Automatic models of first order theories

Pavel Semukhin[a,1], Frank Stephan[b,2]

[a]*Department of Computer Science, University of Regina, Canada.*
[b]*Department of Mathematics and Department of Computer Science, National University of Singapore, Singapore 119076, Republic of Singapore.*

Abstract

Khoussainov and Nerode [14] posed various open questions on model-theoretic properties of automatic structures. In this work we answer some of these questions by showing the following results: (1) There is an uncountably categorical but not countably categorical theory for which only the prime model is automatic; (2) There are complete theories with exactly 3,4,5,... countable models, respectively, and every countable model is automatic; (3) There is a complete theory for which exactly 2 models have an automatic presentation; (4) If *LOGSPACE* = *P* then there is a uncountably categorical but not countably categorical theory for which the prime model does not have an automatic presentation but all the other countable models are automatic; (5) There is a complete theory with countably many countable models for which the saturated model has an automatic presentation but the prime model does not have one.

1. Introduction

This paper is devoted to the study of automatic structures from the model-theoretic point of view. Automatic structures are the algebraic structures whose functions and relations can be recognized by finite automata. Historically, this notion was introduced in the work of Hodgson [12], and later in Khoussainov and Nerode [13] and Blumensath and Grädel [2].

Automatic structures are famous in theoretical computer science because of their decidability properties. Namely, the *model checking problem* for automatic structures is decidable. In other words, there is an algorithm that, given an automatic structure $\mathcal{A}$, a first order formula $\varphi(\bar{x})$ and a tuple $\bar{a}$ of elements from $\mathcal{A}$, it decides whether $\mathcal{A} \vDash \varphi(\bar{a})$. In particular, automatic structures are decidable structures (a structure $\mathcal{A}$ is *decidable* if there is an algorithm that

Email addresses: `pavel@semukhin.name` (Pavel Semukhin), `fstephan@comp.nus.edu.sg` (Frank Stephan)

[1]P. Semukhin was supported by NUS grant R146-000-114-112 and later by a Post-doctoral Fellowship from the Pacific Institute for the Mathematical Sciences and in part by IMS programme "Automata Theory and Applications" during his visit at NUS.

[2]Supported in part by NUS grants R146-000-114-112 and R252-000-420-112.

decides whether $\mathcal{A} \models \varphi(\bar{a})$ holds, for any first order formula $\varphi(\bar{x})$ and any tuple $\bar{a} \in \mathcal{A}$). This, in turn, implies that the first order theory of any automatic structure is decidable. One can use this property to prove the decidability of first order theories for many mathematical structures, for example, the Presburger arithmetic $(\mathbb{N}, +)$.

A lot of work has been devoted to study the question as to which structures have automatic presentations [3, 5, 15, 20, 21, 23]. In some cases we have an elegant characterization of automatic structures in a given class, for instance [5, 15]:

- An ordinal α is automatic if and only if $\alpha < \omega^\omega$.
- The additive semigroup of an ordinals ω^α is automatic if and only if $\alpha < \omega$.
- A Boolean algebra $\mathcal{B}$ is automatic if and only if $\mathcal{B}$ is either finite or isomorphic to B_ω^n, where B_ω is the algebra of all finite and co-finite subsets of the natural numbers.

One the other hand, it is still an open problem to describe automatic Abelian groups and automatic linear orders. Another important question that attracted attention is how difficult are the isomorphism problems for various classes of structures [15, 18, 19].

Recently, Khoussainov and Nerode [14] linked automatic structures to model theory and posed a list of important questions arising from this connection. Some of that questions are answered in the present work. The interested reader is referred to Khoussainov and Rubin [17, 22] for general surveys on automatic structures.

To describe in more detail what it means for a structure to be automatic, we need a notion of convolution. A *convolution* of two strings u and v in an alphabet Σ is a string $Conv(u, v)$ in alphabet $(\Sigma \cup \{\Box\})^2$ which is obtained by putting u above v. If the strings have different length, then we use a new padding symbol $\Box$ to fill in the shorter string. For example, if $u = ab$ and $v = bbaa$, then

$$\otimes(ab, bbaa) = \binom{a}{b}\binom{b}{b}\binom{\Box}{a}\binom{\Box}{a}.$$

Similarly, one can define a convolution of several finite strings.

An n-ary relation R on finite strings is called *automatic* if the set of convolutions of all tuples from R is recognizable by a finite automaton. A function is called *automatic* if its graph is automatic. Intuitively, an automatic function can be thought of as a one whose result can be computed "locally", using a fixed finite amount of external memory. A typical example of an automatic function is the addition operation on natural numbers in binary presentation. In this case, an automaton computes the addition bitwise, remembering only one carry bit when necessary.

An algebraic structure $\mathcal{A}$ is called *automatic* if its domain, functions and relations are automatic. A structure is said to have *an automatic presentation* if it is isomorphic to an automatic structure. Typical examples of automatic structures are [13, 15, 21]:

- the ordering of the natural numbers $(\omega, \leq)$ and the ordering of the rationals $(\mathbb{Q}, \leq)$;
- finitely generated Abelian groups, in particular, the ordered group $(\mathbb{Z}, +, \leq)$;
- Prüfer groups $\mathbb{Z}(p^{\infty})$ for every prime p;
- countably dimensional vector spaces over finite fields;
- the Boolean algebra of all finite and co-finite subsets of $\mathbb{N}$.

On the other hand, the following structures do not have automatic presentations [5, 15]:

- the natural numbers with multiplication $(\mathbb{N}, \times)$;
- the free group over more than one generator;
- the random graph;
- the atomless Boolean algebra.

There was a long standing open question whether the group of rationals under addition $(\mathbb{Q}, +)$ is automatic. This question was recently solved by Tsankov [23] who showed that $(\mathbb{Q}, +)$ does not have an automatic presentation.

The topic of our work was inspired by computable model theory. Computable structures are generalizations of automatic structures in the sense that their operations are computable by Turing machines, rather than by finite automata. One of the general questions studied in computable model theory can be stated as follows: given a complete first order theory T, which models of T have computable presentations? In particular, when the prime or the saturated model of T is computable?

The latter question is especially interesting in the case of uncountably categorical theories (recall that T is called *uncountably categorical* if any two models of T of cardinality $\aleph_1$ are isomorphic). In [9, 10, 16], examples of such theories were constructed with the following properties:

(1) only the prime model of a theory is computable;

(2) only the saturated model of a theory is computable;

(3) all models except the prime one are computable;

(4) all models except the saturated one are computable.

For our purpose, we will reformulate the above mentioned question as follows: *given a complete first order theory T, describe the automatic models of T.* This general question can be divided into a number of more specific subquestions. In our paper we consider the following questions, which are stated as Question 3.2 and Question 3.3 by Khoussainov and Nerode [14].

Question 3.2. *Let $n \geq 1$ be a natural number. Does there exist a theory with exactly n automatic models up to isomorphism? (When $n = 1$, the theory should not be $\aleph_0$-categorical.)*

Question 3.3. *Let T be a decidable complete first order theory such that T has only countably many countable models. Is any of the following true?*

1. *If T has an automatic model, then all countable models of T are automatic.*
2. *If T is $\aleph_1$-categorical and has an automatic model, then all countable models of T are automatic.*
3. *If T has finitely many countable models one of which is automatic, then all countable models of T are automatic.*

We also consider a question mentioned after Question 3.4 in [14]: *does the existence of an automatic saturated model of T imply that the prime model of T is also automatic?* This question is interesting since if one considers decidable models instead of automatic ones, then the answer is positive: the existence of a decidable saturated model implies the existence of a decidable prime model. However, we will show that in the automatic case the answer is negative.

The paper is organized as follows. The next section contains necessary preliminaries. In Section 3 we provide a negative answer to parts 1 and 2 of Question 3.3 from [14] by constructing an $\aleph_1$-categorical theory for which only the prime model is automatic. In Section 4 we give another example of a theory for which only the prime model is automatic. In Section 5 we prove under the assumption $LOGSPACE = P$ that there is an $\aleph_1$-categorical theory for which all countable models except the prime one are automatic (Theorem 10). Next we show that without the above assumption we can construct a complete theory that has countably many countable model and for which the saturated model is automatic but the prime one is not (Theorem 11). In Section 6 we give examples of theories with finitely many automatic models. Namely, for every $n \geq 3$ we construct a complete theory with n countable models all of which are automatic (Theorem 13 and Corollary 14). Then in Theorem 15 we construct a complete theory which has exactly two automatic models, however the theory itself will have uncountably many countable models.

2. Preliminaries

A *(nondeterministic) finite automaton* is a tuple $A = (S, \Sigma, I, T, F)$, where

- S is a finite set of states,

- Σ is a finite alphabet,
- $I \subseteq S$ is the set of initial states,
- T is the transition function $T : S \times \Sigma \to \mathcal{P}(S)$, where $\mathcal{P}(S)$ is the power set of S, and
- $F \subseteq S$ is the set of final states.

An automaton A accepts a word $w = a_1 \dots a_n \in \Sigma^*$ if there is a sequence of states $s_0 s_1 \dots s_n$ such that $s_0 \in I$, $s_n \in F$ and for every $i < n$, $s_{i+1} \in T(s_i, a_{i+1})$. The set of all words accepted by A is called the *language* of A and denoted $L(A)$.

Given k strings $w_1, \dots, w_n$ in alphabet Σ, a *convolution* of the tuple $(w_1, \dots, w_n)$ is the word $Conv(w_1, \dots, w_n)$ of length $\max_i |w_i|$ in alphabet $(\Sigma \cup \Box)^n$, where $\Box$ is a new padding symbol, defined as follows: the kth symbol of $Conv(w_1, \dots, w_n)$ is $(\sigma_1, \dots, \sigma_n)$, where σ_i is the kth symbol of w_i if $i \leq |w_i|$ and $\sigma_i = \Box$, otherwise.

A convolution of an n-ary relation $R \subseteq (\Sigma^*)^n$ is defined as $Conv(R) = \{Conv(\bar{a}) : \bar{a} \in R\}$. A relation R is called *automatic* (or *regular*) if its convolution is recognized by a finite automaton. An n-ary function $f : (\Sigma^*)^n \to \Sigma^*$ is called *automatic* if its graph is automatic.

