On C-Degrees, H-Degrees and T-Degrees

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Abstract

Following a line of research that aims at relating the computation power and the initial segment complexity of a set, the work presented here investigates into the relations between Turing reducibility, defined in terms of computation power, and C-reducibility and H-reducibility, defined in terms of the complexity of initial segments.

The global structures of all C-degrees and of all H-degrees are rich and allows to embed the lattice of the powerset of the natural numbers under inclusion. In particular, there are C-degrees, as well as H-degrees, that are different from the least degree and are the meet of two other degrees, whereas on the other hand there are pairs of sets that have a meet neither in the C-degrees nor in the H-degrees; these results answer questions in a survey by Nies and Miller. There are r.e. sets that form a minimal pair for C-reducibility and Σ_2^0 *sets that form a minimal pair for H-reducibility, which answers questions by Downey and Hirschfeldt. It is also shown that one usual construction for minimal Turing degrees can easily be extended to obtain that the resulting degree is also a minimal C-degree.*

Furthermore, the following facts on the relation between C-degrees, H-degrees and Turing degrees hold. Every C-degree contains at most one Turing degree and this bound is sharp since there are C-degrees that do contain a Turing degree. For the comprising class of complex sets, neither the C-degree nor the H-degree of such a set can contain a Turing degree, in fact, the Turing degree of any complex set contains infinitely many C-degrees. Similarly the Turing degree of any set that computes the halting problem contains infinitely many H-degrees, while the H-degree of any 2-random set R *is never contained in the Turing degree of* R*. By the latter, H-equivalence of Martin-Löf random sets does not imply their Turing equivalence. The structure of the C-degrees contained in the Turing degree of a complex sets is rich and allows to embed any countable distributive lattice; a corresponding statement is true for the structure of H-degrees that are contained in the Turing degree of a set that computes the halting problem.*

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1. Introduction

Kolmogorov complexity measures the complexity of an object (a number or a string) in term of the length of the shortest binary description of this object. For this, one identifies a strings $a_0 a_1 \ldots a_n$ with the natural number that is one less than the number with binary expansion $1a_0a_1 \ldots a_n$, for example, λ corresponds to 0 and 00 to 3. In the following, let $\mathbb{N}$ denote the set of natural numbers, which is identified it with the set of binary strings in the way just mentioned. There are two types of Kolmogorov complexity, the plain complexity C and the prefix-free complexity H. Both are defined using suitable universal machines U_{C} and U_{H} with $\mathrm{C}(x)$ being the length of the string-equivalent of the least number p such that U_{C} is defined and outputs x; $U_{\mathrm{H}}(x)$ is defined in a similar fashion. These two functions are not recursive; in fact their Turing degree is the same as that of the halting problem $\mathbb{K}$; but the two functions can be effectively approximated from above (for an introduction to Kolmogorov complexity and related concepts the reader is referred to Li and Vitányi [12] and to the manuscript of an upcoming monograph by Downey and Hirschfeldt [4]; recursion-theoretic background can be found in the latter reference and in the books by Odifreddi [19, 20] and by Soare [26]).

One can analyse the complexity of a set A by looking at the functions

$$\begin{aligned} n &\mapsto \mathrm{C}(A(0)A(1)\ldots A(n)), \\ n &\mapsto \mathrm{H}(A(0)A(1)\ldots A(n)), \end{aligned}$$

which assign to each n the plain and prefix-free Kolmogorov complexity, respectively, of the initial segment of A from position 0 to n. These functions have been used to characterise several natural classes of sets:

- Schnorr [23] showed that a set is Martin-Löf random iff

  $$\mathrm{H}(A(0)A(1)\ldots A(n)) \geq n$$

  for almost all n; originally Martin-Löf [13] introduced his notion of randomness based on tests, but Schnorr's characterisation is easier and is often used as an alternative definition.

- Analogously, a set A has been called Kolmogorov random iff it satisfies

  $$\mathrm{C}(A(0)A(1)\ldots A(n)) \geq n - c$$

  for some constant c and infinitely many n. Miller [14] as well as Nies, Stephan and Terwijn [18] demonstrated that these sets coincide with the 2-random sets, that is, with the sets that are Martin-Löf random relative to the halting problem $\mathbb{K}$.

- A set A is called H-trivial if its initial segments have minimum prefix-free Kolmogorov complexity in the sense that

  $$\mathrm{H}(A(0)A(1)\ldots A(n)) \leq \mathrm{H}(n) + c$$

  for some constant c and all n. Furthermore, a set A is called low for Martin-Löf randomness if for any set X, the set X is Martin-Löf random relative to A iff X is unrelativized Martin-Löf random. A set A is called low for prefix-free complexity if H and the relativized prefix-free complexity H^A differ at most by an additive constant. Nies [17] discovered the coincidence of H-triviality with these two lowness notions. The three notions themselves date back to work of Chaitin [1, 2, 3], Solovay [27], Kučera and Terwijn [11], Muchnik [5] and Zambella [30].

- A set A is called complex if there is a computable nondecreasing and unbounded function g such that for all n

$$g(n) \leq \mathrm{C}(A(0)A(1)\dots A(n)) ,$$

and similarly, a set A is autocomplex if there is such a function g that is not necessarily computable but is computable with oracle A [10]. The classes of autocomplex and complex sets coincide with the classes of sets which allow to compute a fixed-point free function in the oracle Turing machine model and in the weak truth-table model, respectively [10].

Given the importance of the Kolmogorov complexity of initial segments of a set, it was natural to define and investigate the following reducibilities between sets of natural numbers:

$$\begin{aligned} A \leq_{\mathrm{C}} B &\Leftrightarrow \exists c \forall n\, (\mathrm{C}(A(0)A(1)\dots A(n)) \leq \mathrm{C}(B(0)B(1)\dots B(n)) + c); \\ A \leq_{\mathrm{H}} B &\Leftrightarrow \exists c \forall n\, (\mathrm{H}(A(0)A(1)\dots A(n)) \leq \mathrm{H}(B(0)B(1)\dots B(n)) + c). \end{aligned}$$

For both of these reducibilities there are least sets, for example, the empty set; such least sets are called C-trivial and H-trivial and in both cases the least degree is called the trivial degree. Chaitin [2] already observed that the C-trivial sets are just the recursive ones, whereas Solovay [27] proved that there are nonrecursive H-trivial sets. Subsequent to the initial results of Chaitin and Solovay, these reducibilities and their degree structures became widely studied. Nevertheless, for both, the C-degrees and the H-degrees several questions remained open.

- If A, B are Martin-Löf random, is it then true that $A =_{\mathrm{H}} B$ implies $A =_{\mathrm{T}} B$?
- Is there a minimal pair of r.e. degrees?
- Is there a nontrivial branching degree, that is, a nonzero degree that is the meet of two other degrees?
- Does every pair of degrees have a meet?

The first question is due to Yu [29] and has also been incorporated into a list of open questions by Miller and Nies [15]. The restriction to Martin-Löf random sets is due to the fact that the implication is known to be false in general and is further motivated by the known fact that the H-degree of every Martin-Löf random set is countable.

