Automata on Ordinals and Linear Orders*

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Abstract. We investigate structures recognizable by α-automata with running time a limit ordinal α. The domain of such a structure consists of finite α-words with gaps. An α-automaton resembles a finite automaton but has a limit rule which maps the set of states which appear cofinally often before the limit to a limit state. We determine the suprema of the α-automatic ordinals and the ranks of α-automatic linear orders. The power of α-automata increases with every power of ω.

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1 Introduction

Various types of automata indexed by ordinals have been studied for instance by Büchi, Choueka [1], Wojciechowski [6]. Let us consider a finite automaton which has read an infinite input string. The run of the automaton can be extended to the ordinals by specifying the behaviour at limits via a transition function; this is the extra component that distinguishes automata on ordinals from finite automata. We consider the following type of automaton with running time a limit ordinal α.

Definition 1. Suppose α is a limit ordinal and Σ is a finite alphabet. An *α-automaton* consists of

- a finite set S of states,
- an initial state,
- a set of accepting states,
- a successor transition function $S \times (\Sigma \cup \{\diamond\}) \to S$, and
- a limit transition function $\mathcal{P}(S) \to S$.

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The input of such an α-automaton is a word of length α in a given finite alphabet extended by an extra symbol $\diamond$. The automaton will run for α steps and read all letters of the α-word successively. At each limit $\lambda \leq \alpha$ the limit state is determined by a transition function which maps the set of states occurring cofinally often before λ to the state at λ. After running for α steps, the transition function is applied again and the input is accepted if the state at α is an accepting state. Note that in comparison, for a Büchi automaton there is only one limit transition rule at stage ω which checks if at least one of the accepting states occurred infinitely often. Thus a Büchi automaton can be simulated by a ω-automaton in the sense of this paper. Since we do not repeat the definition and the basic properties of finite automata, we refer to [3] for an introduction to the theory of automata and automatic structures.

Example 2. *Suppose an ω^n-automaton moves into state 0 in successor steps and into state $m+1$ in limits, where m is the maximum of the states appearing cofinally often before the limit. This detects the limit type of the current step: in any step of the form $\omega^{n-1}n_{n-1} + \omega^{n-2}n_{n-2} + \ldots + \omega^m n_m$ with $n_m \neq 0$, the state is $m < n$.*

Any ω^n-automaton can be converted into an ω^n-automaton accepting the same words whose state varies depending on the limit type of the current step, by forming the product with an automaton detecting the limit type.

An *α-automatic presentation* of a relational structure M in a finite language is a structure $N \cong M$ whose domain consists of α-words together with α-automata for the domain and each relation in N accepting exactly the words in the domain or the respective relation of N. Functions will be replaced by their graphs, thus converting any structure into a relational structure. Let us say an α-word is *finite* if all but finitely many of its letters are equal to $\diamond$.

Definition 3. A structure is *finite word α-automatic*, or *α-automatic* for short, if it is has an α-automatic presentation whose domain consists of finite α-words.

Note that if necessary, we can mark the end of every finite string in the domain of a presentation by attaching an extra symbol, and thus obtain another α-automatic presentation of the same structure. It is not hard to prove that the finite word ω-automatic structures are exactly the automatic structures, i.e. structures with a presentation by finite automata which do not run for ω steps but halt at the end of the finite input word.

Comparison of the current input letter with the limit type yields an ω^n-automatic presentation of the following set.

Example 4. *The set of finite ω^n-words whose letter at place $\omega^{n-1}n_{n-1} + \omega^{n-2}n_{n-2} + \ldots + \omega^m n_m$ where $n_m \neq 0$ is either n_m or $\diamond$, is ω^n-automatic.*

A natural question is: what is the supremum of ordinals β so that $(\beta, <)$ is α-automatic? Delhommé [2] proved that the supremum of the automatic ordinals is ω^ω and we first conjectured that the supremum of the α-automatic ordinals is ω^α, however this is false for any ϵ with $\omega^\epsilon = \epsilon$. Since $(\alpha, <)$ is easily α-automatic and any finite product of α-automatic structures is again α-automatic, the supremum is at least α^ω.