Let $\mathcal{A} = (A; R_1, \dots, R_n, f_1, \dots, f_m, c_1, \dots, c_k)$ be a structure in a finite language. We say that $\mathcal{A}$ is *automatic* if for some finite alphabet Σ, the domain A is an automatic subset of Σ^*, and the relations $R_1, \dots, R_n$ together with the functions $f_1, \dots, f_m$ are automatic. We say that a structure is *finite automata presentable* if it is isomorphic to an automatic structure. To simplify the terminology, we will call finite automata presentable structures just *automatic*.

We will often use the following well-known facts about automatic languages and relations.

Lemma 1 (The Pumping Lemma)**.** *Let L be a language which is recognizable by an automaton with p states. Then for every $w \in L$ with $|w| \geq p$, there are strings x, y, z such that $w = xyz$, $y \neq \epsilon$ and $xy^k z \in L$ for all $k \geq 0$.*

An n-ary relation $R \subseteq (\Sigma^*)^n$ is called *locally finite* if there exist k, ℓ such that $k + \ell = n$ and for every $\bar{x} \in (\Sigma^*)^k$ the set $\{\bar{y} \in (\Sigma^*)^\ell : (\bar{x}, \bar{y}) \in R\}$ is finite.

Lemma 2 (Constant Growth Lemma [15])**.** *Let the relation $R(\bar{x}, \bar{y}) \subseteq (\Sigma^*)^n$ be locally finite. If $Conv(R)$ is recognizable by an automaton with p states, then for every $(\bar{x}, \bar{y}) \in R$,*

$$\max_i |y_i| \leq \max_j |x_j| + p.$$

Lemma 3 ([15])**.** *Let L be an infinite regular language. Then there are constants C and c such that*

$$L^{\leq n+1} \leq C|L^{\leq n}| \quad \textit{for all } n, \textit{ and}$$
$$|L^{\leq n}| \geq n/c \quad \textit{for all sufficiently large } n.$$

Here, $L^{\leq n}$ denote $L \cap \Sigma^{\leq n}$, the set of all strings from L of length less than or equal to n.

Lemma 4 (Hodgson [12]; Khoussainov and Nerode [13])**.** *If a relation R can be defined by a first order formula from other automatic relations (possibly using additional parameters and quantifiers 'there exists infinitely many' and 'there exists finitely many'), then R is itself automatic.*

A *theory* is a consistent set of sentences (first order formulas without free variables) in some language $\mathcal{L}$. A theory T is complete if for every sentence φ in language $\mathcal{L}$, either $\varphi \in T$ or $\neg\varphi \in T$. The first order theory of a structure $\mathcal{A}$ is $\mathrm{Th}(\mathcal{A}) = \{\varphi : \mathcal{A} \models \varphi\}$, the set of sentences that hold in $\mathcal{A}$. Two structures $\mathcal{A}$ and $\mathcal{B}$ are called *elementary equivalent* if $\mathrm{Th}(\mathcal{A}) = \mathrm{Th}(\mathcal{B})$.

A model $\mathcal{A}$ of a complete theory T is called *prime* if it is elementarily embeddable into every model of T. A countable model $\mathcal{A}$ of T is called *countable saturated* (or just *saturated* for short) if $\mathcal{A}$ realizes all complete types of T over finite sets of parameters. The prime and saturated models are unique up to isomorphism (in the case when they exist). Moreover, every countable model of T can be elementarily embedded into the countable saturated model. It is well-known that if T has at most countably many countable models, then T has both the countable saturated and the prime models (for more details, see Hodges' textbook on model theory [11]).

A theory T is called *κ-categorical* if any two models of T of cardinality κ are isomorphic. An $\aleph_1$-categorical theory is also called *uncountably categorical.* The countable models of $\aleph_1$- but not $\aleph_0$-categorical theory can be listed in an $\omega+1$-chain of elementary embeddings

$$\mathcal{A}_0 \preccurlyeq \mathcal{A}_1 \preccurlyeq \cdots \preccurlyeq \mathcal{A}_\omega,$$

where $\mathcal{A}_0$ is the prime and $\mathcal{A}_\omega$ is the countable saturated model of the theory (see [1]). The first order theories of the following structures provide typical examples of uncountably categorical theories: the successor structure $(\mathbb{N}, S)$, where $S(x) = x + 1$, vector spaces, algebraically closed fields, etc.

To show the elementary equivalence of some structures, we will use the method of Ehrenfeucht–Fraïssé games. The definition follows the textbook of Hodges [11]; an interested reader can find more details on Ehrenfeucht-Fraïssé games in chapters 3.2 and 3.3, including a proof of Theorem 6 below.

Definition 5. Let $\mathcal{A}$, $\mathcal{B}$ be the structures of the same language $\mathcal{L}$ and let k be a natural number. Then $\mathrm{EF}_k[\mathcal{A}, \mathcal{B}]$, the *unnested Ehrenfeucht–Fraïssé game of length k on $\mathcal{A}$ and $\mathcal{B}$*, is defined as follows. There are two players $\forall$ and $\exists$. The game is played in k steps. At the ith step of the play, player $\forall$ takes one of the structures $\mathcal{A}$, $\mathcal{B}$ and chooses an element of this structure; then player $\exists$ chooses an element of the other structure. Each player is allowed to see and remember all previous moves in the play. At the end of the play, sequences $\bar{x} = (x_i : i < k) \in \mathcal{A}^k$ and $\bar{y} = (y_i : i < k) \in \mathcal{B}^k$ have been chosen. The pair $(\bar{x}, \bar{y})$ is known as the *play.* We say that player $\exists$ *wins the play* $(\bar{x}, \bar{y})$ if and only if for every unnested atomic formula φ with m variables and for every $i_0, i_1, \ldots, i_{m-1} < k$ it holds that $\mathcal{A} \models \varphi(x_{i_0}, x_{i_1}, \ldots, x_{i_{m-1}}) \iff \mathcal{B} \models \varphi(y_{i_0}, y_{i_1}, \ldots, y_{i_{m-1}})$.

We write $\mathcal{A} \approx_k \mathcal{B}$ to mean that player $\exists$ has a winning strategy for the game $\mathrm{EF}_k[\mathcal{A}, \mathcal{B}]$.

Theorem 6 (Ehrenfeucht [6] and Fraïssé [7, 8])**.** *Let $\mathcal{L}$ be a finite first order language and let $\mathcal{A}$ and $\mathcal{B}$ be $\mathcal{L}$-structures. The structures $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent if and only if $\mathcal{A} \approx_k \mathcal{B}$ for every $k \in \omega$.*

3. Automatic prime model and non-automatic saturated model

Khoussainov and Nerode [14] asked whether for a $\aleph_1$-categorical theory it is true that whenever one model of the theory has an automatic presentation than all countable models have automatic presentations (see Question 3.3). The next result provides a negative answer to this question.

For a prime p, let $R_p = \{m/p^n : m, n \in \mathbb{Z},\ n \geq 0\}$ be the subgroup of rationals with denominators powers of p. The *Prüfer group* is defined as the quotient group $\mathbb{Z}(p^\infty) = R_p/\mathbb{Z}$.

Theorem 7. *The Prüfer group $\mathbb{Z}(p^\infty)$ has an $\aleph_1$-categorical but not $\aleph_0$-categorical theory such that the prime model is the only automatic model of the theory.*

Proof. Let T_p be the first order theory of $\mathbb{Z}(p^\infty)$. It is well-known that T_p is $\aleph_1$-categorical and all countable models of T_p are $\mathbb{Z}(p^\infty) \oplus \mathbb{Q}^n$ for $n \in \omega \cup \{\omega\}$ [11, Appendix A.2]. Nies and Semukhin [21] provide an automatic presentation of the group R_p in which the set $\mathbb{Z}$ of integers is also automatic. This gives us an automatic presentation of the Prüfer group since $\mathbb{Z}(p^\infty) = R_p/\mathbb{Z}$. So it remains to show that none of the models $\mathbb{Z}(p^\infty) \oplus \mathbb{Q}^n$ with $n > 0$ is automatic. The proof follows by invoking more general Proposition 8 below and observing that for every $n > 0$ the group $\mathbb{Z}(p^\infty) \oplus \mathbb{Q}^n$ is a supergroup of the additive group of the rational numbers.

Proposition 8. *Let $(E, +)$ be an Abelian supergroup of the rational numbers such that for every $p \in \{1, 2, 3, \ldots\}$ and every $y \in E$ there are only finitely many $x \in E$ with $p \cdot x = y$. Then the group $(E, +)$ is not automatic.*

Proof. Tsankov [23, Theorem 7] showed that for any constant C and a function $h : \mathbb{N} \to \mathbb{N}$, there is no sequence of finite subsets $A_0, A_1, \ldots$ of the set of rational numbers such that

(I) $0 \in A_0$ and $|A_0| \geq 2$;

(II) $A_n + A_n \subseteq A_{n+1}$;

(III) $|A_{n+1}| \leq C \cdot |A_n|$;

(IV) $\{m \in \mathbb{Q} : p \cdot m \in A_n\} \subseteq A_{n+h(p)}$ for all $p \in \{1, 2, 3, \ldots\}$.

Here we are using the following notations: $|D|$ is the cardinality of a finite set D and $|d|$ is the length of a string d.

Now assume by way of contradiction that the above $(E, +)$ is automatic and identify its automatic presentation with the group. Then the domain E is regular, and every relation definable from $+$ and finitely many elements of the group is automatic [12, 13]. Let A be a subset of E such that $(A, +)$ is isomorphic to a subgroup of the rationals. Let c_0 be the first length such that $-1, 0, +1$ are represented in A by strings shorter than c_0 and let c_1 be a positive constant such that

$$\forall x, y \in E \; |x \pm y| \leq \max\{|x|, |y|\} + c_1.$$

Now let $A_n = \{e \in A : |e| \leq c_0 + c_1 \cdot n\}$. Note that (I) is satisfied by the choice of c_0, and (II) by the choice of c_1.

The proof for (III) goes over the corresponding property for E. Let $E_n = \{e \in E : |e| \leq c_0 + c_1 \cdot n\}$, and let D_n be a maximal subset of A_{n+1} such that $0 \in D_n$ and for all $u \neq v$ from D_n, we have that $u - v \notin E_n$.

First, we show that there is a constant C such that $|D_n| \leq C$ for all n. It is enough to prove this for all $n \geq 1$. In this case note that

$$\forall x, y \in E_{n-1} \; \forall u, v \in D_n \; (x + u = y + v \;\rightarrow\; x = y \text{ and } u = v).$$

Indeed, if $u = v$ this is obvious. Otherwise, $u - v = y - x \in E_n$ contradicting to the choice of D_n. Since $E_{n-1} + D_n \subseteq E_{n+2}$, we have

$$|E_{n-1}| \cdot |D_n| \leq |E_{n+2}|.$$

By one of the basic properties of regular languages, there is a constant C such that $|E_{n+2}| \leq C \cdot |E_{n-1}|$. Thus $|D_n| \leq C$.