The second question is a stronger form of a question by Downey and Hirschfeldt [4], who review the construction of a minimal pair of H-degrees by Csima and Montalbán [6] and argue that the construction yields sets that are Σ^0_4. Then they ask whether this bound for the complexity of the sets in the minimal pair can be improved to Σ^0_3, left-r.e. or something similar.

The third and fourth questions are part of Question 9.1 (where in addition the so-called van Lambalgen reducibility is considered) in a list of open questions by Miller and Nies [15], who remark that by results of Miller and Yu almost all pairs of sets have a join neither in the C-degrees nor in the H-degrees, whereas little is known about meets. The third question is restricted to nontrivial degrees, because the minimal pair constructed by Csima and Montalbán shows that the trivial H-degree is branching in the sense that it is equal to the meet of two nontrivial H-degrees. Since their construction works also for C-degrees, the trivial C-degree is branching, too.

In what follows, three of the four open questions stated above are answered explicitly; the second question is answered for C-degrees but remains open in the more popular case of H-degrees, however, for the case of H-degrees the complexity of the sets in a minimal pair is brought down to Σ^0_2.

Theorem 2 provides an example of two H-equivalent Martin-Löf random sets in different Turing degrees. Theorem 14 asserts that there is a minimal pair of r.e. sets for C-degrees. Theorem 20 shows that the lattice of the powerset of $\mathbb{N}$ can be embedded into the C-degrees and H-degrees; thus there are nonzero branching degrees. Furthermore, Theorem 21 shows that there is a pair of sets which has neither a meet in the C-degrees nor in the H-degrees.

Besides the open questions explicitly listed above, other related issues about the relations between Turing degrees on one hand and C-degrees and H-degrees on the other hand are investigated. A major result is the proof of the existence of a minimal C-degree in Theorem 17. Furthermore, it is shown that every C-degree contains at most one Turing degree and that this bound is sharp since there are C-degrees that do contain a Turing degree. For the comprising class of complex sets, neither the C-degree nor the H-degree of such a set can contain a Turing degree, in fact, the Turing degree of any complex set contains infinitely many C-degrees and the structure of the C-degrees contained in the Turing degree of a complex sets is so rich that it allows indeed to embed any countable distributive lattice; the latter assertion is also true for the structure of H-degrees that are contained in the Turing degree of any set that computes the halting problem.

Many of the results shown in the sequel exploit the close relations between C-degrees and H-degrees on one hand and the Turing degrees on the other hand that are established in Propositions 7 and 8. More precisely, these propositions assert that C- and H-reductions to sets of certain form imply Turing reductions to the set, hence some known results on Turing degrees can be translated into results on C-degrees and H-degrees.

2. A question of Yu

For any nonrecursive H-trivial set A and any recursive set B holds $A =_{\mathrm{H}} B$, whereas $A =_{\mathrm{T}} B$ and $A =_{\mathrm{C}} B$ are both false. Yu [15, 29] asked whether at least on the Martin-Löf random sets H-equivalence would imply Turing equivalence. Theorem 2 answers this question in the negative. The proof of the theorem will use the following observation which is based on a result of Sacks [24]; the observation was made explicit by Downey, Nies, Weber and Yu [7].

Proposition 1 (Downey, Nies, Weber and Yu [7]). *Let $A \leq_{\mathrm{T}} \mathbb{K}$ be a nonrecursive set and let $X \geq_{\mathrm{T}} A$. Then X is not 2-random, that is, X is not Martin-Löf random relative to $\mathbb{K}$.*

Proof. Recall that by a result of Sacks [24, Section 10, Theorem 1] the upper Turing cone of the nonrecursive set A has Lebesgue measure 0. Hence for any fixed oracle Turing machine M, the Lebesgue measure of the Σ^0_1 classes

$$C_n = \{X\colon A(m) = M^X(m) \text{ for all } m \leq n\}$$

goes to 0 when n goes to infinity. Now let $b(n)$ be the least index such that the measure of $C_{b(n)}$ is at most 2^{-n}; as $A \leq_{\mathrm{T}} \mathbb{K}$, the mapping $n \mapsto b(n)$ is $\mathbb{K}$-recursive and the sequence $C_{b(0)}, C_{b(1)}, \ldots$ of classes is a uniformly $\mathbb{K}$-r.e. sequence of classes. As each $C_{b(n)}$ has at most measure 2^{-n}, the $C_{b(n)}$ form a Martin-Löf test relative to $\mathbb{K}$ that covers all sets to which A is reducible via M. Then the set X is covered by a $\mathbb{K}$-r.e. Martin-Löf test, hence X is not Martin-Löf random relative to $\mathbb{K}$. □

Theorem 2. *Let X be any 2-random set; that is, let X be any set which is Martin-Löf random relative to $\mathbb{K}$. Then there is a set Y such that $X =_{\mathrm{H}} Y$ but $X \not\leq_{\mathrm{T}} Y$ and $Y \not\leq_{\mathrm{T}} X$.*

Proof. Muchnik [5] demonstrated that there is a nonrecursive set $A \leq_{\mathrm{T}} \mathbb{K}$ that is low for prefix-free Kolmogorov complexity, that is, for any word w the H-complexity of w with and without oracle A differ at most by an additive constant that does not depend on w. Fix such a set A and let $Y = X \Delta A$. Then up to an additive constant, the initial segments of X and Y of identical length have identical H-complexity with oracle A. Hence, up to an additive constant, these initial segments have also identical H-complexity without oracle A. So, $X =_{\mathrm{H}} Y$.

Note that X is Martin-Löf random relative to $\mathbb{K}$ by choice of X. Then the set Y is also Martin-Löf random relative to $\mathbb{K}$. The reason is that $A \leq_{\mathrm{T}} \mathbb{K}$, hence any Martin-Löf test that is recursive in $\mathbb{K}$ and covers Y can be transformed into a Martin-Löf test that is recursive in $\mathbb{K}$ and covers X by replacing any word w that is enumerated in some component of the test by $w\Delta(A(0)\dots A(|w|-1))$. Now, by Proposition 1, $A \not\leq_{\mathrm{T}} X$ and $A \not\leq_{\mathrm{T}} Y$. As $A \leq_{\mathrm{T}} X \oplus Y$, it follows that X and Y are incomparable with respect to Turing reduction. □

As the 2-random sets have measure 1, one obtains from Theorem 2 the following corollary.

Corollary 3. *For almost all sets X, hence in particular for almost all Martin-Löf random sets, there is a set Y such that $X =_{\mathrm{H}} Y$ but $X \neq_{\mathrm{T}} Y$.*

3. Translating Reducibilities

Propositions 7 and 8 below assert that for sets Y of a certain special form any set that is H- or C-reducible to Y is also Turing reducible to Y. While these propositions are slightly technical, they yield easily several interesting results about H-degrees and C-degrees and their relations to Turing degrees. The proofs of Propositions 7 and 8 make use of the following result of Owings [9, 21]. Recall that a tree T is a set of words that is closed under taking prefixes and that a tree has constant width k for some constant k, if for all n, the tree T contains at most k words of length n.