Example 5. *Let $(n_0, ..., n_k) <^* (m_0, ..., m_l)$ if $k < l$ and $n_i = m_i$ for all $i \leq k$, or there is $i \leq \min\{k, l\}$ with $n_i < m_i$ and $n_j = m_j$ for all $j < i$. This is a well ordering of type ω^ω. The representation of the finite sequence $(n_0, ..., n_k)$ by the finite ω^2-word $0^{n_0}1 \diamond^\omega ... 0^{n_k}1 \diamond^\omega \diamond^{\omega^2}$ leads to an ω^2-automatic presentation of $<^*$.*

Thus $(\omega^\omega, <)$ is ω^2-automatic and the supremum of the ω^2-automatic ordinals is at least ω^{ω^2}. A straightforward generalization of this example shows that ω^β is $\omega \cdot \beta$-automatic and the supremum of the $\omega \cdot \beta$-automatic ordinals is at least $\omega^{\beta \cdot \omega}$.

To prove that ω^{ω^2} is not ω^2-automatic, we will follow Delhommé's proof [2, 3] that ω^ω is not automatic. In generalizing this proof, the step of splitting the domain of an automatic structure of order type ω^n into finitely many pieces is a problem. For automatic structures the pieces of the partition are parametrized by finite words of fixed length so that one of the pieces necessarily has order type ω^n. However, in this setting, the words have infinite length so that the domain is split into infinitely many pieces and the argument breaks down. We will find a piece of the partition of the same order type when ω^n is replaced by a sufficiently closed ordinal such as $\omega^{\omega \cdot n}$. To do this, note that there are only finitely many possibilities, or types, in which two pieces can be arranged relative to each other. Thus the ordinals occurring in α-automatic structures can be characterized via a commutative product of ordinals. This leads to the computation of the supremum of the α-automatic ordinals and the supremum of ranks of α-automatic linear orders for limit ordinals α.

2 Ordinals

Let us introduce a *finite type product* of ordinals which naturally occurs in automatic structures. We fix the following notation. When C is a set of ordinals, let $tp(C)$ denote its order type and $C(\mu)$ its μ^{th} element. When C, D are sets of ordinals, let $tp(C, D)$ denote the isomorphism type of $(C \cup D, <, C, D)$.

Definition 6. $\alpha \cdot_{fin} \beta$ is the supremum of the ordinals γ such that there is $(C_\delta : \delta < \epsilon)$ with $\gamma = \bigcup_{\delta<\epsilon} C_\delta$ and

- $\forall \delta < \epsilon \; tp(C_\delta) \leq \alpha$,
- there are only finitely many $tp(C_\delta, C_\eta)$ for $\delta, \eta < \epsilon$, and
- $tp(Tr_\mu) \leq \beta$ for $Tr_\mu = (C_\delta(\mu) : \delta < \epsilon)$ (the *trace* of μ) where $\mu < \alpha$.

To define $tp(Tr_\mu)$, we identify Tr_μ with its image. Note that the length ϵ of the sequence can be arbitrarily large. We will see that there is always such a sequence $(C_\delta : \delta < \epsilon)$ with $\alpha \cdot_{fin} \beta = \bigcup_{\delta<\epsilon} C_\delta$. Let us call such a sequence an *instance* of $\alpha \cdot_{fin} \beta$.

Remark 7. If $(C_\delta : \delta < \epsilon)$ is as above, then there are only finitely many $tp(Tr_\mu)$.

Proof. There are finitely many $tp(C_\gamma, C_\delta)$ and hence there is a finite set F of ordinals such that $tp(C_\gamma \restriction F, C_\delta \restriction F)$ determines $tp(C_\gamma, C_\delta)$ for all γ, δ. To each η we associate the function f_η which maps each type $tp(G, H)$ for $|G| = |H| = |F|$ to $0, 1$ or 2 depending on whether $C_\gamma(\eta) < C_\delta(\eta)$, $C_\gamma(\eta) = C_\delta(\eta)$, or $C_\gamma(\eta) > C_\delta(\eta)$ for all γ, δ with $tp(C_\gamma \restriction F, C_\delta \restriction F) = tp(G, H)$. We partition α into finitely many pieces so that f is constant on each piece. Then $tp(Tr_\eta) = tp(Tr_\zeta)$ for η, ζ in the same piece.

The finite type product is the same as the well-known commutative product.