Now we will show that every $v \in A_{n+1}$ is of the form $u \pm x$ for some $u \in D_n$ and $x \in A_n$, that is $A_{n+1} \subseteq D_n \pm A_n$. So consider any $v \in A_{n+1}$. If $v \in D_n$ then v is of the given form as every member in D_n is the sum of 0 and itself. If $v \notin D_n$ then, as D_n is a maximal sets of its form, the set $D_n \cup \{v\}$ violates the constraint on the choice of D_n; so there are $u \in D_n$ and $x \in E_n$ such that $v - u = x$ or $v - u = -x$. Note that x is actually in A_n as both $v, u \in A$. Hence v is of required form.

Therefore, we obtain that

$$|A_{n+1}| \leq |D_n \pm A_n| \leq 2|D_n| \cdot |A_n| \leq 2C \cdot |A_n|.$$

This completes the proof that (III) holds.

To show (IV), fix $p \in \{1, 2, 3, \ldots\}$ and consider the relation

$$R(x, y) \iff p \cdot x = y.$$

Note that R is automatic, and by our assumption the set $\{x : R(x, y)\}$ is finite for every $y \in E$. Therefore, due to the pumping lemma there is a constant $h(p)$

such that $|x| \leq |y| + h(p)$ for all $(x, y) \in R$. So if $m \in A$ and $p \cdot m \in A_n$, then $R(m, p \cdot m)$ holds and $|m| \leq |p \cdot m| + h(p)$. Hence we obtain that $m \in A_{n+h(p)}$.

Therefore, in the case that $(E, +)$ is automatic there is a family $A_0, A_1, \ldots$ of finite subsets of the rationals which satisfies the four conditions (I), (II), (III) and (IV) given above. Since Tsankov [23] showed that such a family of sets cannot exist, $(E, +)$ cannot be an automatic group. $\square$

The version of Proposition 8 when E is assumed to be torsion-free is also a corollary of a result of Braun and Strüngmann [3].

4. Another theory with automatic prime model

Here is another example of a complete theory for which only the prime model has an automatic presentation.

Theorem 9. *A countable model of the elementary theory of* $(\mathbb{Z}, +, <)$ *is automatic if and only if it is the prime model.*

Proof. Let $T = \mathrm{Th}(\mathbb{Z}, +, <)$; it is well-known that $(\mathbb{Z}, +, <)$ is an automatic prime model of T. Now assume that $(A, +, <)$ is an automatic model of T which is different from $(\mathbb{Z}, +, <)$. Consider the relation $\equiv$ given by

$$x \equiv y \text{ iff } \{z : x < z < y \vee y < z < x\} \text{ is finite.}$$

In other words, $x \equiv y$ iff $x - y \in \mathbb{Z}$ when $\mathbb{Z}$ is considered as a subgroup of A. Note that this relation is automatic since it is first order definable from other automatic relations using quantifier 'there exists finitely many'. Furthermore, if $x < y$, $x \not\equiv y$, $v \equiv x$ and $w \equiv y$ then $v < w$. Also if $x \equiv y$ and $v \equiv w$ then $x + v \equiv y + w$. This can be seen by considering the differences $a = x - y$ and $b = v - w$. Both a, b are in $\mathbb{Z}$ and since $x + v = y + w + a + b$, we have that $x + v$ is the sum of $y + w$ and an element from $\mathbb{Z}$. Hence $x + v \equiv y + w$.

So we can consider the quotient structure $(B, +, <)$ of $(A, +, <)$ modulo $\equiv$. Note that the model $(B, +, <)$ is also automatic. Now for every $x \in B$, there exists a $y \in B$ with $y + y = x$. The reason is that since $(A, +, <)$ is elementary equivalent to $(\mathbb{Z}, +, <)$, x as an element of A is either $y + y$ or $y + y + 1$ for some $y \in A$. Therefore, x as an element of A is $\equiv$-equivalent to $y + y$. In the same way one can show that every $x \in B$ can be divided by any nonzero integer from $\mathbb{Z}$. Hence the group $(B, +, <)$ is divisible. Note that $(B, +, <)$ is nontrivial as we supposed that $(A, +, <)$ is different from $(\mathbb{Z}, +, <)$. Due to Tsankov [23, Theorem 2 (i)] the group $(B, +)$ cannot be automatic since it is torsion-free (by being linearly ordered) and p-divisible for infinitely many primes. $\square$

In this context it is interesting to point out that $(\mathbb{Z}, +)$ has an automatic presentation but its theory does not have a prime model. Therefore, the theory of an automatic model may have and may not have a prime model; so both ways are possible, answering a question of Khoussainov and Nerode [14] in the comment after Question 3.4.

5. Automatic saturated models and non-automatic prime models

Khoussainov and Nerode [14] asked in the comment after Question 3.4 whether the existence of an automatic (saturated) model of a theory implies the existence of an automatic prime model of the theory. In this section, a negative answer will be provided. First, in Theorem 10 a stronger answer is given under an unresolved complexity-theoretic hypothesis $LOGSPACE = P$ and then a weaker answer to the same question without a complexity-theoretic assumption is provided in Theorem 11.

Theorem 10. *If $LOGSPACE = P$ then there is an $\aleph_1$-categorical but not $\aleph_0$-categorical theory such that all countable models of the theory except the prime one have an automatic presentation.*

Proof. Let L be a variant of Kolmogorov complexity based on a machine U which is universal by adjunction such that $L(n)$ is the length of the shortest binary string x such that U with input x outputs 0^n and uses at most space n for the computation. Here U is assumed to be a three-tape machine with a read-only input-tape, a work-tape and a write-only unidirectional output-tape. The head on the input-tape can only move within the boundaries of x so that there are $|x| + 1$ many possible head-positions. Note that U is universal with a constant factor of space usage among all Turing machines; furthermore, "universal by adjunction" means that a the a simulated Turing machine with corresponding input are coded by a self-delimiting prefix containing the Turing table of the simulated machine followed by its input, where the same prefix can be used for any computation of that Turing machine. For all m, let

$$b_0 = 1,\ b_{m+1} = 2^{2^{2^{b_m}}} \quad \text{and}$$
$$a_m = \min\{n \geq b_m : L(n) \geq b_m\}.$$

Note that $a_m \leq 2^{b_m}$ for all m.

The models of the theory will be directed graphs. The prime model consists of two copies of finite circles of length $2a_m + 4$, for each m, in the form of a successor relation connecting each element of the circle with the next one. The further models are obtained by adding to the prime model finitely or infinitely many copies of the integers with the successor relation from x to $x + 1$, that is, chains that are infinite in both directions. We will show that the prime model is not automatic while the other models are automatic under the complexity-theoretic assumption given above.

Suppose that the prime model is automatic. First note that no circle of length $2a_m + 4$ can have an element shorter than $\log(b_m)$: To see this, assume by way of contradiction that this is not true. Note that the set of all convolutions of x and 0^n where x is part of a circle of length n is in P; this is because one can start from x and compute n times the successor of the current tape value and compare whether the result is indeed again equal to x after each round. If the number of rounds needed is exactly n then the convolution of x and 0^n is accepted, else the convolution of x and 0^n is rejected. Note that each successor of a string is at most a constant longer so that the longest string in the computation has the length $|x|+c\cdot n$ for some constant c; furthermore, each construction of a successor is done in linear time as the corresponding function giving the successor of a string is automatic. Hence the whole process is in polynomial time.

By assumption the same set is recognized by a *LOGSPACE*-machine V. One can modify V such that V instead of the convolution of x and 0^n takes as input a prefix-free coding of the two parameters x and $bin(n)$ which are coded in the form $0^{|x|}1x0^{|bin(n)|}1bin(n)$ where $bin(n)$ is the binary representation of n. Now V can check in linear space whether x belongs to a circle of length n.

In the next step one can replace V by a three-tape machine W which searches for n that corresponds to given x. On input x, W writes the part $0^{|x|}1x0^{|bin(n)|}1bin(n)$ onto one half of the work tape and starts with the initial value $n=0$. Whenever $n<2$ or the simulation shows that V does not accept $0^{|x|}1x0^{|bin(n)|}1bin(n)$ then W increases n by 1, adjusts the input on the left side of the work tape, writes a 0 onto the output tape and reruns the simulation. Once a pair is accepted, the simulation stops. Note that the work space used by W is linear in $0^{|x|}\,1\,x0^{|bin(n)|}\,1\,bin(n)$ for the final value of n, hence, for almost all n generated by a small x it is clearly below n.

It follows that whenever a circle of length $2a_m+4$ is generated by a very small x, one could compute 0^{a_m} from this x by a suitable modification of W using space below a_m and would get that $L(a_m)$ is linear in x. This modification can be translated into U and the work-space usage would increase only by a linear factor and therefore stay below a_m for infinitely many m. In other words, $L(a_m) < b_m$ for infinitely many m, a contradiction. Hence, for almost all m, only the circles belonging to $a_0, a_1, \ldots, a_{m-1}$ can contribute to strings shorter than $\log(b_m)$ and these are roughly $O(a_{m-1})$ many strings. That is, for almost all m, the number $5\cdot 2^{b_{m-1}}$ would be a safe upper bound on this number of strings although every regular set (as the domain of the automatic presentation) has at least $\log(b_m)/c$ strings below the length $\log(b_m)$ which is, by the choice, at least $2^{2^{b_{m-1}}}/c$. This gives a contradiction and thus the prime model is not automatic under the assumption $LOGSPACE = P$.

Now it is shown that the next model after the prime model is automatic. For this, one needs the following Turing machine V.

The machine V is defined such that it halts on input x iff $x=0^n$ (n times the symbol 0) and $n=a_m$ for some m; one can choose the tape alphabet size of V large enough to make sure that V uses space $|x|$ when checking whether $U(x)$ is equal to some a_m or not. If $x \notin 0^*$ then V runs forever. On input $x=0^n$, V computes the unique m with $b_m \leq n < b_{m+1}$ which can be done in space n, provided that the alphabet size is large enough. Afterwards V computes 0^{b_m}. Having this, V marks for each x with $|x| \leq b_m$ using the domain-alphabet of U how much space $U(x)$ uses in the case that $U(x)$ terminates and uses at most space n; these markings can be done in form of a word $w = w_1w_2\ldots w_n$ such that $w_k = 1$ iff there is a some terminating computation among the tested once using space k. Note that given x with $|x| \leq n$ the machine V can find out in space n whether the computation of $U(x)$ has to be considered; the reason is that V just discards the computation whenever it at some time requests more space as n and whenever it runs longer than d^n steps for some suitable constant d as such a computation runs in an infinite loop. In the case that the outcome of these tests is that $w = 1^{n-1}0$ then $n = a_m$ and V halts; otherwise V runs forever in an infinite loop.