Fact 4. *If T is a tree that has constant width and is recursively enumerable relative to a set Y, then every infinite path X on T is recursive relative to Y, that is, $X \leq_T Y$.*

Proof. Given any infinite path X on T, let $k' \leq k$ be maximum such that there are infinitely many initial segments X that are extended by k' different words in T of identical length. Let τ be an initial segment of X that is so long that τ is not an initial segment of any of the finitely many other infinite pathes on T and such that τ is not extended by strictly more than k' words in T of identical length. Then $X(n)$ can be computed for any $n \geq |\tau|$ with access to oracle Y by enumerating T until one has enumerated k' distinct extensions of τ of identical length that all agree on the bit that corresponds to $X(n)$. □

For further use in the proofs of Propositions 7 and 8, the following variant of the usual concept of an H-trivial or C-trivial set is defined.

Definition 5. Given a set I of natural numbers and a constant c, a set X is called (H,c)-trivial on I if for all $i \in I$ the initial segment u of X of length i satisfies

$$\mathrm{H}(u) \leq \mathrm{H}(|u|) + c \,. \tag{1}$$

Furthermore, a word w is (H,c)-trivial on a set I if (1) is satisfied for any prefix u of w where $|u| \in I$; a word w is extendably (H,c)-trivial on I if there is an extension v of w that is (H,c)-trivial on I such that $|v| \in I$. Corresponding notions of being (C,c)-trivial or extendably (C,c)-trivial on I are defined similarly with the function C in place of the function H.

Proposition 6. *For any infinite set I of natural numbers and for any constant c, the set of words that are extendably (H, c)-trivial on I is a tree of constant width. A similar statement holds with (C, c)-trivial in place of (H, c)-trivial.*

Proof. Let $I \subseteq \mathbb{N}$ be infinite and c be a constant. By definition, the sets of words that are (H,c)-trivial on I and are extendably (H,c)-trivial on I, respectively, are both closed under taking prefixes. As a consequence, in case a word w is extendably (H,c)-trivial on I then for the minimum $i \in I$ where $|w| \leq i$, the word w is a prefix of a word v of length i such that v is (H,c)-trivial on I. But there are at most constantly many such words w', because by the coding theorem for prefix-free Kolmogorov complexity for any constant c there is a constant k such that for any length i there are at most k words u of length i that satisfy (1). In summary, the set of words that are extendably (H,c)-trivial on I is a tree of constant width. A corresponding statement for the set of words that are (C,c)-trivial on I follows by almost literally the same argument, now using the coding theorem for plain Kolmogorov complexity. □

First one shows Proposition 7 on H-reducibility, which is slightly easier but also weaker than the corresponding result for C-reducibility given in Proposition 8. Observe that the assertion of Proposition 7 remains true with H-reducibility replaced by C-reducibility.

Proposition 7. *Let Y be any subset of $\{2^n : n \in \mathbb{N}\}$. Then for any set X,*

$$X \leq_{\mathrm{H}} Y \quad \textit{implies} \quad X \leq_{\mathrm{T}} Y \oplus \mathbb{K}\ .$$

In particular, in case Y computes the halting problem $\mathbb{K}$, then

$$X \leq_{\mathrm{H}} Y \quad \textit{implies} \quad X \leq_{\mathrm{T}} Y\ .$$

Proof. Let g be the function defined by

$$g(n) = 2^n + \sum_{k<n} 2^k \cdot Y(2^{k+2})$$

and observe that the value $g(n)$ is always between the numbers 2^n and $2^{n+1} - 1$. As the values of $Y(2^0)$ and $Y(2^1)$ are constant, one can compute the initial segment $Y(0)Y(1)\dots Y(2^{n+1})$ of Y from $g(n)$. As a consequence, for all n and up to an additive constant c_1, the initial segment of Y of length $g(n)+1$ has the same complexity as $g(n)$ itself, that is,

$$\mathrm{H}(Y(0)Y(1)\dots Y(g(n))) \leq \mathrm{H}(g(n)) + c_1\ . \tag{2}$$

Now let X be any set where $X \leq_{\mathrm{H}} Y$. Then (2) remains true when Y is replaced by X and c_1 is replaced by some appropriate constant c_2. This means that the set X is (H,c_2)-trivial on the set $I = \{g(n) \colon n \in \mathbb{N}\}$. Hence, in particular, any initial segment of X is extendably (H,c_2)-trivial on I. By Proposition 6, the set T of all words that are extendably (H,c_2)-trivial on I is a tree of constant width. Furthermore, the set T can be computed with oracle $Y \oplus \mathbb{K}$, because for given n one can compute $g(n)$ as well as all the words u of this length such that $\mathrm{H}(u) \leq \mathrm{H}(|u|) + c_2$. So X is on a tree of constant width that is recursive in the oracle $Y \oplus \mathbb{K}$, hence the set X itself can be computed with this oracle according to Remark 4. □

While Proposition 7 remains true with C-reducibility in place of H-reducibility, for C-reducibility one obtains indeed the following improved variant where the Turing reduction does not require the additional oracle $\mathbb{K}$.

Proposition 8. *Let Y be a subset of $\{2^{2^n} : n \in \mathbb{N}\}$. Then, for any set X, it holds that*

$$X \leq_{\mathrm{C}} Y \quad \textit{implies} \quad X \leq_{\mathrm{T}} Y \ .$$

Proof. Let Y be as above and assume that $X \leq_{\mathrm{C}} Y$. In the following, the values of $Y(2^{2^0})$ and $Y(2^{2^1})$ are constants which need not to be computed; furthermore, it is used that $Y(\ell) = 0$ whenever ℓ is not of the form 2^{2^k}. Let

$$g(n) = 2^n + \sum_{k<n} 2^k \cdot Y(2^{2^{k+2}}) \ .$$

Note that $g(n) < 2^{n+1}$, hence $2^{g(n)+1} \leq 2^{2^{n+1}}$ for all n. Thus, given a number m in the interval

$$J_n = \{2^{g(n)}, \ldots, 2^{g(n)+1} - 1\}$$

for some n, one can first compute $g(n)$ as this is obtained by rounding down the logarithm of m. Then one can compute $Y(2^{2^t})$ for all t with $t \leq n$ from $g(n)$. As $\mathrm{C}(m) \leq g(n)$ up to an additive constant, there is a constant c_1 such that for all n and all m in J_n holds

$$\mathrm{C}(Y(0)Y(1)\ldots Y(m)) < g(n) + c_1$$

and hence, by $X \leq_{\mathrm{C}} Y$, for some constant c_2,

$$\mathrm{C}(X(0)X(1)\ldots X(m)) < g(n) + c_2 \ . \tag{3}$$

Now let T be the set of all words w such that $w \in J_n$ for some n and w has an extension w' of length equal to $\max J_{n+1}$ such that for all prefixes u of w' and all m,

$$u \in J_m \ \text{ implies } \ \mathrm{C}(u) < g(m) + c_2 \ .$$

Then T is a tree and X is an infinite path on T. Furthermore, the tree T is recursively enumerable in Y, hence by Remark 4 in order to obtain $X \leq_{\mathrm{T}} Y$ it suffices to show that T has constant width.

