A cohesive set which is not high

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Abstract

We study the degrees of unsolvability of sets which are cohesive (or have weaker recursion-theoretic "smallness" properties.) We answer a question raised by the first author in 1972 by showing that there is a cohesive set A whose degree $\mathbf{a}$ satisfies $\mathbf{a}'' = \mathbf{0}''$ and hence is not high. We characterize the jumps of the degrees of r-cohesive sets, and we show that the degrees of r-cohesive sets coincide with those of the cohesive sets. We obtain analogous results for strongly hyperimmune and strongly hyperhyperimmune sets in place of r-cohesive and cohesive sets respectively. We show that every strongly hyperimmune set whose degree contains either a Boolean combination of Σ_2 sets or a 1-generic set is of high degree. We also study primitive recursive analogues of these notions and in this case we characterize the corresponding degrees exactly.

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1 Introduction

A set A of natural numbers is *cohesive* if A is infinite but is not split nontrivially by any recursively enumerable set, *i.e.* for every r.e. set B either $A \subseteq^* B$ or $A \subseteq^* \overline{B}$ holds. A set A is *maximal* if it is r.e. and its complement is cohesive. The maximal sets have been extensively studied within the lattice of r.e. sets (see [15, p. 187]), and furthermore MARTIN [12] characterized the degrees of maximal sets as precisely the high r.e. degrees. (For our purposes, we define a (Turing) degree $\mathbf{a}$ to be *high* if $\mathbf{a}' \geq \mathbf{0}''$, without requiring that $\mathbf{a} \leq \mathbf{0}'$.) Martin's result made it reasonable to conjecture that the degrees of the cohesive sets (which we shall call the *cohesive degrees*) are precisely the high degrees. Some progress on this conjecture was made by JOCKUSCH [3, 5] and by COOPER [1]. Specifically, JOCKUSCH [3, 5] showed that every high degree is cohesive and that the cohesive degrees are closed upwards, while COOPER [1] showed that every cohesive degree below $\mathbf{0}'$ is high.

The general problem remained open and seemed very resistant. This is partially because natural constructions of cohesive sets tend to produce high sets (*i.e.* sets of high degree). For example, one natural way to make a cohesive set A is by forcing with pairs (F, W), where F is a finite set and W is an infinite r.e. set. (Such a pair (F, W) imposes the condition that the constructed cohesive set A satisfies $F \subseteq A \subseteq F \cup W$.) It is easily seen that all generic sets for this forcing are high. (Given any recursive function f and any such condition (F, W) there is an extension (F, V) $(V \subseteq W)$ such that for any set A with $F \subseteq A \subseteq F \cup V$ the principal function p_A of A eventually majorizes f. Thus if A is sufficiently generic for this forcing, then p_A majorizes all recursive functions, so A is high by [12].)

In this paper, we show nonetheless that there is a cohesive degree which is not high. We show in fact that there is a cohesive degree $\mathbf{a}$ which is low$_2$, *i.e.* $\mathbf{a}'' = \mathbf{0}''$. We do this by combining several results rather than with a direct forcing construction. The intermediate results we use involve weakenings of the notion of cohesiveness. Call a set A *r-cohesive* (respectively *p-cohesive*) if it is infinite but not split nontrivially by any recursive, (respectively primitive recursive) set. The first step in the proof is to characterize the p-cohesive degrees as those degrees $\mathbf{a}$ such that $\mathbf{a}' \in \mathbf{C}$, where $\mathbf{C}$ is a certain class of degrees to be defined below. ($\mathbf{C}$ is analogous to the degrees of complete extensions of Peano arithmetic, but relativized to $\mathbf{0}'$.) The Low Basis Theorem [7, Theorem 2.1] is used to show that there are degrees $\mathbf{c} \in \mathbf{C}$ with $\mathbf{c}' = \mathbf{0}''$, and thus that there are low$_2$ p-cohesive degrees. Another use of the Low Basis theorem produces a set A such that every recursive set is primitive recursive in A and this, in combination with the previous result, produces a low_2 r-cohesive degree. The final step uses the surprising result that every non-high r-cohesive set is in fact cohesive. From this it follows that the r-cohesive degrees coincide with the cohesive degrees, and thus there is a low_2 cohesive degree.