In the following let (x, y, z) denote the convolution of x, y, z. The basic cycle looks as follows, only x and y are given and the component z is left out as it is always either 0 or 1 or 2 throughout the basic cycle.

```
Tuple Conv(x,y,z)                                Comment
f 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8          Rotate Forward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

8 f 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8          Rotate Forward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

8 8 f 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0          Rotate Forward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

0 8 8 f 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 #          Rotate Forward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

# 0 8 8 f 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3          Check 1,
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8          Change f -> b

# 0 8 8 b 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3          Rotate Backward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

0 8 8 b 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 #          Rotate Backward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

8 8 b 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0          Rotate Backward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

8 b 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8          Rotate Backward
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8

b 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8          Check 2,
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8          Change b -> f

f 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8          Origin,
# 0 2 2 # 0 2 3 # 0 2 4 # 0 3 3 # 0 8 8          starting configuration
```

Now let A be the set of all such configurations where x is a sequence of Turing configurations separated by #, so is y and the special separator “f” or “b” is between the two first occurrences of # in y. Furthermore, y starts always with # and x and y have always the same length. Furthermore, $z \in \{0, 1, 2\}$.

Let B_1 be the set of all such configurations where Check 1 is done and B_2 be the set of all such configurations where Check 2 is done. For each pair $(x, y, z) \in B_1 \cup B_2$, let (x', y', z) be the next member in the cycle. Furthermore, for $(x, y, 2) \in B_1$, let "outgoing" be the length-lexicographically next pair in B_1 after $(x, y, 2)$. Note that (x, y, z) and "outgoing" belong to different cycles and that "outgoing" is of the form $(x'', y'', 0)$ for some x'', y''. Now one modifies the successor relation on elements in B_1 and B_2 according to the following tables; in the first table, $(x, y, z) \in B_1$ and "incoming" is the last node done in a set of cycles before reaching the cycle.

tuple (x, y, z) from B_1	Successor of (x, y, z) if Check 1 true	Successor of (x, y, z) if Check 1 false
incoming	$(x', y', 0)$	$(x', y', 0)$
$(x, y, 0)$	outgoing	$(x', y', 1)$
$(x, y, 1)$	$(x', y', 1)$	$(x', y', 2)$
$(x, y, 2)$	$(x', y', 2)$	outgoing

The corresponding table for Check 2 is the following; here $(x, y, z) \in B_2$:

tuple (x, y, z) from B_2	Successor of (x, y, z) if Check 2 true	Successor of (x, y, z) if Check 2 false
$(x, y, 0)$	$(x', y', 0)$	$(x', y', 1)$
$(x, y, 1)$	$(x', y', 1)$	$(x', y', 2)$
$(x, y, 2)$	$(x', y', 2)$	$(x', y', 0)$

So the main thing is that if Check 1 or Check 2 turns out to be false, that is, if the underlying (x, y) do not code an accepting computation in V, then in at least one of these checks there is a change of the level by 1. This then connects the cycles of level 1 and level 2 with the main thread going through level 0 to change the levels and to run through all 3 cycles: in each cycle the level advances by 1 or 2 modulo 3 and it takes 3 rounds to get to the outgoing node. If Check 1 and Check 2 are both true, then the underlying (x, y) codes an accepting computation of V and the thread goes only through level 0 while the cycles on levels 1 and 2 are not connected but cycles of their own. Hence these two cycles exist iff their length is a_m for some m and their origin code the corresponding accepting computation of V.

One further feature has to be added: If $(x, y, 0)$ is the length-lexicographic least among all nodes in B_1 then this cycle has not yet an incoming node; hence one adds a descending chain $\ldots \to 3333 \to 333 \to 33 \to 3 \to (x, y, 0)$ to the model for making the thread starting at $(x, y, 0)$ to become a chain of type $(\mathbb{Z}, u \mapsto u + 1)$.

Higher models are obtained by joining copies of the integers with successor to the model, hence all countable models except the prime model have an automatic presentation. □

Theorem 11. *There is a complete theory with countably many countable models, a countable prime model and a countable saturated model such that the saturated model is automatic but the prime model is not.*

Proof. We define the theory by constructing an automatic presentation of its saturated model. The language of the theory consists of two symbols for binary relation, S and R, where S defines an equivalence relation on the elements of the model and R defines a symmetric, irreflexive relation that respects S. In other words, if $R(x, y)$ holds and x, y are S-equivalent to x', y', respectively, then $R(x', y')$ holds. Each countable model of the theory consists of possibly infinite chains of the form $a - b - \ldots - c - d$, where a, b, c, d denote the number of elements in corresponding S-equivalence classes, and all members of each equivalence class are R-connected to the members of the neighbouring equivalence classes. The saturated model of the theory consists of the following chains:

- One of the form $2 - 1 - 1 - \ldots - 1 - 2^{2^m \cdot (m+1)}$ of length $m + 2$ for each $m \geq 2$;
- Infinitely many of the form $2^n - 1 - 1 - 1 - 1 - \ldots$ for each $n \geq 1$;
- Infinitely many of the form $\ldots - 1 - 1 - 1 - 1 - \ldots$;
- Infinitely many of the form $\omega - 1 - 1 - 1 - 1 - \ldots$.

The prime model consists of the following chains:

- One of the form $2 - 1 - 1 - \ldots - 1 - 2^{2^m \cdot (m+1)}$ of length $m + 2$ for each $m \geq 2$;
- Infinitely many of the form $2^n - 1 - 1 - 1 - 1 - \ldots$ for each $n \geq 2$.

Note that if $\mathcal{A}$ is any countable model of the theory, then in addition to the chains that exist in the prime model, $\mathcal{A}$ may contain a finite or infinite number of the chains of the following forms:

- $2 - 1 - 1 - 1 - 1 - \ldots$;
- $\ldots - 1 - 1 - 1 - 1 - 1 - \ldots$; and
- $\omega - 1 - 1 - 1 - 1 - \ldots$.

These chains are limits of the chains that exist in the prime model. So adding them to the prime model does not change the first order theory of the structure. This can be shown more formally using the technique of Ehrenfeucht–Fraïssé games. On the other hand, $\mathcal{A}$ cannot contain chains of any other form since that would violate the first order theory of the structure. Therefore, all countable models of the theory must have the form described above.

We now explain how to construct an automatic presentation of the saturated model. The prototype of the finite chains would be the following nodes given as convolutions of two strings x and y (an example is given for the case of $m = 3$):

```
Node (convolution of x and y)           Equivalence Class Size
#000#001#010#011#100#101#110#111                     2
#000#001#010#011#100#101#110#111

1#000#001#010#011#100#101#110#11                     1
#000#001#010#011#100#101#110#111

11#000#001#010#011#100#101#110#1                     1
#000#001#010#011#100#101#110#111

111#000#001#010#011#100#101#110#                     1
#000#001#010#011#100#101#110#111

#111#000#001#010#011#100#101#110             4294967296
#000#001#010#011#100#101#110#111
```

To obtain multiple copies of the same node, we convolute it with a third string z, which is equal to 0^{32} or 1^{32} in the case of the first node (to obtain a class of size 2) and which is equal to any string from $\{0,1\}^{32}$ in the case of the last node (to obtain a class of size 2^{32}). For the intermediate nodes, z is equal to 0^{32}.

Here is a formal construction of an automatic presentation of the saturated model: Let S denote a regular expression for the set of strings in $\{0,1\}^*$ that contain at least one 0 and at least one 1, e.g., $S = 1^+0(0+1)^* + 0^+1(0+1)^*$. Then the domain of the structure contains the following convolutions:

(1) $Conv(x, y, z)$ for all $x, y, z \in \{0, 1, \#\}^*$ with $|x| = |y| = |z|$ such that y is of the form $\#0^+(\#S)^*\#1^+$, and either

  (a) x is of the form $\#0^+(\#S)^*\#1^+$. In this case, $z \in \{0^{|x|}, 1^{|x|}\}$ if $x = y$, and $z = 0^{|x|}$ otherwise.

  (b) x is of the form $1^+\#0^+(\#S)^*\#1^*$ and $z = 0^{|x|}$.

  (c) x is of the form $\#1^+\#0^+(\#S)^*$. In this case, $z \in \{0,1\}^{|x|}$ if the following condition is satisfied: the $\#$'s of x and y are located at the same positions above each other, and every entry of the digits $a_1a_2\dots a_m$ between two $\#$'s of x or at the end of x is obtained from the corresponding entry $b_1b_2\dots b_m$ of y by subtracting 1 modulo 2^m; here note that m depends on the position of the corresponding entries in x and y. If this condition does not hold, then $z = 0^{|x|}$.

(2) $Conv(x0^k, y, z)$ where $k \in \omega$, $|x| = |y| = |z|$, $z = 0^{|x|}$ and either

  (a) x and y are of the form $\#0^+(\#S)^*\#1^+$, and $x \neq y$; or

(b) x is of the form $\#1^+\#0^+(\#S)^*$, y is of the form $\#0^+(\#S)^*\#1^+$, and the condition described in the item (1c) above does not hold.

In this presentation, two convolutions $Conv(x_1, y_1, z_1)$ and $Conv(x_2, y_2, z_2)$ are S-equivalent if $x_1 = x_2$ and $y_1 = y_2$. The relation R is defined as follows: $Conv(x_1, y_1, z_1)$ is R-connected to $Conv(x_2, y_2, z_2)$ if $y_1 = y_2$ and either

- $|x_1| = |x_2|$ and x_1 is a cyclic shift of x_2 by one symbol to the left or to the right; or
- $x_1 = x_2 0$ or $x_2 = x_1 0$.