For every n there are $2^{g(n)}$ many numbers in I_n; hence for any n there is a number $i_n \in I_n$ such that for some constant c_3 holds

$$g(n) \leq \mathrm{C}(i_n) + c_3 \ . \tag{4}$$

Let $I = \{i_n \colon n \in \mathbb{N}\}$ and let $c_4 = c_2 + c_3$. Then by inequalities (3) and (4) for all i_n, one has that

$$C(X(0)X(1)\ldots X(i_n)) < \mathrm{C}(i_n) + c_4 \tag{5}$$

(that is, the set X is (C,c_4)-trivial on the set I). By construction and choice of c_4, the tree T is contained in the set of all words that are extendably (C,c_4)-trivial on I, where by Proposition 6 the latter set is a tree of constant width, hence T has constant width, too. $\square$

4. C- and H-degrees inside Turing degress

In this section, the results of Section 3 are applied in order to prove that Turing degrees can completely contain C-degrees and H-degrees. For a start, a similar result is derived on lower cones.

Proposition 9. *For any set A there is a set $X =_{\mathrm{m}} A$ such that the lower C-cone of X is contained in the lower Turing cone of X, that is, such that*

$$\{Y : Y \leq_{\mathrm{C}} X\} \subseteq \{Y : Y \leq_{\mathrm{T}} X\} .$$

For any set A that computes the halting problem there is a set $X =_{\mathrm{m}} A$ such that the lower H-cone of X is contained in the lower Turing cone of X, that is, such that

$$\{Y : Y \leq_{\mathrm{H}} X\} \subseteq \{Y : Y \leq_{\mathrm{T}} X\} .$$

Proof. For given A, let $X = \{2^{2^n} : n \in A\}$; then A and X are m-equivalent and the first assertion follows by Proposition 8. Similarly, the second assertion follows from Proposition 7. □

The following Theorem 10 states that the Turing degree of any set that computes the halting problem contains a C-degree and a H-degree completely. By using different and somewhat more involved methods, this result will be extended in two directions by Theorems 22 and 23 in Section 9. There it is shown that, first, one can embed any countable distributive lattice into the C-degrees or H-degrees contained in the Turing degree of any set that computes the halting problem. Second, such embeddings into C-degrees are possible for all sets that are complex, a notion introduced by Kjos-Hanssen, Merkle and Stephan [10], which is considerably more general than computing the halting problem.

Theorem 10. *Every Turing degree above the Turing degree of the halting problem contains a set X such that the C-degree and the H-degree of X are both contained in the Turing degree of X.*

Proof. Let $A \geq_{\mathrm{T}} \mathbb{K}$. Define inductively a function g such that $g(0) = 1$ and $g(n+1) = 2^{16g(n)}$. Furthermore, let $f(n)$ be the least natural number x with

$$g(n) < x \ , \ \ 3g(n) < \mathrm{C}(x) \ \text{ and } \ 3g(n) < \mathrm{H}(x) \ .$$

It is easy to see that $f(n) < g(n+1)$ and $f \leq_{\mathrm{T}} A$. Now define the set

$$X = \{2^{2^{f(2n+A(n))}} : n \in \mathbb{N}\}$$

where $A(n) = 0$ if $n \notin A$ and $A(n) = 1$ if $n \in A$. Let n be a sufficiently large natural number and $b \in \{0, 1\}$. Then X has an element between $2^{2^{g(2n+b)}}$ and $2^{2^{g(2n+b+1)}}$ iff $A(n) = b$. In the case that this element exists, it is $2^{2^{f(2n+b)}}$ and one can compute $f(2n + b)$ from the initial segment of X up to $2^{2^{g(2n+b+1)}}$; hence the Kolmogorov complexity of this initial segment is at least $2g(2n + b)$. In the case that this element does not exist, one knows that all elements of X are of the form $2^{2^{f(k)}}$ for $k < 2n$ and one get that the Kolmogorov complexity of X up to the bound is much below $2g(2n+b)$. This argument works for both types of Kolmogorov complexity. Hence, if Y is C-equivalent to A then one has for all sufficiently large n that

- $\mathrm{C}(Y(0)Y(1)...Y(2^{2^{g(n+1)}})) < 2g(n)$ iff $n \in A$;

- $\mathrm{C}(Y(0)Y(1)\ldots Y(2^{2^{g(n+2)}})) < 2g(n+1)$ iff $n \notin A$.

Similarly, if Y is H-equivalent to A, one has for all sufficiently large n that

- $\mathrm{H}(Y(0)Y(1)...Y(2^{2^{g(n+1)}})) < 2g(n)$ iff $n \in A$;
- $\mathrm{H}(Y(0)Y(1)\ldots Y(2^{2^{g(n+2)}})) < 2g(n+1)$ iff $n \notin A$.

As the C-complexity and H-complexity of strings can be approximated from above, one can see which case holds and gets that $A \leq_{\mathrm{T}} Y$.

On the other hand, if $Y \leq_{\mathrm{C}} X$ then $Y \leq_{\mathrm{T}} X$ by Proposition 8 and $Y \leq_{\mathrm{T}} A$ by choice of X. If $Y \leq_{\mathrm{H}} X$ then $Y \leq_{\mathrm{T}} X \oplus \mathbb{K}$ by Proposition 7. As $A \geq_{\mathrm{T}} \mathbb{K}$ and $A \geq_{\mathrm{T}} X$, this gives $Y \leq_{\mathrm{T}} A$. Hence all sets in the C-degree and the H-degree of X are inside the Turing degree of A. □

5. Turing degrees inside C- and H-degress

The previous section showed that there are C-degrees and H-degrees which are completely contained in a Turing degree. In this section, the investigation is continued in the reverse direction, that is, it is asked how many Turing degrees can be completely contained in a C-degree or a H-degree. For a start, it is shown in Proposition 11 and Theorem 12 that for complex sets the answer is 0, while the answer is at most 1 for C-degrees in general. In this connection, note again that in Section 9 it will be shown that the Turing degree of any complex set contains infinitely many C-degrees and that any Turing degree above the Turing degree of the halting problem contains infinitely many H-degrees.

Proposition 11. *Neither the C-degree nor the H-degree of a complex set contain any Turing degree completely.*

Proof. Let a complex set A be given. Recall that A is complex iff there is an increasing recursive function h such that $\mathrm{C}(A(0)A(1)\ldots A(h(n))) > n$ for all n [10]. Fix such a function h. Let B be any set which is C-equivalent or H-equivalent to A and let

$$D = \{h(4n) : n \in B\} .$$

It is easy to see that the following statements hold for almost all n:

$$\begin{aligned}
\mathrm{C}(B(0)B(1)\ldots B(h(4n))) &> 3n; \\
\mathrm{H}(B(0)B(1)\ldots B(h(4n))) &> 3n; \\
\mathrm{C}(D(0)D(1)\ldots D(h(4n))) &< 2n; \\
\mathrm{H}(D(0)D(1)\ldots D(h(4n))) &< 2n.
\end{aligned}$$

Hence the sets B and D are neither C-equivalent nor H-equivalent, however the sets B and D are Turing equivalent. □

Theorem 12. *Every C-degree contains at most one Turing degree.*

Proof. Assume that the Turing degrees of sets A and B are both contained in the same C-degree and consider the sets $\{2^{2^n} : n \in A\}$ and $\{2^{2^n} : n \in B\}$. The two latter sets are Turing equivalent to A and B, respectively, and are C-equivalent. By the latter and Proposition 8, these sets are Turing equivalent to each other, hence $A =_{\mathrm{T}} B$. As a consequence, every C-degree contains at most one Turing degree. □

Note that Theorem 12 does not extend to H-reducibility as the least H-degree contains all H-trivial sets and thus contains countably infinitely many Turing degrees.

