Theorem: If $L$ is accepted by a $T(n)$ time bounded $k$-tape machine then $L$ is accepted by $T(n)\log T(n)$ time bounded 2 tape machine.

Proof sketch: We will simulate a $k$-tape machine $M$ accepting $L$ using a two tape machine $M'$.

The first tape of $M'$ will use two tracks to simulate each tape of $M$. The second tape of $M'$ is used as a scratch tape, useful for copying parts of the first tape in the simulation below.

In the proof we show how to simulate one move of $M$. Let us call this simulation as a basic move of $M'$. We will give the proof for claimed time bound after the simulation.

Think of the first tape of $M'$ as being divided into blocks, ..., $B_{-2}, B_{-1}, B_0, B_1, B_2, ...$, (see figure below). $B_0$ consists of one cell. $B_i$ and $B_{-i}$ consist of $2^{i-1}$ cells. In the presentation below we will assume that the boundaries of the different blocks are marked, though the markers will actually be placed only when the blocks are first used.

![Tape Blocks](image)

Figure 1: Tape Blocks

Let us concentrate on the simulation of one tape of $M$ by the corresponding 2 tracks on the first tape of $M'$. The simulation is similar for all the other tapes. $M'$ uses a special symbol called *empty* (this is different from blank).

Suppose at the start of any step the contents of cell being read by $M$ on the tape being simulated is $a_0$, and the contents of cells on the right are $a_1,a_2, ..., $ and the contents of the cells on the left are $a_{-1}, a_{-2}, ...$ (where some of $a_i$’s may be blank).

At the beginning of any basic step we will have the following invariant.

(A) For any $i > 0$, either

(A1) lower tracks of $B_i$ and $B_{-i}$ are full (i.e. none of the symbols are empty), and upper tracks of $B_i$ and $B_{-i}$ are empty (i.e. all the symbols are empty).

(A2) Both tracks of $B_i$ are full, whereas both tracks of $B_{-i}$ are empty.

(A3) Both tracks of $B_{-i}$ are full, whereas both tracks of $B_i$ are empty.

Note that this invariant implies that for each $i > 0$, $B_i$ and $B_{-i}$ together have exactly $2^i$ empty symbols and $2^i$ non-empty symbols.

(B) The lower track of $B_0$ contains the content of the cell being scanned by $M$, that is $a_0$. The head of $M'$ is at $B_0$.

(C) Suppose we read the contents of the cells to the right of $B_0$, in the order of their distance from $B_0$, upper track first and lower track next, then we get $a_1, a_2, ...$.

(D) Similarly if we read the contents of the cells to the left of $B_0$ we get $a_{-1}, a_{-2}, ...$.
Initially the upper track of first tape of $M'$ is all empty and the lower track contains $\ldots, a_{-2}, a_{-1}, a_0, a_1, a_2$, where $a_0$ is in block $B_0$. Note that the invariant is satisfied in the beginning.

We now show how to do the basic step of $M'$ simulating a step of $M$ and maintain the invariant.

Clearly, $M'$ can determine the symbol being scanned by $M$ (for all the tapes) since they are at $B_0$, and thus determine the symbol to be written on the cell being scanned and whether the head of $M$ on the corresponding tape moves left or right. Thus, the symbol to be written on the scanned tape can be written at $B_0$. We now consider how to simulate the left move of $M$. The right move can be simulated similarly.

(1) $M'$ first moves to the right until it finds the first block such that at least one of its upper/lower tracks is empty. Let this block be $B_i$. Note that this implies that the cells, $B_1, B_2, \ldots, B_{i-1}$ have both tracks full. Let the case of $B_i$ having both tracks empty be called Case 1, and $B_i$ having only the upper track empty be called Case 2.

(2) Now we want to rearrange the contents of blocks $B_{-i}, \ldots, B_0, \ldots, B_i$ to maintain the invariant.

Case 1: In this case $B_{-i}$ and $B_1, \ldots, B_{i-1}$ have both tracks full, whereas, $B_i$ and $B_{-1}, \ldots, B_{-(i-1)}$ have both tracks empty. Thus the symbols in these blocks are $a_{-2i+1}, \ldots, a_0, \ldots, a_{2i(2^{i-1}-1)}$.

After the rearrangement, we want: $a_{-2i+1}, \ldots, a_2$ to the left of $B_0$, $a_{-1}$ at $B_0$ and $a_0, \ldots, a_{2i(2^{i-1}-1)}$ to the right of $B_0$.

We can do this by first copying these symbols to tape 2 (in sequential order), then copying back to tape 1 by filling lower track of $B_{-i}, \ldots, B_0, \ldots, B_i$. Note that this leaves the tape contents as required above.

Case 2: In this case $B_{-i}$ and $B_i$ have their lower tracks full, upper track empty, $B_1, \ldots, B_{i-1}$ have both tracks full, $B_{-1}, \ldots, B_{-(i-1)}$ have both tracks empty. Thus the symbols in these blocks are $a_{-2i-1}, \ldots, a_0, \ldots, a_{2i(2^{i-1}-1)+2^{i-1}}$.

After the rearrangement we want: $a_{-2i-1}, \ldots, a_{2i}$ to the left of $B_0$, $a_{-1}$ at $B_0$ and $a_0, \ldots, a_{2i(2^{i-1}-1)+2^{i-1}}$ to the right of $B_0$.

We can do this by first copying these symbols to tape 2 (in sequential order), then copying back to tape 1 by filling lower track of $B_{-(i-1)}, \ldots, B_0, \ldots, B_{i-1}$, and then filling both tracks of $B_i$. Note that this leaves the tape contents as required above.

(3) The head of $M'$ then returns to the block $B_0$

This completes the basic move of $M'$. Note that for simulating $k$ tapes we can do the steps 1, 2, and 3 above for each of the tape in a serial fashion (since after the simulation for each tape the head is back at $B_0$).

We now consider the time required to simulate each tape. When $B_i$ is selected as in step 1 above, we call the basic step (with respect to the corresponding tape) a $B_i$-basic step.

Time required to do a $B_i$-basic step (steps 1 to 3) is proportional to $2^i$ (where the constant of proportionality is independent of $i$). Note that a $B_i$ basic step needs to be done at most once every $2^{i+1}$ steps of $M$. This is so since, for a $B_i$-basic step to be done, $B_1, \ldots, B_{i-1}$ must have to be full on both tracks; after $B_i$-basic step, $B_1, \ldots, B_{i-1}$ are empty in the upper track. It will take at least $2^{i+1} - 1$ steps for them to become full again.) Also the first $B_i$-basic step cannot be done until at least $2^{i-1}$ steps of $M$ are completed since we initially started with all blocks having empty upper track.

Thus the time for simulation is bounded by:
\[ k \sum_{i=1}^{1+\log T(n)} c \cdot 2^i \frac{T(n)}{2^i-1} \]

for some constant \( c \). This is bounded by \( c' T(n) \log T(n) \), for some constant \( c' \).

This proves the time bound for the simulation.