This structure is then enriched by adding infinitely many copies of the chains $2^n - 1 - 1 - 1 - \ldots$ for each $n \geq 1$ and infinitely many copies of $\omega - 1 - 1 - 1 - \ldots$ and $\ldots - 1 - 1 - 1 - 1 - \ldots$. For example, this can be done by adding the following convolutions to the structure (S-equivalence on these new elements is defined in the same way as above):

- $Conv(0^k, 10^m, z)$, where $k,\ m \geq 1$, and $z \in 0^+$, if $k = 1$, and $z = 0$, otherwise. The relation R is defined such that $Conv(0^k, 10^m, z_1)$ is connected to $Conv(0^{k+1}, 10^m, z_2)$ for all z_1, z_2. This produces infinitely many copies of $\omega - 1 - 1 - 1 - \ldots$.
- $Conv(0^k, 1^n 0^m, z)$, for every $n \geq 2$ and $k,\ m \geq 1$, such that $z \in \{0,1\}^{n-1}$, if $k = 1$, and $z = 0$, otherwise. The relation R is defined such that $Conv(0^k, 1^n 0^m, z_1)$ is connected to $Conv(0^{k+1}, 1^n 0^m, z_2)$ for all z_1, z_2. This produces infinitely many copies of $2^n - 1 - 1 - \ldots$ for each $n \geq 1$.
- $Conv(0^k, 0^m, 0)$, where $k,\ m \geq 1$, and $Conv(0^k, 0^m, 0)$ is connected to $Conv(0^\ell, 0^m, 0)$ if either $|k - \ell| = 2$ or $\{k, \ell\} = \{1, 2\}$. This produces infinitely many copies of $\ldots - 1 - 1 - 1 - 1 - \ldots$.

It is a routine exercise to check that the domain of the structure and the relations defined above can be recognized by finite automata.

It is quite straightforward to verify that the constructed model contains the same chains as the saturated model. For instance, every chain added in the first part of the construction contains a node $Conv(x, y, z)$ such that $|x| = |y| = |z|$ and x, y are both of the form $\#0^+(\#S)^*\#1^+$. Now consider the following cases:

- $x = y$ and y is of the form $\#bin_m(0)\#bin_m(1)\#\ldots\#bin_m(2^m - 1)$ (here $bin_m(k)$ is the m-digit binary number representing k). In this case, condition (1c) is satisfied after shifting x to the right until it has the form $\#1^+\#0^+(\#S)^*$. Therefore, this node belongs to a chain $2 - 1 - 1 - \ldots - 1 - 2^{2^m \cdot (m+1)}$ of length $m + 2$ as both x and y have length $2^m(m + 1)$.
- $x = y$ but y is not of the form $\#bin_m(0)\#bin_m(1)\#\ldots\#bin_m(2^m - 1)$. In this case the node belongs to an infinite chain of the form $2 - 1 - 1 - \ldots$.

- $x \neq y$. In this case, the node either belongs to an infinite chain of the form $2^n - 1 - 1 - \dots$, for some $n \geq 2$, or of the form $\dots - 1 - 1 - 1 - \dots$, depending on whether condition (1c) is satisfied or not after shifting x to the right until it has the form $\#1^+\#0^+(\#S)^*$.

In the remaining part of the proof we will show that the prime model does not have an automatic presentation. So, assume by way of contradiction that there is an automatic presentation of the prime model. Then every equivalence class consisting of 2 members belongs to a chain of the form $2 - 1 - 1 - \dots - 1 - 2^{2^m \cdot (m+1)}$. Let c_1 be a constant such that there are at most $2^{c_1 n}$ strings of length n in the automatic structure. Hence, the longest string in the equivalence class of the other end of the chain has the length at least $2^m(m+1)/c_1$. By the Pumping Lemma, the length difference between two connected nodes in two adjacent finite equivalence classes is bounded by a constant c_2. So, the length of the two representatives in the first equivalence class with 2 members is at least $2^m(m+1)/c_1 - (m+1)c_2 = (m+1)(2^m/c_1 - c_2)$. This term grows superlinearly in m while the set of all strings which are in an equivalence class with exactly two members is definable by a first order formula and hence regular. This gives a contradiction as by Lemma 3 every infinite regular set has below length n (for sufficiently large n) at least n/c_3 many members for some constant c_3. Therefore, the prime model does not have an automatic presentation. Moreover, every automatic model must have infinitely many copies of the chain $2 - 1 - 1 - 1 - \dots$ which do not exist in the prime model. $\square$

One can easily construct a structure which has uncountably many countable models such that the prime model and the saturated model are both automatic. An example would be to take as a prime model the structure of all chains of the form $n - 1 - 1 - \dots - 1 - 1 - m$ such that there are $n + m$ nodes of the form 1 in the chain between the nodes consisting of equivalence classes with n and m members, respectively. Each of these finite chains exists in one copy. The theory of this prime model also has a saturated model which in addition to the above mentioned finite chains contains infinitely many copies of the infinite chains of the form $\dots - 1 - 1 - 1 - \dots$ and $n - 1 - 1 - \dots$, where $n \in \{1, 2, \dots, \omega\}$. Between these two automatic models, there are uncountably other countable models as one can assign every cardinality from $\{0, 1, 2, \dots, \aleph_0\}$ to the number of occurrences of the chain $n - 1 - 1 - \dots$ which then gives $2^{\aleph_0}$ many countable models. Clearly, most of these models are not automatic. Somehow, it is still an open problem whether the same can be done with countably many countable models.

Open Problem 12. *Is there a complete theory with countably many countable models such that both the prime model and the saturated model of the theory are automatic but some other countable model of the theory is not automatic?*

6. Finitely many automatic models

Khoussainov and Nerode [14] asked in Question 3.2 whether there is for every finite $n \geq 1$ a complete theory with exactly n automatic models. The answer to this question is affirmative. First, the answer is given for the case of $n \geq 3$. Then we will give an answer for the case of $n = 2$. The main difference between these two cases is that the constructed complete theory has non-automatic countable models, as there is no complete theory with exactly two countable models [4, 11]. It should be noted that the second result can be extended to having a complete theory with infinitely many countable models such that exactly n of them are automatic, where $n \geq 2$. For $n = 1$, the Prüfer group already gives the affirmative answer as shown in Theorem 7.

The next result is an adjustment of the well-known construction of a complete theory with exactly three countable models.

Theorem 13. *There is a complete theory with exactly three countable models, all of which are automatic.*

Proof. It is well-known that the theory of the dense linear order (without end-points) enriched by additional constants $x_0, x_1, x_2, \ldots$ ordered as $x_0 < x_1 < x_2 < \ldots$ has three countable models, depending on whether the above sequence $x_0, x_1, x_2, \ldots$ is unbounded, or has a supremum x_ω, or is bounded but without a supremum. We will adopt this construction for the case of automatic structures. As the signature has to be finite, the basic idea is to use a preordering instead of the ordering above and to replace every element by an equivalence class in this preordering. In this case, x_n will be a representative of the least equivalence class containing $n+1$ elements, and every equivalence class which is larger than x_n in the preordering contains at least $n+1$ elements.

Now we will shown how to construct the smallest automatic model of the theory. For this, let us define the lexicographic preorder $\leq$ based on $0 < 1 < 2 \equiv 3$, where the digits 2 and 3 can be interchanged without changing the position of the string in the preordering. For example,

$$01 < 211 \equiv 311 < 220.$$

Let the domain of the structure be 2^*3^*L, where $L = 0 + 0(0+1)^*1$, and let the preorder be defined as above. Note that in this structure, L represents an automatic dense linear order with the least element 0. In this domain, let $x_n = 2^n0$; so $x_0 = 0$, $x_1 = 20$, $x_2 = 220$ and so on. Then each half-open interval $X_n = \{y : x_n \leq y < x_{n+1}\}$ consists of equivalence classes with exactly $n+1$ elements, and x_n represents the least equivalence class in this interval. Also the half-open interval X_n has no greatest element, and it is densely linearly ordered (modulo the equivalence relation).

Now let T be the theory of just constructed structure whose language consists of one binary relation $\leq$ interpreted as a preordering. The following axioms are true in T:

- The preordering gives, modulo its equivalence relation, a dense linear ordering with a least element but without a greatest one.
- For every $n \in \omega$, there is an element x_n such that the equivalence class of a y contains at least $n+1$ elements if and only if $x_n \le y$.
- x_n is not equivalent to x_{n+1} for all $n \in \omega$, that is, $x_0 < x_1 < x_2 < \dots$.

Note that here "For every $n \in \omega$, there is …" is not actually implemented by quantifying over n but by a list of corresponding axioms, one for each $n \in \omega$. Similarly, the third item is also a list of axioms. Note that the x_n are not constants in the theory but defined by the first order formulas given in the second item. To be more precise, these formulas define equivalence classes of x_n's.

If $\mathcal{A}$ is a model of the above theory, then for every $y \in \mathcal{A}$, the equivalence class of y has infinitely many members if and only if y is an upper bound for all x_n's. As the models considered are countable, all such equivalence classes are countable. Furthermore, these y's form a dense linear ordering without a greatest element. So, any model of the theory is either isomorphic to the one constructed above or to a model which is obtained from it by adding a dense interval X_ω consisting of infinite equivalence classes appended after all the intervals $X_0, X_1, X_2, \dots$. Note that X_ω might have or not have a least element x_ω, which gives two additional models of the theory.

Both additional models have automatic presentations. One can construct the model with the least element in X_ω as follows. Add a new symbol 4 to the alphabet, and let the domain of the structure be $2^*3^*L + 4L4^*$. As before, let the strings from 2^*3^*L be ordered according to the lexicographical preordering. Also, for every $u \in 2^*3^*L$ and $v \in 4L4^*$, let $u < v$. On $4L4^*$ the preordering is defined as follows: for every $n, m \in \omega$, let

$$4u4^n \le 4v4^m \text{ if and only if } u \le_{lex} v.$$

Note that this definition implies that every equivalence class in $4L4^*$ is countable. Indeed, the equivalence class of $4u$ is equal to $4u4^*$. In this presentation, 404^* is the least infinite equivalence class. The model without the least infinite equivalence class would then have the domain $2^*3^*L + 4L'4^*$ and the preordering relation defined as before. Here $L' = 0(0+1)^*1$ represents the dense linear order without end points.