The next result mainly establishes that in Theorem 12 the upper bound 1 for the number of Turing degrees contained in a C-degree is sharp.

Theorem 13. *There are uncountably many sets X such that the Turing degree of X is contained in both, the C-degree of X and the H-degree of X.*

More precisely, there is a partial-recursive $\{0,1\}$-valued function ψ such that the co-domain $\{a_0, a_1, a_2, \ldots\}$ of ψ is infinite and every set X of hyperimmune-free Turing degree such that the characteristic function of X extends ψ satisfies that its Turing degree is a subset of both its C-degree and its H-degree.

Proof. With a priority construction, one can satisfy for each e and each $n \geq e$ and each $x \leq a_n$ the following condition: Whenever (the characteristic function of X) satisfies that $\varphi_e^X(x)$ terminates in exactly s steps relative to X then $s \leq a_{s+1}$. This is achieved by making $\psi(x) = Z(x)$ for all $x = a_n + 1, a_n + 2, \ldots, s$ whenever e, n, x, Z are found to violate the above constraint at stage s and no requirement of higher priority needs attention. The resulting function ψ has no recursive extension as there is for every recursive function f an index e such that $\varphi_e^Z(x)$ terminates with use s iff $Z(s) \neq f(s)$ and ψ_s is undefined on more than x places below s. Then some requirement belonging to e will ensure that ψ differs from f somewhere on its domain. Stephan and Yu [25, Proposition 11] present a similar construction in more detail.

Now assume that $Y =_T X$. It follows that $Y =_{tt} X$. Let f be a joint recursive bound function for both translations (from X to Y and from Y to X).

Given n, let m be the maximal natural number with $f(m) \leq n$. If n is sufficiently large, there is at most one undefined place of ψ between m and $n + f(n)$, thus one can get $X(0)X(1)\ldots X(m)$ from $Y(0)Y(1)\ldots Y(n)$ via the truth-table reduction and all but at most one value of $X(m+1), X(m+2), \ldots, X(n)$ from ψ.

The missing value can be given non-uniformly in order to verify that $X \leq_{\mathrm{C}} Y$ and $X \leq_{\mathrm{H}} Y$. Furthermore, one can get $X(n+1), X(n+2), \ldots, X(f(n))$ from ψ plus one bit for the either missing or last defined value; so given $X(0)X(1)\ldots X(n)$ one can extend with constantly much additional information to $X(0)X(1)\ldots X(f(n))$ and compute from this string $Y(0)Y(1)\ldots Y(n)$. Thus $Y \leq_{\mathrm{C}} X$ and $Y \leq_{\mathrm{H}} X$. $\square$

6. Minimal Pairs

Recall that two sets form a minimal pair for a given reducibility, if neither of the two sets is contained in the least degree, while any set that is reducible to both sets is contained in the least degree. For example, sets A and B form a minimal pair for C-reducibility if neither A nor B is C-trivial and for any set X holds that

$$X \leq_{\mathrm{C}} A \quad \text{and} \quad X \leq_{\mathrm{C}} B \quad \text{implies} \quad X \leq_{\mathrm{C}} \emptyset\,.$$

Theorem 14. *There are two recursively enumerable sets that form a minimal pair for C-reducibility.*

Proof. Let A and B be recursively enumerable sets that form a minimal pair with respect to Turing reducibility [20, Theorem X.6.5]. Then the sets

$$X = \{2^{2^n} : n \in A\} \quad \text{and} \quad Y = \{2^{2^n} : n \in B\}$$

are recursively enumerable. By Proposition 8, every set that is C-reducible to both sets is Turing reducible to both sets, hence is recursive. Consequently, the sets X and Y form a minimal pair for C-reducibility. $\square$

The existence of an recursively enumerable minimal pair for C-reducibility improves on a result of Csima and Montalbán [6] who showed the existence of minimal pairs for H-reducibility and C-reducibility. Downey and Hirschfeldt [4] analyse the construction of a minimal pair of H-degrees by Csima and Montalbán [6] and argue that this construction yields sets that are Σ^0_4. Then they ask whether this bound for the complexity of the sets in the minimal pair can be improved to Σ^0_3, left-r.e. or something similar. Theorem 16 below asserts that for H-reducibility there is a minimal pair of Σ^0_2-sets, whereas it is it is still open whether there is a minimal pair of r.e. sets.

In the proof of Theorem 16 the following proposition will be used. Note that the proposition could be shown for C in place of H by roughly the same proof. Note further that every set which is hypersimple relative to $\mathbb{K}$ is a Σ^0_2-set.

Proposition 15. *Assume that the set Y is not H-trivial and that $Y \leq_{\mathrm{H}} X$ for some set X. Then there is a function*

$$f \leq_T Y \oplus \mathbb{K}$$

such that for every n, the set X has at least one element between n and $f(n)$.

Proof. Let $Y \leq_{\mathrm{H}} X$ and let c be a constant such that

$$\mathrm{H}(Y(0)Y(1)...Y(n)) \leq \mathrm{H}(X(0)X(1)...X(n)) + c$$

for all n. Now let $f(n)$ be the least number $m > n$ such that for all finite sets $D \subseteq \{0, 1, \ldots, n\}$ holds

$$\mathrm{H}(Y(0)Y(1)...Y(m)) > \mathrm{H}(D(0)D(1)...D(m)) + c$$

Such a number m exists as all finite sets D satisfy $Y \not\leq_{\mathrm{H}} D$. It is easy to see that the function f can be computed with oracle $Y \oplus \mathbb{K}$. As $Y \leq_{\mathrm{H}} X$ with the given constant c, one knows that X differs below m from all subsets of $\{0, 1, \ldots, n\}$. Thus there is an element x of X with $x \in \{n+1, n+2, \ldots, f(n)\}$. $\square$

Theorem 16. *There is a minimal pair of H-degrees. This minimal pair can be given as a pair of any two sets which are hypersimple relative to $\mathbb{K}$ and form a minimal pair in the Turing degrees relative to $\mathbb{K}$.*

Proof. Let X_0, X_1 be the complements of two $\mathbb{K}$-hypersimple sets which form a minimal pair for computations relative to $\mathbb{K}$; each of them has the same H-degree as its complement. Without loss of generality, X_0 and X_1 are both subsets of $\{2^0, 2^1, 2^2, \ldots\}$. Now consider any set Y which is H-reducible to both, X_0 and X_1. It follows from Proposition 7 that $Y \leq_T X_a \oplus \mathbb{K}$ for $a = 0, 1$. Due to the minimal pair property, $Y \leq_T \mathbb{K}$. If Y would not be H-trivial then X_0, X_1 would not be hyperimmune relative to $\mathbb{K}$ by Proposition 15. Thus Y must be H-trivial and X_0, X_1 form a minimal pair for H-reducibility. The complements of X_0 and X_1 are Σ^0_2-sets which also represent the H-degrees of this minimal pair. $\square$

7. Minimal C-Degrees

In this section it is shown that one can modify the construction of a set of minimal and hyperimmune-free Turing degree such that the resulting set has also a minimal C-degree. Indeed, its C-degree and Turing degree coincide.