Definition 8. [5] Suppose $\alpha = \Sigma_i \omega^{\alpha_i}$ and $\beta = \Sigma_j \omega^{\beta_j}$ are in Cantor normal form. The *commutative sum* $\alpha \oplus \beta$ is defined as the sum of all ω^{α_i} and ω^{β_j} arranged in decreasing order. The *commutative product* $\alpha \otimes \beta$ is defined as the sum of all $\omega^{\alpha_i \oplus \beta_j}$ arranged in decreasing order.

Note that the commutative sum and product are strictly monotone in both coordinates.

Lemma 9. *$\alpha \cdot_{fin} \beta = \alpha \otimes \beta$ for all α, β.*

Proof. We construct an instance of $\alpha \cdot_{fin} \beta$ of order type $\alpha \otimes \beta$ such that $tp(C_\delta) = \alpha$ and $tp(Tr_\mu) = \beta$ for all δ, μ. Suppose $\alpha = \omega^{\bar{\alpha}} = \omega^{\Sigma_i \omega^{\bar{\alpha}_i}}$ and $\beta = \omega^{\bar{\beta}} = \omega^{\Sigma_i \omega^{\bar{\beta}_i}}$. Let us suppose the least term in the Cantor normal form of $\bar{\alpha} \oplus \bar{\beta}$ is $\omega^{\bar{\alpha}_k}$ and suppose $(C_\alpha : \alpha < \epsilon)$ is an instance of $\omega^{\bar{\alpha} - \omega^{\bar{\alpha}_k}} \otimes \omega^{\bar{\beta}} = \omega^{(\bar{\alpha} \oplus \bar{\beta}) - \omega^{\bar{\alpha}_k}}$. We let $C_{\alpha,0} = C_\alpha$ and find sets $C_{\alpha,\gamma}$ for $0 < \gamma < \delta := \omega^{\omega^{\bar{\alpha}_k}}$ such that

- $tp(C_{\alpha,\gamma}, C_{\beta,\gamma}) = tp(C_\alpha, C_\beta)$ for all $\alpha, \beta < \epsilon$ and all $\gamma < \delta$, and
- $\sup(C_{\alpha,\gamma}) < \min(C_{\beta,\eta})$ for all $\alpha, \beta < \epsilon$ and all $\gamma < \eta < \delta$.

Let $C_\alpha := \bigcup_{\gamma<\delta} C_{\alpha,\gamma}$ for $\alpha < \epsilon$. Then $(C_\alpha : \alpha < \epsilon)$ is an instance of $\alpha \cdot_{fin} \beta$ of order type $\alpha \otimes \beta$. If the least term in $\bar{\alpha} \oplus \bar{\beta}$ is $\omega^{\bar{\beta}_k}$, we choose instead an instance $(C_\alpha : \alpha < \epsilon)$ of $\omega^{\bar{\alpha}} \otimes \omega^{\bar{\beta}-\omega^{\bar{\beta}_k}} = \omega^{(\bar{\alpha}\oplus\bar{\beta})-\omega^{\bar{\beta}_k}}$ and define $C_{\alpha,\gamma}$ as before. Then $(C_{\alpha,\gamma} : \alpha < \epsilon, \gamma < \delta)$ is an instance of $\alpha \cdot_{fin} \beta$ of order type $\alpha \otimes \beta$.

Suppose $\alpha = \alpha_0 + \alpha_1$ with $\alpha_1 = \omega^{\bar{\alpha}}$ and $\beta = \beta_0 + \beta_1$ with $\beta_1 = \omega^{\bar{\beta}}$. We can choose (by inductive hypothesis) instances $(C^{i,j}_\delta : \delta < \epsilon_{i,j})$ of $\alpha_i \otimes \beta_j$ and there is freedom to choose the relative position of $C^{i,j}_\gamma$ and $C^{k,l}_\delta$ for $(i,j) \neq (k,l)$. We can assume that $\epsilon_{i,j} = \epsilon$ for all (i,j). It is not difficult to see from the definition of the commutative sum that the relative positions of $C^{i,j}_\delta$ can be arranged so that their union has order type $\alpha \otimes \beta = \Sigma^{\oplus}_{i,j} \alpha_i \otimes \beta_j$. Let $C_\delta = C^{0,0}_\delta \cup C^{1,0}_\delta$ for $\delta < \epsilon$ and $C_\delta = C^{1,0}_{\beta_0+\delta} \cup C^{1,1}_{\beta_0+\delta}$ for $\epsilon \leq \delta < \epsilon \cdot 2$. Then $(C_\delta : \delta < \epsilon \cdot 2)$ is an instance of $\alpha \otimes \beta$.