It remains to show that any two of the above models are elementary equivalent. Let $\mathcal{A}$ and $\mathcal{B}$ be such models. By Theorem 6, it suffices to show that $\mathcal{A} \approx_k \mathcal{B}$ for all $k \in \omega$. Let A and B denote the domains of $\mathcal{A}$ and $\mathcal{B}$, respectively, and let $\le$ denote the preordering in both of these models. Furthermore, let $\tilde{A}$ and $\tilde{B}$ be the equivalence classes of A and B induced by the preordering $\le$. Let $\tilde{a}_0, \tilde{a}_1, \dots, \tilde{a}_{k-1}$ and $\tilde{b}_0, \tilde{b}_1, \dots, \tilde{b}_{k-1}$ be the least equivalence classes in A and B with $1, 2, \dots, k$ elements, respectively. As both sets $\tilde{A}$ and $\tilde{B}$ are dense linear orders with least element $\tilde{a}_0$ and $\tilde{b}_0$, respectively, and as $\tilde{a}_0 < \tilde{a}_1 < \dots < \tilde{a}_{k-1}$ and $\tilde{b}_0 < \tilde{b}_1 < \dots < \tilde{b}_{k-1}$, there is an order isomorphism f from $\tilde{A}$ to $\tilde{B}$ with

$f(\tilde{a}_0) = \tilde{b}_0$, $f(\tilde{a}_1) = \tilde{b}_1$, …, $f(\tilde{a}_{k-1}) = \tilde{b}_{k-1}$. Now the player $\exists$ uses the following winning strategy in the k-round Ehrenfeucht–Fraïssé game: Whenever player $\forall$ picks an element $a \in A$ (or $b \in B$) which has not yet been used, player $\exists$ picks an element $b \in B$ (or $a \in A$, respectively) which has not been used such that f maps the equivalence class of a to that of b. Clearly the ordering between the elements picked in A and the counterparts in B is the same, the only thing to be verified is that player $\exists$ does not end up in a situation where it cannot pick a new element as all elements in the corresponding equivalence class are already used up. To see this, note that the following holds for equivalence classes $\tilde{a}$ and $\tilde{b}$ with $f(\tilde{a}) = \tilde{b}$: either

- $\tilde{a}_\ell \leq \tilde{a} < \tilde{a}_{\ell+1}$ and $\tilde{b}_\ell \leq \tilde{b} < \tilde{b}_{\ell+1}$ for some $\ell < k-1$, in which case $|\tilde{a}| = |\tilde{b}| = \ell + 1$, or
- $\tilde{a}_{k-1} \leq \tilde{a}$ and $\tilde{b}_{k-1} \leq \tilde{b}$, in which case $k \leq \min\{|\tilde{a}|, |\tilde{b}|\}$.

In the first case, we have that at the beginning of each round the same amount of elements are free in $\tilde{a}$ and $\tilde{b}$, respectively, and that during the round either none or one element from each class is selected, so that this property is preserved. In the second case, there are at least k elements in both $\tilde{a}$ and $\tilde{b}$. Therefore, player $\exists$ can always pick a new element even if player $\forall$ selects the same equivalence class in every round. Hence player $\exists$ has a winning strategy for the Ehrenfeucht–Fraïssé game, and, therefore, the two models $\mathcal{A}$ and $\mathcal{B}$ are elementary equivalent. $\square$

One can generalize this result to show that there is a theory with $n+2$ countable models all of which are automatic, where $n \geq 1$. To do so, we enrich the theory by n new predicates $P_1, \ldots, P_n$ which satisfy the following conditions:

- Every x belongs to exactly one P_k, and if x and y are equivalent, then $x \in P_k$ iff $y \in P_k$, for every $k = 1, \ldots, n$.
- For each x, y with $x < y$ there are $z_1, z_2, \ldots, z_n$ with $x < z_1 < z_2 < \ldots < z_n < y$ such that $z_k \in P_k$ for every $k = 1, \ldots, n$.
- Every x_n belongs to P_1.

Then this theory has $2+n$ models. The first one doesn't have infinite equivalence classes. The second one has infinite classes but doesn't have the least class among them. The other n models contain x_ω (a representative from the smallest infinite class), and they are distinguished by the number k for which $x_\omega \in P_k$ holds.

Corollary 14. *For every $n \geq 3$, there is a complete theory with exactly n countable models, all of which are automatic.*

In the next theorem we will give an answer for the case of $n = 2$.

Theorem 15. *There is a complete first order theory that has exactly two automatic models.*

Proof. First, we construct two models $\mathcal{A}$ and $\mathcal{A}'$ of the same first order theory T that have automatic presentations. The language of the theory will consist of one unary symbol U, two binary symbols S, R, and one ternary relation *Plus*. The model $\mathcal{A}$ consists of:

- An infinite set $U^{\mathcal{A}} = \{u_i : i \in \omega\}$.
- An equivalence relation S on the elements of $\mathcal{A}$ that are not in $U^{\mathcal{A}}$.
- On every S-equivalence class we define the relation *Plus* such that it becomes isomorphic to the Prüfer group $\mathbb{Z}(2^\infty)$.
- For every $i < j$, there is a unique equivalence class $S_{i,j}$ and a unique element $s_{i,j} \in S_{i,j}$ such that $R(u_i, s_{i,j})$ and $R(u_j, s_{i,j})$ hold, where $s_{i,j}$ represents an element of order 2^i in the group structure of $S_{i,j}$.
- There are no other S-equivalence classes apart from the $S_{i,j}$ for $i < j$ described above.

The structure $\mathcal{A}'$ is obtained from $\mathcal{A}$ by adding one more element u_ω to $U^{\mathcal{A}}$ and new equivalence classes $S_{i,\omega}$ for every $i \in \omega$ such that

- $U^{\mathcal{A}'} = U^{\mathcal{A}} \cup \{u_\omega\}$,
- $R(u_i, s_{i,\omega})$ and $R(u_\omega, s_{i,\omega})$ hold for every $i \in \omega$, where $s_{i,\omega}$ is an element of order 2^i in $S_{i,\omega}$.
- The relation *Plus* is extended to the new classes $S_{i,\omega}$ so that they become isomorphic to the Prüfer group $\mathbb{Z}(2^\infty)$.

One can think of the structure $\mathcal{A}$ as a graph with labelled edges, where $U^{\mathcal{A}} = \{u_0, u_1, \ldots\}$ is the set of vertices of the graph, and relation R defines the edges with labels from S-equivalence classes. More precisely, one can think that a vertex u is connected to v by an edge labelled with n iff $R(u, s)$ and $R(v, s)$ hold for an element s of order 2^n. For instance, for every $i < j$, u_i is connected to u_j by an edge with label i.

Lemma 16. *The structures $\mathcal{A}$ and $\mathcal{A}'$ are elementary equivalent.*

We will use Ehrenfeucht–Fraïssé games to prove the lemma (see Theorem 6). In the following, $\mathrm{ord}(a)$ denote the order of an element a in a group, that is, $\mathrm{ord}(a)$ is the least $r \in \omega$ such that $r \cdot a = 0$.

Lemma 17. *Let $r > 0$ be a natural number and let $x_0, \ldots, x_{n-1}$ and $y_0, \ldots, y_{n-1}$ be elements of the Prüfer group $\mathbb{Z}(2^\infty)$ such that for every $l_0, \ldots, l_{n-1} \in \mathbb{Z}$ with $|l_i| \leq 2r^2$ and every $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq r$ we have that*

$$\sum_{i \leq n-1} l_i x_i = a \iff \sum_{i \leq n-1} l_i y_i = a.$$

Then

(a) *For every* x_n*, there exist* y_n *such that for every* $l_0, \dots, l_n \in \mathbb{Z}$ *with* $|l_i| \leq r$ *and every* $a \in \mathbb{Z}(2^\infty)$ *with* $\mathrm{ord}(a) \leq r$*, we have that*

$$\sum_{i \leq n-1} l_i x_i + l_n x_n = a \iff \sum_{i \leq n-1} l_i y_i + l_n y_n = a.$$

(b) *For every* y_n*, there exist* x_n *such that for every* $l_0, \dots, l_n \in \mathbb{Z}$ *with* $|l_i| \leq r$ *and every* $a \in \mathbb{Z}(2^\infty)$ *with* $\mathrm{ord}(a) \leq r$*, we have that*

$$\sum_{i \leq n-1} l_i x_i + l_n x_n = a \iff \sum_{i \leq n-1} l_i y_i + l_n y_n = a.$$

Proof. We will prove part (a) since (b) will hold by symmetry. Let us fix some $x_n \in \mathbb{Z}(2^\infty)$ and find y_n that satisfies the required properties. It is enough to consider only the cases when $l_n \neq 0$ since for $l_n = 0$ the equivalences hold trivially for any choice of y_n.

First, suppose that for all $l_0, \dots, l_n \in \mathbb{Z}$ with $|l_i| \leq r$ and $l_n \neq 0$ and for all $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq r$ we have that $\sum_{i \leq n-1} l_i x_i + l_n x_n \neq a$. Note that for any $l_0, \dots, l_n \in \mathbb{Z}$ with $l_n \neq 0$ and any $a \in \mathbb{Z}(2^\infty)$ there are only finitely many $y_n \in \mathbb{Z}(2^\infty)$ satisfying the equation $\sum_{i \leq n-1} l_i y_i + l_n y_n = a$. Since there are only finitely many choices for such $l_0, \dots, l_n$ and a, we can always find y_n such that $\sum_{i \leq n-1} l_i y_i + l_n y_n \neq a$ for all $l_0, \dots, l_n$ with $|l_i| \leq r$ and $l_n \neq 0$ and all a with $\mathrm{ord}(a) \leq r$.

Now suppose that there are $l_0, \dots, l_n \in \mathbb{Z}$ with $|l_i| \leq r$ and $l_n \neq 0$ and $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq r$ such that

$$\sum_{i \leq n-1} l_i x_i + l_n x_n = a. \tag{1}$$

Suppose that the values $l_0, \dots, l_n$ and a in the above equation are chosen such that $\mathrm{pow}_2(l_n)$ has the smallest possible value. Here $\mathrm{pow}_2(n)$ denotes the largest s such that 2^s divides n. Let y_n be any element of $\mathbb{Z}(2^\infty)$ satisfying

$$\sum_{i \leq n-1} l_i y_i + l_n y_n = a. \tag{2}$$

We now show that for every $l'_0, \dots, l'_n$ with $|l'_i| \leq r$ and every a' with $\mathrm{ord}(a') \leq r$,

$$\sum_{i \leq n-1} l'_i x_i + l'_n x_n = a' \iff \sum_{i \leq n-1} l'_i y_i + l'_n y_n = a'.$$

Suppose that

$$\sum_{i \leq n-1} l'_i x_i + l'_n x_n = a'. \tag{3}$$

Again, assume that $l'_n \neq 0$ since otherwise the implication holds trivially by assumption. Let $C = \gcd(l_n, l'_n)$ and let k_n, k'_n be such that $C = l_n k_n = l'_n k'_n$. Multiplying (1) by k_n and (3) by k'_n and subtracting them, we get

$$\sum_{i \leq n-1} (l_i k_n - l'_i k'_n) x_i = a k_n - a' k'_n.$$

Since $|l_i k_n - l'_i k'_n| \leq 2r^2$ and $\mathrm{ord}(ak_n - a'k'_n) \leq r$, we have by our assumption that

$$\sum_{i \leq n-1} (l_i k_n - l'_i k'_n) y_i = ak_n - a'k'_n.$$

Subtracting this from (2) multiplied by k_n and taking into account that $l_n k_n = l'_n k'_n$ we obtain that

$$k'_n \Big(\sum_{i \leq n-1} l'_i y_i + l'_n y_n \Big) = k'_n a',$$

which in turn implies that $\sum_{i \leq n-1} l'_i y_i + l'_n y_n = a'$ since k'_n must be odd number due to the fact that we have chosen l_n with the minimal possible value of $\mathrm{pow}_2(l_n)$.