Theorem 17. *There are minimal* C*-degrees.*

Proof. The construction is done by producing a descending sequence of perfect recursive trees starting with T_0 being the tree whose infinite branches are exactly the subsets of $\{2^{2^n} : n \in \mathbb{N}\}$. Let Q_e be a perfect recursive subtree of T_e such that one of the following three conditions holds:

- For all infinite branches A of Q_e, φ_e^A is recursive.
- For all infinite branches A of Q_e, φ_e^A is partial.
- Q_e is e-splitting, that is, for all infinite branches A, B of Q_e, φ_e^A, φ_e^B are total and if $x = \min\{z : A(z) \neq B(z)\}$ then there is an y with $\varphi_e^A(y) \neq \varphi_e^B(y)$ such that all further infinite branches D which coincide with A, B below x also coincide with one of these two sets up to the use of the computation $\varphi_e^D(y)$.

Furthermore, if φ_e is partial then let $T_{e+1} = Q_e$ else choose T_{e+1} such that the following requirements are all satisfied:

- $T_{e+1} \subseteq Q_e$ and T_{e+1} is a perfect recursive tree;
- φ_e is not an infinite branch of T_{e+1};
- for all x there is at most one branching node σ on T_{e+1} with $x \leq |\sigma| \leq \varphi_e(x)$.

This completes the inductive definition of the sequence $T_0, T_1, \ldots$; let A be an infinite branch of $\bigcap_e T_e$. The set A exists since $T_0 \supseteq T_1 \supseteq T_2 \supseteq \ldots$ and the set of infinite branches of each tree is a nonempty compact subset of $\{0,1\}^\infty$.

Following the arguments given by Odifreddi [19, Section V.5] there is always a subtree Q_e of T_e of the desired property. Furthermore, it is clear that T_{e+1} can be made from Q_e by thinning out the branching nodes. The resulting set A has minimal and hyperimmune-free Turing degree [19, Section V.5].

As $A \subseteq \{2^{2^n} : n \in \mathbb{N}\}$ every set which is C-reducible to A is also Turing-reducible to A. Furthermore, every set B which is Turing reducible to A is in fact Turing reducible to A with a recursive use φ_e as A has hyperimmune-free Turing degree. Without loss of generality φ_e is monotonically increasing. Consider the tree T_{e+1}. For any x, $B(0)B(1)\ldots B(x)$ can be computed from $A(0)A(1)\ldots A(\varphi_e(x))$. As there is only one branching node of T_{e+1} between x and $\varphi_e(x)$, there are at most two strings σ, τ on T_{e+1} of length $\varphi_e(x)+1$ which extend $A(0)A(1)\ldots A(x)$. Hence it follows that $\mathrm{C}(B(0)B(1)\ldots B(x)) \leq \mathrm{C}(A(0)A(1)\ldots A(x)) + c$ for some constant c and all x.

Thus $B \leq_{\mathrm{T}} A \Leftrightarrow B \leq_{\mathrm{C}} A$ for all sets B. As the recursive sets form the trivial (= least) C-degree, it remains to show that the Turing degree of A is indeed exactly one C-degree.

Now assume that $B = \varphi_e^A$ is not recursive; assume without loss of generality that the use of $\varphi_e^D(z)$ for all z is at least z and is monotonically increasing. Note that $B =_{\mathrm{T}} A$ as A has minimal Turing degree. The tree Q_e is e-splitting. Given $B(0)B(1)\ldots B(x)$, consider any infinite branches D, E of

Q_e such that $\varphi_e^D(z) = \varphi_e^E(z) = B(z)$ whenever the corresponding use is at most x. Assume that $y = \min(\{z : D(z) \neq A(z)\}) \leq x$. Then, by the property of Q_e being e-splitting, the set E coincides with either D or A up to x. Hence there are at most two strings of length $x+1$ on Q_e which are consistent with $B(0)B(1)\ldots B(x)$ with respect to φ_e and therefore one can conclude that $A \leq_{\mathrm{C}} B$. □

Remark 18. Lacombe [19, Exercise V.5.12] showed that there are uncountably many minimal degrees and Odifreddi's proof ot this would directly generalise to Theorem 17. Furthermore, one can show that all sets in the Turing degree of A are also H-equivalent although there are also H-equivalent sets outside this Turing degree, namely $A \oplus B =_{\mathrm{H}} A$ and $A \oplus B >_{\mathrm{T}} B$ for any nonrecursive and H-trivial set B. Hence one gets the following result: *There are uncountably many sets A such that the Turing degree of A coincides with the* C*-degree of A and is contained in the* H*-degree of A.* This result slightly improves Theorem 13. It is still open whether there are minimal H-degrees.

8. Existence and Nonexistence of Meets

In the following it is shown that the lattice of the powerset of $\mathbb{N}$ with operations union and intersection can be embedded into the C-degrees and into the H-degrees, from which then the existence of nontrivial branching degrees is immediate. The proof of this embedding result exploits Proposition 19, which is a straightforward modification of a result by Reimann and Stephan [22] and asserts that given any function with growth rate sufficiently near to $1/2$, one can construct noneffectively a set whose initial segment complexity agrees up to a constant with the given function.

Proposition 19. *There is a constant c such that for every function f from $\mathbb{N}$ to the rational numbers $\mathbb{Q}$ where for all x, y holds*

$$f(x) + y/2 - y/c \leq f(x+y) \leq f(x) + y/2 + y/c\ ,$$

there are a constant c' and sets X and Y such that $C(X(0)X(1)\ldots X(n))$, $H(Y(0)Y(1)\ldots Y(n))$ and $f(n)$ differ by at most c' for all n.

Theorem 20. *The lattice of the power set of $\mathbb{N}$ can be embedded into the C-degrees and the H-degrees. In particular, there are nonzero branching degrees.*

Proof. Partition $\mathbb{N}$ into intervals $I_{x,y}$ of length $2y$ for all $x, y \in \mathbb{N}$. Now, for any set $A \subseteq \mathbb{N}$, let on $I_{x,y}$ the function $f^A(n)$ take the value $n/2$ if $x \notin A$ and $n/2 + m/c$ if $x \in A$ and m is the distance of n to the boundary of $I_{x,y}$ and c is the constant from Proposition 19. It is easy to see that $f^{A\cup B}$ is the pointwise maximum of f^A and f^B, $f^{A\cap B}$ is the pointwise minimum of f^A and f^B. As one can by Proposition 19 build sets X^A and Y^A whose C-complexity and H-complexity follows up to a constant the function f^A, respectively, one gets that the embeddings $A \mapsto X^A$ and $A \mapsto Y^A$ from the subsets of $\mathbb{N}$ into the C-degrees and H-degrees, respectively, preserve meets and joins.

