x'_i$, determine the matrix $T$ that minimizes the sum-squared error $E$:

$$E = \sum_{i=1}^{n} \|x'_i - Tx_i\|^2.$$  \hspace{1cm} (5)

Note:

- To completely define image mosaicking problem, need to define sub-problems 1 and 3 as well. (Exercise)
Constrained Mapping Problem

In some applications, there are constraints that must be satisfied.

Consider the generic mapping problem in Section 1:
- The problem definition does not prohibit multiple points in \( S \) to map to a single point in \( S' \), i.e., it allows for many-to-one mapping.
- Suppose we need to impose one-to-one mapping.
- Then, the problem definition can be re-formulated as follows:

Given a set \( S \) of \( m \) points \( x_i \), a set \( S' \) of \( n \) points \( x'_j \), and a difference measure \( d(x_i, x'_j) \), determine the function \( f : S \rightarrow S' \) that minimizes the sum-squared error \( E \):

\[
E = \sum_{i=1}^{m} d^2(x_i, f(x_i)) \quad (6)
\]

subject to the constraint that \( f \) is a one-to-one function.
Another way to describe the constraint is:

subject to the constraint that $f(x_i) \neq f(x_k)$ for any $x_i, x_k \in S$
such that $x_i \neq x_k$.

The above problem is called a constrained optimization problem.
Summary

- A good problem definition is precise and it states the problem requirements and objectives.
- Many multimedia analysis problems can be formulated as optimization problems.
- Before you try to solve a problem, first study it carefully and then formulate the problem.
- Usually, you need to revise your problem formulations several times to make it more precise and more correctly describe the problem.
(1) Define sub-problems 1 and 3 of image mosaicking.