Now suppose that $\sum_{i \leq n-1} l'_i y_i + l'_n y_n = a'$. If $l'_n = 0$ then there is nothing to prove. Otherwise, let $C = \gcd(l_n, l'_n)$ and let k_n, k'_n be such that $C = l_n k_n = l'_n k'_n$. By a similar argument as above we obtain that

$$k'_n \Big(\sum_{i \leq n-1} l'_i x_i + l'_n x_n \Big) = k'_n a'.$$

If $\mathrm{pow}_2(l'_n) \geq \mathrm{pow}_2(l_n)$ then k'_n must be odd and we have that $\sum_{i \leq n-1} l'_i x_i + l'_n x_n = a'$. If $\mathrm{pow}_2(l'_n) < \mathrm{pow}_2(l_n)$ then

$$\sum_{i \leq n-1} l'_i x_i + l'_n x_n = a' + d,$$

where $\mathrm{ord}(d) \leq 2^{\mathrm{pow}_2(k'_n)} \leq |k'_n| \leq r$. Thus $\mathrm{ord}(a' + d) \leq r$ and $\mathrm{pow}_2(l'_n) < \mathrm{pow}_2(l_n)$ which contradicts the choice of l_n. Hence this case is impossible. $\square$

Lemma 18. *Consider $k \in \omega$ and two tuples $x_0, \ldots, x_{n-1}$, $y_0, \ldots, y_{n-1}$ from the Prüfer group $\mathbb{Z}(2^\infty)$ such that for every $l_0, \ldots, l_{n-1} \in \mathbb{Z}$ with $|l_i| \leq 2^{2^{k+1}-1}$ and every $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq 2^{2^{k+1}-1}$ we have that*

$$\sum_{i \leq n-1} l_i x_i = a \iff \sum_{i \leq n-1} l_i y_i = a.$$

Then $(\mathbb{Z}(2^\infty), x_0, \ldots, x_{n-1}) \approx_k (\mathbb{Z}(2^\infty), y_0, \ldots, y_{n-1})$.

Proof. The proof is by induction on k. Let $k = 0$; we need to show that the tuples $(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$ satisfy the same unnested atomic formulas. Each unnested atomic formula in the language of groups has the form $x = y$ or $x + y = z$. By assumption, for every $l_0, \ldots, l_{n-1} \in \mathbb{Z}$ with $|l_i| \leq 2$ and every $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq 2$, we have that $\sum_{i \leq n-1} l_i x_i = a$ iff $\sum_{i \leq n-1} l_i y_i = a$. This is clearly enough to guarantee that the same unnested atomic formulas are true on $(x_0, \ldots, x_{n-1})$ and $(y_0, \ldots, y_{n-1})$.

Now assume that $k > 0$ and that the statement of the lemma holds for $k-1$. Applying Lemma 17 for $r = 2^{2^k-1}$, one can show that:

- For every x_n, there exist y_n such that for every $l_0, \ldots, l_n \in \mathbb{Z}$ with $|l_i| \leq 2^{2^k-1}$ and every $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq 2^{2^k-1}$, we have that

$$\sum_{i \leq n-1} l_i x_i + l_n x_n = a \iff \sum_{i \leq n-1} l_i y_i + l_n y_n = a.$$

- For every y_n, there exist x_n such that for every $l_0, \ldots, l_n \in \mathbb{Z}$ with $|l_i| \leq 2^{2^k-1}$ and every $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq 2^{2^k-1}$, we have that

$$\sum_{i \leq n-1} l_i x_i + l_n x_n = a \iff \sum_{i \leq n-1} l_i y_i + l_n y_n = a.$$

By inductive hypothesis, we obtain the following:

- For every x_n, there exist y_n such that

$$(\mathbb{Z}(2^\infty),\, x_0, \ldots, x_{n-1}, x_n) \approx_{k-1} (\mathbb{Z}(2^\infty),\, y_0, \ldots, y_{n-1}, y_n).$$

- For every y_n, there exist x_n such that

$$(\mathbb{Z}(2^\infty),\, x_0, \ldots, x_{n-1}, x_n) \approx_{k-1} (\mathbb{Z}(2^\infty),\, y_0, \ldots, y_{n-1}, y_n).$$

Therefore, $(\mathbb{Z}(2^\infty),\, x_0, \ldots, x_{n-1}) \approx_k (\mathbb{Z}(2^\infty),\, y_0, \ldots, y_{n-1})$. □

Corollary 19. *Let $c, d \in \mathbb{Z}(2^\infty)$ be such that either one of the following holds*

(a) $\mathrm{ord}(c) \leq 2^{2^{k+2}-2}$ *and* $\mathrm{ord}(c) = \mathrm{ord}(d)$,

(b) $\mathrm{ord}(c),\ \mathrm{ord}(d) > 2^{2^{k+2}-2}$.

Then $(\mathbb{Z}(2^\infty),\, c) \approx_k (\mathbb{Z}(2^\infty),\, d)$.

Proof. In the first case, $(\mathbb{Z}(2^\infty),\, c)$ is isomorphic to $(\mathbb{Z}(2^\infty),\, d)$, and hence it is obvious that $(\mathbb{Z}(2^\infty),\, c) \approx_k (\mathbb{Z}(2^\infty),\, d)$.

In the second case, for every $l \in \mathbb{Z}$ such that $|l| \leq 2^{2^{k+1}-1}$ and $l \neq 0$, we have that $\mathrm{ord}(lc) > 2^{2^{k+1}-1}$ and $\mathrm{ord}(ld) > 2^{2^{k+1}-1}$. Thus, for every l with $|l| \leq 2^{2^{k+1}-1}$ and every $a \in \mathbb{Z}(2^\infty)$ with $\mathrm{ord}(a) \leq 2^{2^{k+1}-1}$, we have that $lc = a$ iff $lc = a$. Lemma 18 now implies that $(\mathbb{Z}(2^\infty),\, c) \approx_k (\mathbb{Z}(2^\infty),\, d)$. □

To describe the main strategy for the $\exists$ player in $\mathrm{EF}_k[\mathcal{A}, \mathcal{A}']$, we first introduce two axillary functions

$$F[\bar{p}, \bar{q}] : \omega \to \omega \cup \{\omega\} \text{ and } F'[\bar{p}, \bar{q}] : \omega \cup \{\omega\} \to \omega$$

which depend on parameters $\bar{p} = (p_0, \ldots, p_t) \in \omega$ and $\bar{q} = (q_0, \ldots, q_t) \in \omega \cup \{\omega\}$ with the property that for any $i, j \leq t$,

- $p_i = p_j$ iff $q_i = q_j$,

- $p_i > 2^{k+2} - 2$ iff $q_i > 2^{k+2} - 2$,
- if $p_i \leq 2^{k+2} - 2$ then $p_i = q_i$.

Let $n \in \omega$; define $F[\bar{p}, \bar{q}](n)$ as follows:

- if $n = p_i$ for some $i \leq t$, then let $F[\bar{p}, \bar{q}](n) = q_i$;
- otherwise, consider the following two cases:
  (a) if $n \leq 2^{k+2} - 2$ then let $F[\bar{p}, \bar{q}](n) = n$;
  (b) if $n > 2^{k+2} - 2$ then let $F[\bar{p}, \bar{q}](n)$ be the least $m > 2^{k+2} - 2$ such that $m \notin \{q_0, \dots, q_t\}$.

The value of $F'[\bar{p}, \bar{q}](n)$ is defined in a similar way by replacing all p_i's with q_i's and vice versa in the definition above.

For every x from $\mathcal{A}$ or $\mathcal{A}'$, define $R_0(x)$ and $R_1(x)$ as follows:

- If $x \in U$, that is, $x = u_i$ for some $i \in \omega \cup \{\omega\}$, then let $R_0(x) = R_1(x) = i$.
- If $x \in S_{i,j}$ for some $i, j \in \omega \cup \{\omega\}$, then let $R_0(x) = i$ and $R_1(x) = j$.

Let us fix $k \in \omega$ and show that $\mathcal{A} \approx_k \mathcal{A}'$, that is, player $\exists$ has a winning strategy for the game $\mathrm{EF}_k[\mathcal{A}, \mathcal{A}']$. Suppose that the players have already made $n < k$ steps in the game, and the tuples $\bar{x} = (x_0, \dots, x_{n-1}) \in \mathcal{A}$ and $\bar{y} = (y_0, \dots, y_{n-1}) \in \mathcal{A}'$ have been chosen by the players such that the following conditions are satisfied:

(1) The tuples $\bar{x}$ and $\bar{y}$ satisfy the same unnested atomic formulas, that is, for every $s, t, r \leq n - 1$

- $x_s = x_t$ iff $y_s = y_t$;
- $U^{\mathcal{A}}(x_s)$ iff $U^{\mathcal{A}'}(y_s)$;
- $R^{\mathcal{A}}(x_s, x_t)$ iff $R^{\mathcal{A}'}(y_s, y_t)$;
- $S^{\mathcal{A}}(x_s, x_t)$ iff $S^{\mathcal{A}'}(y_s, y_t)$;
- if x_s, x_t, x_r are in the same S-equivalence class, then $Plus^{\mathcal{A}}(x_s, x_t, x_r)$ iff $Plus^{\mathcal{A}'}(y_s, y_t, y_r)$.

(2) For every $s, t \leq n - 1$,

$$|\{R_0(x_s), R_1(x_s)\} \cap \{R_0(x_t), R_1(x_t)\}| =$$
$$|\{R_0(y_s), R_1(y_s)\} \cap \{R_0(y_t), R_1(y_t)\}|.$$

(3) For every $s \leq n - 1$ and $e \in \{0, 1\}$, either one of the following holds:

- $R_e(x_s) \leq 2^{k+2} - 2$ and $R_e(x_s) = R_e(y_s)$;
- $R_e(x_s)$, $R_e(y_s) > 2^{k+2} - 2$.