In the finite type product, each C_δ comes with its increasing enumeration. We define a variant of the finite type product by replacing the enumerations of the sets C_δ with arbitrary partial functions from α onto C_δ. Let us call this product $\alpha \cdot_p \beta$. We prove $\alpha \cdot_p \beta \leq \alpha \otimes \beta$ by induction and this implies $\alpha \cdot_{fin} \beta = \alpha \cdot_p \beta = \alpha \otimes \beta$ for all α and β. It is sufficient to prove this for $\alpha = \omega^{\bar{\alpha}}$ and $\beta = \omega^{\bar{\beta}}$, since it can be extended to all α and β using the fact that $\alpha \otimes \beta$ is the maximal order type of sets which are unions of two sets of order types α and β. There exists $\theta < tp(C_\delta)$ for every $\delta < \epsilon$ such that for all ζ, η, if $\sup C_\zeta < \sup C_\eta$ and $C_\eta(\theta)$ is defined, then $\sup C_\zeta < C_\eta(\theta)$. We can assume that $\sup C_\gamma < \sup C_\eta$ implies $\sup C_\gamma < \min C_\eta$ since this can be achieved by splitting C into finitely many pieces. We may also assume that $tp(C_\gamma) = tp(C_\delta)$ for all $\gamma, \delta < \epsilon$. If there are $\gamma \neq \delta$ such that $\sup C_\gamma \neq \sup C_\delta$, then every proper initial segment of $\bigcup_{\gamma<\epsilon} C_\gamma$ is bounded by $\alpha \cdot_{fin} \bar{\beta} = \alpha \otimes \bar{\beta} < \alpha \otimes \beta$ for some $\bar{\beta} < \beta$. If $\sup C_\gamma = \sup C_\delta$ for all $\gamma, \delta < \epsilon$, then every proper initial segment is bounded by $\bar{\alpha} \cdot_{fin} \beta = \bar{\alpha} \otimes \beta$ for some $\bar{\alpha} < \alpha = tp(C_\gamma)$.

If α is a limit, then every $\alpha \cdot n$-automaton can be simulated by an α-automaton and thus every $\alpha \cdot n$-automatic structure is α-automatic. Although not every α-automatic structure is necessarily ω^γ-automatic for $\omega^\gamma \leq \alpha < \omega^{\gamma+1}$, the following arguments easily adapt to show that the supremum of the α-automatic ordinals is the same as that of the ω^γ-automatic ordinals. If x is an α-word, let $|x|$ denote the least $\beta \leq \alpha$ such that $x(\gamma) = \diamond$ for all $\beta \leq \gamma < \alpha$.

Proposition 10. *Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\omega^{\beta \cdot \omega}$ is the supremum of the α-automatic ordinals.*

Proof. The ordinals $\omega^{\beta \cdot n}$ are α-automatic as in Example 5. Let us suppose there is an α-automatic structure of order type at least $\omega^{\beta \cdot \omega}$. We follow the proof for automatic ordinals [3, Theorem 17]. We choose an element u_n of rank $\omega^{\beta \cdot n}$ for each $n \geq 1$ and write the set of its predecessors as

$$u_n \downarrow = Y \sqcup \bigsqcup_{|v|=|u_n|} X_v^{u_n},$$

where $X_v^u = \{vw : vw < u\}$ and $Y = \{x : |x| < |u_n| \& x < u_n\}$. Since $|u_n| < \alpha$, every $|u_n|$-automatic ordinal is strictly below ω^β by the induction hypothesis and $\gamma := tp(Y) < \omega^\beta$. The number of types $tp(X_v^u, X_w^u)$ is bounded by the product of the numbers of states of the automata on inputs (u_n, v) and (v, v), since the type depends only on the states reached after reading u_n and v. Hence the sets $X_v^{u_n}$ form a finite type product. Since each Tr_δ is ϵ-automatic for some $\epsilon < \alpha$, $tp(Tr_\delta) < \omega^\beta$. Let δ be the maximal $tp(X_v^{u_n})$. Then $\omega^{\beta \cdot n} \leq \gamma \otimes \delta$ and thus $\delta = \omega^{\beta \cdot n}$. This implies that for any n there are u_n, v_n with $|u_n| = |v_n|$ and $tp(X_{v_n}^{u_n}) = \omega^{\beta \cdot n}$, however there are only finitely many types.