As the set $\{0,1\}$ is the meet of the sets $\{0,1,2\}$ and $\{0,1,3\}$, one has that the image $X^{\{0,1\}}$ is a nonzero branching degree which is the meet of the images $X^{\{0,1,2\}}$ and $X^{\{0,1,3\}}$ in the C-degrees. The same holds for $Y^{\{0,1\}}$, $Y^{\{0,1,2\}}$ and $Y^{\{0,1,3\}}$ in the H-degrees. □

The embeddings asserted in Theorem 20 witness in particular the embeddability of all countable distributive lattices into the C-degrees and into the H-degrees, however least elements are not preserved by these embeddings.

In the H-degrees the join of any pair of left-r.e. sets exists and coincides with the addition of the sets when viewed as expansions of reals, however joins fails to exist for pairs of H-degrees in general. In fact, Miller and Yu [16] showed that if sets X and Y are relative Martin-Löf random to each other, then the two sets do not have a common upper bound in the H-degrees. The situation for meets was much less clear for both, C-degrees and H-degrees. While the embeddability of all countable distributive lattice witness the existence of meets (and also of joins), Theorem 21 asserts that for a pair of sets in general their meet exists neither in the C-degrees nor in the H-degrees. Together these results answer corresponding questions by Nies and Miller [15, Question 9.1].

Theorem 21 can be viewed as a translation to the setting of C-degrees and H-degrees of the following fact on the upper semilattice of Turing degrees. There are countable nonprincipal ideals in the Turing degrees, and such ideals are generated by an ascending chain $A_0, A_1, A_2, \ldots$ of sets. These ideals have an exact pair, that is, for any such ideal there are two sets Z_0 and Z_1 such that a set Y is Turing reducible to some A_k iff it is Turing reducible to both, Z_0 and Z_1. As a consequence, the Turing degrees of Z_0 and Z_1 do not have a meet.

Theorem 21. *There is a pair X_0 and X_1 of sets which does not have a meet neither in the C-degrees nor in the H-degrees.*

Proof. Odifreddi [19, Section V.4] constructs for a given non-principal Turing ideal $A_0, A_1, A_2, \ldots$ an exact pair Z_0, Z_1 by defining finite variants B_n of A_n such that

- $Z_0 = \{\langle x, y\rangle : x \in A_y\}$,
- $Z_1 = \{\langle x, y\rangle : x \in B_y\}$,
- if $Y \leq_T Z_0$ and $Y \leq_T Z_1$ then $Y \leq_T A_0 \oplus A_1 \oplus \ldots \oplus A_n$ for some n,

where $\langle ., . \rangle$ is the usual effective pairing function. In particular, Z_0 and Z_1 do not have a meet in the Turing degrees. One can easily choose the A_n such that $A_n \geq_{\mathrm{T}} \mathbb{K}$ for all n. Then also $Z_0, Z_1 \geq_{\mathrm{T}} \mathbb{K}$.

There is a $\mathbb{K}$-recursive function f and a recursive partition of the natural numbers into intervals $I_0, I_1, I_2, \ldots$ such that

- $f(n) \in I_n$ and $f(n)$ is a power of 2 for all n;
- for every set $E \subseteq \mathbb{N}$, for $F = \{f(x) : x \in E\}$ and for every $x \notin E$: $\mathrm{C}(F(0)F(1)\ldots F(\max(I_x))) \leq \min(I_x)$ and $\mathrm{H}(F(0)F(1)\ldots F(\max(I_x))) \leq \min(I_x)$.
- for every set $E \subseteq \mathbb{N}$, for $F = \{f(x) : x \in E\}$ and for every $x \in E$: $\mathrm{C}(F(0)F(1)\ldots F(\max(I_x))) \geq 3\min(I_x)$ and $\mathrm{H}(F(0)F(1)\ldots F(\max(I_x))) \geq 3\min(I_x)$.

Now let
$$X_a = \{f(x) : x \in Z_a \oplus \overline{Z_a}\}$$
for $a = 0, 1$. It is now shown that X_0 and X_1 do not have a meet neither in the C-degrees nor in the H-degrees. For every n, one can consider the set

$$F_n = \{f(2\langle x, n\rangle) : x \in A_n\} \cup \{f(2\langle x, n\rangle + 1) : x \notin A_n\}$$

which is the intersection of X_0 with the recursive set $R_n = \bigcup_x (I_{2\langle x,n\rangle} \cup I_{2\langle x,n\rangle+1})$. It is also a finite variant of $X_1 \cap R_n$. Hence $F_n \leq_{\mathrm{C}} X_a$ and $F_n \leq_{\mathrm{H}} X_a$ for $a = 0, 1$.

Assume now by contradiction that Y would be the meet of X_0 and X_1 in the C-degrees. Then $F_n \leq_{\mathrm{C}} Y$ for all n. If $x \in A_n$ and x is sufficiently large then

- $f(2\langle x,n\rangle) \in F_n$;
- $\mathrm{C}(F_n(0)F_n(1)\ldots F_n(\max(I_{2\langle x,n\rangle}))) \geq 3\min(I_{2\langle x,n\rangle})$;
- $\mathrm{C}(Y(0)Y(1)\ldots Y(\max(I_{2\langle x,n\rangle}))) > 2\min(I_{2\langle x,n\rangle})$;
- $f(2\langle x,n\rangle + 1) \notin X_0$;
- $\mathrm{C}(X_0(0)X_0(1)\ldots X_0(\max(I_{2\langle x,n\rangle+1}))) \leq \min(I_{2\langle x,n\rangle+1})$;
- $\mathrm{C}(Y(0)Y(1)\ldots Y(\max(I_{2\langle x,n\rangle+1}))) < 2\min(I_{2\langle x,n\rangle+1})$.

Furthermore, if $x \notin A_n$ and x is sufficiently large then

- $f(2\langle x,n\rangle + 1) \in F_n$;
- $\mathrm{C}(F_n(0)F_n(1)\ldots F_n(\max(I_{2\langle x,n\rangle+1}))) \geq 3\min(I_{2\langle x,n\rangle+1})$;
- $\mathrm{C}(Y(0)Y(1)\ldots Y(\max(I_{2\langle x,n\rangle+1}))) > 2\min(I_{2\langle x,n\rangle+1})$;
- $f(2\langle x,n\rangle) \notin X_0$;
- $\mathrm{C}(X_0(0)X_0(1)\ldots X_0(\max(I_{2\langle x,n\rangle}))) \leq \min(I_{2\langle x,n\rangle})$;
- $\mathrm{C}(Y(0)Y(1)\ldots Y(\max(I_{2\langle x,n\rangle}))) < 2\min(I_{2\langle x,n\rangle})$.