(4) For every $s \leq n-1$ such that $x_s \in S^{\mathcal{A}}_{i_0,i_1}$ and $y_s \in S^{\mathcal{A}'}_{j_0,j_1}$ for some $i_0, i_1, j_0, j_1 \in \omega \cup \{\omega\}$, the following condition holds: Let $\bar{u}$ and $\bar{v}$ be the subsequences of $\bar{x}$ and $\bar{y}$ consisting of elements belonging to $S^{\mathcal{A}}_{i_0,i_1}$ and $S^{\mathcal{A}'}_{j_0,j_1}$, respectively. Then the play $(\bar{u}, \bar{v})$ agrees with a winning strategy for player $\exists$ in the game

$$\mathrm{EF}_k[(S^{\mathcal{A}}_{i_0,i_1}, s^{\mathcal{A}}_{i_0,i_1}), (S^{\mathcal{A}'}_{j_0,j_1}, s^{\mathcal{A}'}_{j_0,j_1})].$$

Such a winning strategy exists due to condition (3) above and Corollary 19.

Note that these conditions hold trivially at the beginning of the play.

We will show that for every $x_n \in \mathcal{A}$ there exists $y_n \in \mathcal{A}'$ and for every $y_n \in \mathcal{A}'$ there exists $x_n \in \mathcal{A}$ such that the tuples $(x_0, \ldots, x_{n-1}, x_n)$ and $(y_0, \ldots, y_{n-1}, y_n)$ satisfy the conditions above (with $n-1$ being replaced by n). This is enough to ensure that after k many steps player $\exists$ wins the game.

Suppose that player $\forall$ have chosen $y_n \in \mathcal{A}'$ (the other case is similar). Let

$$\bar{p} = (R_0(x_0), R_1(x_0), \ldots, R_0(x_{n-1}), R_1(x_{n-1}))$$

and

$$\bar{q} = (R_0(y_0), R_1(y_0), \ldots, R_0(y_{n-1}), R_1(y_{n-1})).$$

Consider the following possibilities.

(1) Suppose $y_n \in U^{\mathcal{A}'}$, that is, $y_n = u^{\mathcal{A}'}_j$, for some $j \in \omega \cup \{\omega\}$. In this case, let x_n to be equal to $u^{\mathcal{A}}_i$, where $i = F'[\bar{p}, \bar{q}](j)$.

(2) Suppose $y_n \in S^{\mathcal{A}'}_{j_0,j_1}$ for some $j_0, j_1 \in \omega \cup \{\omega\}$. Define

$$i_0 = F'[\bar{p}, \bar{q}](j_0) \quad \text{and} \quad i_1 = F'[(\bar{p}, i_0), (\bar{q}, j_0)](j_1).$$

If i_0 happens to be greater than i_1, then swap their values to ensure that $i_0 < i_1$. Consider the game

$$\mathrm{EF}_k[(S^{\mathcal{A}}_{i_0,i_1}, s^{\mathcal{A}}_{i_0,i_1}), (S^{\mathcal{A}'}_{j_0,j_1}, s^{\mathcal{A}'}_{j_0,j_1})].$$

Let $\bar{u}$ and $\bar{v}$ be the subsequences of $\bar{x}$ and $\bar{y}$ consisting of elements belonging to $S^{\mathcal{A}}_{i_0,i_1}$ and $S^{\mathcal{A}'}_{j_0,j_1}$, respectively. By assumption, the play $(\bar{u}, \bar{v})$ agrees with a winning strategy for player $\exists$ for that game. In this case, let player $\exists$ chooses $x_n \in S^{\mathcal{A}}_{i_0,i_1}$ according to his winning strategy taking into account the moves $(\bar{u}, \bar{v})$ that have been made earlier. Since by assumption $n < k$, player $\exists$ can always make such a move.

This completes the description of step n of the play. It can be verified that whenever all conditions hold at the beginning of step n then they also hold at the end of step n. Therefore, by induction, the player $\exists$ has a winning strategy and the two models $\mathcal{A}$ and $\mathcal{A}'$ are elementary equivalent. $\square$

Let T be the first order theory of either $\mathcal{A}$ or $\mathcal{A}'$. The following lemma completes the proof of Theorem 15.

Lemma 20. *The theory T has only two automatic models, namely $\mathcal{A}$ and $\mathcal{A}'$.*

Proof. First, let us show that the structures $\mathcal{A}$ and $\mathcal{A}'$ are automatic and then prove that T does not have any other automatic models other than these two. We construct an automatic presentation for the larger model $\mathcal{A}'$ since the presentation for $\mathcal{A}$ is similar.

Define the following automatic presentation of $\mathbb{Z}(2^\infty)$. Let the alphabet be $\{0,1\}$ and let the string $a_0 \dots a_n$ represent an element $0.a_0 \dots a_n \in \mathbb{Z}(2^\infty)$ written in binary form. Thus we assume that the domain of this presentation consists of all string ending in 1 apart from the string 0, which represents the neutral element of $\mathbb{Z}(2^\infty)$.

Let the alphabet an automatic presentation of $\mathcal{A}'$ be $\Sigma = \{0,1,\#\}^3$. For every $i \in \omega \cup \{\omega\}$, define α_i as follows

$$\alpha_i = \begin{cases} 0^{i+1} & \text{if } i < \omega; \\ 1 & \text{if } i = \omega. \end{cases}$$

Let the elements of $U = \{u_i\}_{i \in \omega \cup \{\omega\}}$ be represented as $Conv(\alpha_i, \epsilon, \epsilon)$, where ϵ is the empty string. Let the elements from an equivalence class $S_{i,j}$, for $i,j \in \omega \cup \{\omega\}$, be represented as $Conv(\alpha_i, \alpha_j, p)$, where p is a string from the domain of $\mathbb{Z}(2^\infty)$. In this case, the binary relation R consists of the pairs

$$\big(Conv(\alpha_k, \epsilon, \epsilon),\ Conv(\alpha_i, \alpha_j, 0^{i-1}1)\big),$$

where $i,j \in \omega \cup \{\omega\}$, $i < j$ and $k \in \{i,j\}$. It is not hard to verify that in the presentation defined above the domain of $\mathcal{A}'$ and the predicates U, S and R can be recognized by finite automata.

We now show that T does not have any other automatic models apart from $\mathcal{A}$ and $\mathcal{A}'$. Consider the following list of sentences from the theory T:

(1) S is an equivalence relation on the complement of U.

(2) Every equivalence class of S equipped with the predicate *Plus* has the same first order theory as $(\mathbb{Z}(2^\infty), +)$, with $Plus(x,y,z)$ being interpreted as $x + y = z$.

(3) For every u and s, if $R(u,s)$ holds then $u \in U$, $s \notin U$ and s is the unique element from its equivalence class with the property that there are exactly two elements $u, v \in U$ for which $R(u,s)$ and $R(v,s)$ hold.

(4) For every $u, v \in U$ such that $u \neq v$, there exists s for which $R(u,s)$ and $R(v,s)$ hold.

(5) For every $i < \omega$, there are $u_0, \dots, u_{i-1} \in U$ and a unique $u_i \in U$ such that $u_0, \dots, u_i$ are pairwise different and

$$\forall v \in U - \{u_0, \dots, u_{i-1}\}\ \exists s\ (R(v,s) \wedge R(u_i,s) \wedge \mathrm{ord}(s) = 2^i).$$

(6) The u_i defined above is also unique with the property that there are two different $u, v \in U$ and $s, t \notin U$ such that $\mathrm{ord}(s) = \mathrm{ord}(t) = 2^i$ and $R(u_i, s) \wedge R(u, s)$ and $R(u_i, t) \wedge R(v, t)$ hold.

Let $\mathcal{B}$ be an automatic model of T. Then every S-equivalence class in $\mathcal{B}$ must be isomorphic to $\mathbb{Z}(2^\infty)$ since otherwise Theorem 7 and (2) above would imply that $\mathcal{B}$ is not automatic. By definition, the domain of $\mathcal{B}$ contains a set U of vertices that are R-connected with each other via elements from S-equivalence classes. By (5), U contains an infinite sequence $\{u_i\}_{i\in\omega}$ such that for every $i \neq j$, we have that u_i and u_j are connected via an element s of order $2^{\min\{i,j\}}$.

If there are no other elements in U apart from u_i's, then $\mathcal{B} \cong \mathcal{A}$. If U contains just one extra element, say u_ω, then $\mathcal{B} \cong \mathcal{A}'$. If U contains at least two extra elements, say u_ω and u'_ω, then they must be connected with each other via some element $s \notin U$. Note that s has a finite order since every S-equivalence class is isomorphic to $\mathbb{Z}(2^\infty)$. Let $\mathrm{ord}(s) = 2^i$. Then there are three elements in $\mathcal{B}$ that satisfy property (6) above, namely u_i, u_ω and u'_ω. Therefore, this case is impossible, and $\mathcal{B}$ can only be isomorphic either to $\mathcal{A}$ or to $\mathcal{A}'$. $\square$

Open Problem 21. *Note that the theory constructed in the last theorem has uncountably many countable models. We leave it as an open question whether there exists a complete theory with countably many (or even finitely many) countable models such that only two of them are automatic.*

7. Conclusion

The present paper answers several open problems posed by Khoussainov and Nerode on the way to establish an automatic model theory, which would be a counterpart of the well-developed recursive model theory. While previous work mainly asked which structures are automatic or determined the complexity of the isomorphism problem for certain types of automatic structures, the approach of Khoussainov and Nerode seeks more to locate the automatic models of a theory inside the collection of its countable models. So they asked natural questions like the following: How many automatic models can a complete first-order theory have? If a theory with an automatic model has a prime model and a saturated model, are these necessarily automatic? While most questions could be answered, some natural questions remain open: Does the result on the existence of an $\aleph_1$-categorical but not $\aleph_0$-categorical theory whose all countable models except the prime one are automatic depend on any complexity-theoretic assumption? This result is proven under the assumption $\mathit{LOGSPACE} = P$ and could be proven under a bit weaker assumption, but can we get rid of it completely? Furthermore, if a theory has countably many countable models and both the prime and the saturated models are automatic, are then all countable models of the theory automatic? Is there a theory with only countably many (or finitely many) countable models such that exactly two of them have automatic presentations?

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