In particular, the supremum of the ω^n-automatic ordinals is ω^{ω^n}.

3 Linear orders

In a similar fashion as for ordinals, we analyze the ranks of α-automatic linear orders following [4]. Since there is no trace function for families of linear orders, let us define a finite type product of linear orders as follows. If $(f_n : \alpha \to C_n : n < m)$ is a tuple of partial onto functions and $C = \bigcup_{n<m} C_n$, let $tp(f_0, ..., f_{m-1})$ denote the isomorphism type of the two-sorted structure $(C, \alpha, <, f_0, ..., f_{m-1})$.

Definition 11. A linear order C is a *finite type product* of linear orders A and B if there is are sequences $(C_\gamma : \gamma < \epsilon)$ of subsets of C and $(f_\gamma : \alpha \to C_\gamma : \gamma < \epsilon)$ of partial onto functions with

- $C = \bigcup_{\gamma<\epsilon} C_\gamma$,
- $C_\gamma \hookrightarrow A$ for all $\gamma < \epsilon$,
- for each n and $\gamma_i < \epsilon$, there are only finitely many $tp(f_{\gamma_0}, ..., f_{\gamma_n})$, and
- $ran(g_\mu) \hookrightarrow B$ for all $\mu < \alpha$, where $g_\mu(\gamma) = f_\gamma(\mu)$.

If L is a linear order and $(L_i : i \in L)$ is a family of linear orders, the L-sum of $(L_i : i \in L)$ is defined as the lexicographic order on pairs (i, j) where $i \in L$ and $j \in L_i$.

Definition 12. Suppose L is a linear order. Let $rk(L) = 0$ if L is finite. Let $rk(L) \leq \alpha$ if L is a $\mathbb{Z} \cdot n$-sum of linear orders of rank below α for some n.

The linear orders into which $(\mathbb{Q}, <)$ does not embed are called *scattered*. Easily every scattered linear order has a rank.

Lemma 13. *Suppose C is a scattered linear order and C is a finite type product of A and B. Then $rk(C) \leq rk(A) \oplus rk(B)$.*

Proof. Suppose $(C_\gamma : \gamma < \epsilon)$ and $(f_\gamma : \alpha \to C_\gamma : \gamma < \epsilon)$ witness the finite type product. Let us split A and B into finitely many $\mathbb{Z}$-sums of linear orders of rank below $rk(C)$ and accordingly C into finitely many finite type products. We can assume that $\sup C_\gamma < \sup C_\delta$ implies $C_\gamma < C_\delta$ and $\inf C_\gamma < \inf C_\delta$ implies $C_\gamma < C_\delta$ and that each f_γ is total, since we can achieve this by splitting C into finitely many pieces and the rank is closed under finite unions. Let $A = \Sigma_{z \in \mathbb{Z}} A_z$ and $B = \Sigma_{z \in \mathbb{Z}} B_z$ such that $rk(A_z) < rk(A)$ and $rk(B_z) < rk(B)$. We partition C into $C^0 < C^1 < C^2$ such that

- $\sup C_\gamma = \sup C_\delta$ and $C_\gamma \subseteq C^i$ imply $C_\delta \subseteq C^i$,
- if $\{\inf C_\gamma : C_\gamma \subseteq C^i\}$ has a minimum and $C_\gamma, C_\delta \subseteq C^i$, then $\inf C_\gamma = \inf C_\delta$, and
- if $\{\sup C_\gamma : C_\gamma \subseteq C^i\}$ has a maximum and $C_\gamma, C_\delta \subseteq C^i$, then $\sup C_\gamma = \sup C_\delta$.