So it follows for all n that $A_n \leq_{\mathrm{T}} Y$ and $\mathbb{K} \leq_{\mathrm{T}} Y$. Furthermore, by the choice of X_0, X_1 as sets of powers of 2, it follows that $Y \leq_{\mathrm{T}} X_0$ and $Y \leq_{\mathrm{T}} X_1$. So $Y \leq_{\mathrm{T}} Z_0$ and $Y \leq_{\mathrm{T}} Z_1$ in contradiction to the construction given by Odifreddi [19, Section V.4]. In the same way one can show that there is also no meet Y for X_0 and X_1 in the H-degrees. □

9. Lattices of C-degrees and H-degrees inside a Turing degree

In this section, a lattice embedding into the C-degrees inside a single Turing degree of some set A is constructed such that each C-degrees in the range of the mapping is completely contained in the Turing degree of A. The embedding preservers meets and joins with respect to the structure of all degrees, not just with respect to the degrees in the range of the embedding or in the given Turing degree. A similar result holds for lattice embedding into the H-degrees that are contained in a Turing degree is. Note that the first embedding results in the following theorem actually holds for the truth-table degree of A.

Theorem 22. *If a set A is complex, then any countable distributive lattice can be embedded into the C-degrees that are contained in the Turing degree of A.*

Proof. Let A be any complex set. One can assume that whenever the set A or its complement is r.e. in a set Y then A is actually recursive in Y, because otherwise one can replace A by the join of A with the complement of A, where this join is in the Turing degree of A and has the assumed property. Fix a recursive order h that is a lower bound for the C-complexity of the initial segments of A, that is, for all n holds

$$h(n) \leq \mathbf{C}(A(0) \ldots A(n-1)) \ . \tag{6}$$

Let $S = \{2^{2^n} : n \in \mathbb{N}\}$. Partition the natural number into consecutive intervals $I_1, I_2, \ldots$ such that

$$|I_j| + j \leq h(|I_{j+1} \cap S|) \ . \tag{7}$$

Let A_S be the set such that for all j the initial segment of A of length $|I_{j+1} \cap S|$ is coded in the natural way at the places in the intersection of I_j and S, that is, for all j and all $m \in I_j$, let $A_S(m) = 0$ in case $m \notin S$ and otherwise, let

$$A_S(m) = A(|I_j \cap S \cap \{0, \ldots, m-1\}|) \ .$$

Then define a mapping f from sets to sets where bit $X(i)$ is coded into the restriction of $f(X)$ to the interval I_j as follows

$$f(X) \cap I_j = \begin{cases} A_S \cap I_j & \text{if } X(i) = 1 \ , \\ \emptyset & \text{if } X(i) = 0 \ . \end{cases}$$

As usual, let $=^*$ and $\subseteq^*$ denote equality and inclusion of sets up to finitely many places, that is, $X_0 \subseteq^* X_1$ holds if $X_1 - X_0$ is finite, $X_0 =^* X_1$ holds if $X_0 \subseteq^* X_1$ and $X_1 \subseteq^* X_0$. Recall that the relation $\subseteq^*$ induces a partial ordering $<$ on the equivalence classes of the recursive sets with respect to $=^*$ and that the equivalence classes of the recursive sets under the relation $<$ is the (up to isomorphism a unique) countable atomless Boolean algebra [8]. Recall further that any countable distributive lattice can be embedded as a lattice into the countable atomless Boolean algebra [8]. Since lattice embeddings are closed under composition, it suffices to show that the mapping

$$X \mapsto f(\mathbb{N} \oplus A \oplus X)$$

is a lattice embedding of the equivalence classes under $=^*$ of the recursive sets under $\subseteq^*$ into the Turing degree of A.

First, let X be a recursive set and assume that Y is C-equivalent to $f(\mathbb{N} \oplus A \oplus X)$. By construction, the complement of A is r.e. in Y, hence $A \leq_{\mathrm{T}} Y$ by choice of A. On the other hand, one has

$$Y \leq_{\mathrm{T}} f(\mathbb{N} \oplus A \oplus X) \leq_{\mathrm{T}} \mathbb{N} \oplus A \oplus X \leq_{\mathrm{T}} A$$

by $f(\mathbb{N} \oplus A \oplus X) \subseteq S$, by the definition of f and because X is recursive. As a consequence, for any recursive set X, the C-degree of $f(\mathbb{N} \oplus A \oplus X)$ is contained in the Turing degree of A.

Next it is argued that all recursive sets X_0 and X_1 satisfy

$$X_0 \subseteq^* X_1 \ \text{ iff } \ f(\mathbb{N} \oplus A \oplus X_0) \leq_{\mathrm{C}} f(\mathbb{N} \oplus A \oplus X_1) \ . \tag{8}$$

Indeed, in case $X_0 \subseteq^* X_1$, then knowing a program that decides X_0, for any m one can compute the initial segment of $f(\mathbb{N} \oplus A \oplus X_0)$ of length m from the initial segment of $f(\mathbb{N} \oplus A \oplus X_1)$ of the same length, hence $f(\mathbb{N} \oplus A \oplus X_0)$ is C-reducible to $f(\mathbb{N} \oplus A \oplus X_1)$. Conversely, in case $X_0 \not\subseteq^* X_1$, then by construction there are infinitely many intervals I_j such that the C-complexity of the restrictions of $f(\mathbb{N} \oplus A \oplus X_0)$ to the first j intervals exceeds the C-complexity of the restrictions of $f(\mathbb{N} \oplus A \oplus X_1)$ to the first j intervals by more than $j - c_1$ for some constant c_1, hence $f(\mathbb{N} \oplus A \oplus X_0)$ is not C-reducible to $f(\mathbb{N} \oplus A \oplus X_1)$.

Finally, the meet and join of recursive sets X_0 and X_1 are

$$X_{\text{meet}} = X_0 \cap X_1 \quad \text{and} \quad X_{\text{join}} = X_0 \cup X_1 \ ,$$

respectively. By (8), the set $f(\mathbb{N} \oplus A \oplus X_{\text{meet}})$ is a lower bound for $f(\mathbb{N} \oplus A \oplus X_0)$ and $f(\mathbb{N} \oplus A \oplus X_1)$; similarly, the set $f(\mathbb{N} \oplus A \oplus X_{\text{join}})$ is an upper bound for $f(\mathbb{N} \oplus A \oplus X_0)$ and $f(\mathbb{N} \oplus A \oplus X_1)$. So it suffices to show that these sets are in fact greatest lower and least upper bounds, which follows because by construction for any m and up to an additive constant , the C-complexities of the length m prefixes of $f(\mathbb{N} \oplus A \oplus X_{\text{meet}})$ and of $f(\mathbb{N} \oplus A \oplus X_{\text{join}})$ is equal to the minimum and the maximum, respectively, of the C-complexities of the length m prefix of $f(\mathbb{N} \oplus A \oplus X_0)$ and of $f(\mathbb{N} \oplus A \oplus X_1)$. □

Theorem 23. *If a set A computes the halting problem, then any countable distributive lattice can be embedded into the H-degrees that are contained in the Turing degree of A.*

Proof. Any set A that computes the halting problem is complex; the concept complex remains the same if in its definition one replaces C-complexity by H-complexity [10]. Accordingly the proof of the theorem can be done along the lines of the proof of Theorem 22, where however, now Proposition 7 is used instead of Proposition 8. This way one gets only that $Y \leq_{\mathrm{T}} A \oplus \mathbb{K}$. For that reason, $A \geq_{\mathrm{T}} \mathbb{K}$ had been required, so that the halting problem can be omitted in the former reduction and one obtains $Y \leq_{\mathrm{T}} A$. □

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