If there exist $\gamma \neq \delta$ with $\sup C_\gamma \neq \sup C_\delta$, we may choose $C^0, C^2 \neq \emptyset$. It suffices to prove for all $i \leq 2$ and $c < d$ in C^i that $rk([c, d]) < rk(A) \oplus rk(B)$. If $\{\inf C_\gamma : C_\gamma \subseteq C^0\}$ has no minimum, then $[c, d]$ is a subset of a finite type product of A and $\Sigma_{z \in F} B_z$ for some finite $F \subseteq \mathbb{Z}$ and hence $rk([c, d]) \leq rk(A) \oplus rk(\Sigma_{z \in F} B_z) < rk(A) \oplus rk(B)$ by the induction hypothesis. If $C_\gamma, C_\delta \subseteq C^0$ implies $\inf C_\gamma = \inf C_\delta$ and hence $\sup C_\gamma = \sup C_\delta$, then $[c, d]$ is a subset of a finite type product of $\Sigma_{z \in F} A_z$ and B for some finite $F \subseteq \mathbb{Z}$ and hence $rk([c, d]) \leq rk(\Sigma_{z \in F} A_z) \oplus rk(B) < rk(A) \oplus rk(B)$. The argument for $c, d \in C^2$ is similar. For all $c < d$ in C^1, $[c, d]$ is a subset of a finite type product of A and $\Sigma_{z \in F} B_z$ for some finite $F \subseteq \mathbb{Z}$ and hence $rk([c, d]) < rk(A) \oplus rk(B)$.

If $\sup C_\gamma = \sup C_\delta$ and hence $\inf C_\gamma = \inf C_\delta$ for all γ, δ, let $C^0, C^2 = \emptyset$. For all $c < d$ in C^1, $[c, d]$ is a subset of a finite type product of $\Sigma_{z \in F} A_z$

and B for some finite $F \subseteq \mathbb{Z}$ and hence $rk([c,d]) \leq rk(\Sigma_{z\in F} A_z) \oplus rk(B) < rk(A) \oplus rk(B)$.

This is sufficient to determine the supremum of ranks of α-automatic scattered linear orders, similar to ordinals.

Proposition 14. *Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\beta \cdot \omega$ is the supremum of ranks of scattered α-automatic linear orders.*

Proof. Instead of choosing an element u_n of rank $\omega^{\beta \cdot n}$ as we did for ordinals, we choose intervals $[u_n^0, u_n^1]$ of rank $\omega^{\beta \cdot n}$ for each n, and proceed as in the proof of Proposition 10.

The rank can be extended to non-scattered linear orders via a finite condensation function c_{FC} which amalgamates elements of a partial order with only finitely many elements in between. Let c_{FC}^{α} be the α^{th} iterate of c_{FC} and $rk(L)$ the least α so that $c_{FC}^{\alpha}(L)$ does not contain a convex suborder isomorphic to ω or ω^*. Note that this implies $c_{FC}^{\alpha+1}(L) = c_{FC}^{\alpha+2}(L)$.

Proposition 15. *Suppose $\alpha = \omega \cdot \beta = \omega^\gamma$. Then $\beta \cdot \omega$ is the supremum of ranks of α-automatic linear orders.*

Proof. Note that every linear order A is a dense sum of scattered linear orders. It is sufficient to prove that there is a uniform bound below $\beta \cdot \omega$ on the rank of scattered intervals. Suppose $[u_0, u_1]$ is a scattered interval in A and $\nu = \max\{|u_0|, |u_1|\}$. Let us write

$$[u_0, u_1] = Y \sqcup \bigsqcup_{|v|=\nu} X_v^{u_0,u_1},$$

where $X_v^{u_0,u_1} = \{vw \in A : u_0 < vw < u_1\}$ and $Y = \{x : u_0 < x < u_1 \& |x| < \nu\}$. Then Y is μ-automatic for some $\mu < \omega \cdot \beta$ and hence $rk(Y) < \beta$ by the induction hypothesis. The number of isomorphism types of $X_v^{u_0,u_1}$ for $u_0, u_1, v \in A$ is bounded by the product of the numbers of states of the automata on inputs (u_0, v), (u_1, v), and (v, v). Hence there is m such that $rk(X_v^{u_0,u_1}) \leq \beta \cdot m$ for all $u_0, u_1, v \in A$. Since the sets $X_v^{u_0,u_1}$ form a finite type product, $rk(\bigsqcup_{|v|=\nu} X_v^{u_0,u_1}) \leq \beta \cdot m \oplus \beta$.

We leave open the following questions:

- Is the theory of any α-automatic structure decidable?
- Is the isomorphism problem for α-automatic ordinals decidable if α is computable?
- Is every countable infinite word α-automatic structure finite word α-automatic?

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