Once the conjecture that the cohesive degrees coincide with the high degrees is refuted, it is natural to try to find *some* characterization of the cohesive degrees. We do not fully succeed in this, although we show that the *jumps* of the cohesive degrees are precisely the degrees in $\mathbf{C}$. We also prove that the cohesive degrees (unlike the p-cohesive degrees) cannot be characterized purely in terms of their jumps by showing that there is a cohesive degree $\mathbf{a}$ and a noncohesive degree $\mathbf{b}$ with $\mathbf{a}' = \mathbf{b}'$. In addition we show that the degrees of cohesive sets are closely related to the degrees of diagonally nonrecursive (DNR) functions. (A function g is *DNR* if $g(e) \neq \varphi_e(e)$ for all e. See, for example, [6, 8, 10, 11].) By results mentioned above, to characterize the cohesive degrees in general it suffices to characterize the cohesive degrees in $\mathbf{Q}$, where $\mathbf{Q}$ is the set of all non-high degrees whose jump is in $\mathbf{C}$. We show that all cohesive degrees in $\mathbf{Q}$ contain DNR functions and, as a partial converse, that all degrees in $\mathbf{Q}$ which contain $\{0,1\}$–valued DNR functions are cohesive. (Note: The degrees of DNR functions are the same as the degrees of fixed-point free functions f (*i.e.* $\forall e[W_{f(e)} \neq W_e]$) [8, Lemma 4.1], and the degrees of $\{0,1\}$–valued functions in DNR are the same as the degrees of complete extensions of Peano arithmetic [6, Proposition 2], due to JOCKUSCH-SOARE and SOLOVAY.)

We compare p-cohesiveness (which has not been studied previously) with more standard "immunity" properties, and we study the degrees of sets with these properties. We show that every p-cohesive set is hyperimmune, but there exist p-cohesive degrees which are not strongly hyperimmune. On the other hand, there are hyperhyperimmune degrees which are not p-cohesive. Thus there are few implications between p-cohesiveness and more standard immunity properties. We will also develop analogues of some of the above results in which cohesiveness is replaced by strong hyperhyperimmunity. For example, we show that the jumps of the strongly hyperhyperimmune degrees are precisely the degrees which are DNR relative to $\mathbf{0}'$. This implies that the jumps of the cohesive degrees do not coincide with the jumps of the strongly hyperhyperimmune degrees. We also extend Cooper's result [1] that cohesive degrees below $\mathbf{0}'$ are high by showing that any p-cohesive degree which contains a Boolean combination of Σ_2 sets is high. We also show that every non-high cohesive set is contained in the complement of some effectively simple set, and is thus effectively immune. We obtain various further results showing that non-high cohesive sets have special properties.

Notation. We use the notation of Soare's textbook [15]. If P is a property of sets, we say that a degree $\mathbf{a}$ has property P (or is a P-degree) if at least one set of degree $\mathbf{a}$ has property P. $\mathbf{C}$ denotes the set of all degrees $\mathbf{c}$ such that every partial $\mathbf{0}'$- recursive $\{0,1\}$-valued function has a total extension recursive in $\mathbf{c}$. Using the Low Basis Theorem relative to $\mathbf{0}'$ as in Exercise 3.9 of [15, page 258], it is easy to show that there is a degree $\mathbf{c} \in \mathbf{C}$ such that $\mathbf{c}' = \mathbf{0}''$. Note that $\mathbf{C}$ is closed upward and that $\mathbf{c} \geq \mathbf{0}'$ for any degree $\mathbf{c} \in \mathbf{C}$.

2 Degrees of Cohesive Sets

In this section we study the degrees of p-cohesive, r-cohesive, and cohesive sets. It turns out that the p-cohesive degrees are the best behaved of these.

Theorem 2.1 *For any degree* $\mathbf{c}$*,* $\mathbf{c}$ *is p-cohesive iff* $\mathbf{c}' \in \mathbf{C}$*.*

Proof: (*only if*) Let $X = \{x_0, x_1, \ldots\}$, where $x_0 < x_1 < \ldots$, be a p-cohesive set of degree $\mathbf{c}$. Now take an arbitrary partial $\{0,1\}$-valued $\mathbf{0}'$-recursive function γ. By the proof of the Limit Lemma, γ has a primitive recursive $\{0,1\}$-valued approximation g; let $\hat{g}(x) = \lim_n g(x, x_n)$. For fixed x, the sets $G_0 = \{m : g(x,m) = 0\}$ and $G_1 = \overline{G_0} = \{m : g(x,m) = 1\}$ are primitive recursive and therefore, either $(\forall^\infty n)[x_n \in G_0]$ or $(\forall^\infty n)[x_n \in G_1]$. Thus the limit always exists and $\hat{g}$ is a total $\mathbf{c}'$-recursive extension of γ. Hence $\mathbf{c}' \in \mathbf{C}$.

(*if*) Let $A_0, A_1, \ldots$ be the usual uniformly recursive listing of all primitive recursive sets. Thus $\{\langle i,j \rangle : j \in A_i\}$ is recursive and the A_i's are closed under Boolean operations and in fact there are recursive functions on the indices which induce intersection and complementation. The following function is partial $\mathbf{0}'$-recursive:

$$\gamma(i,j) = \begin{cases} 0 & \text{if } |A_i \cap A_j| < |A_i \cap \overline{A_j}|, \\ 1 & \text{if } |A_i \cap \overline{A_j}| < |A_i \cap A_j|, \\ \uparrow & \text{otherwise.} \end{cases}$$

Since $\mathbf{c}' \in \mathbf{C}$, γ has a $\mathbf{c}'$-recursive total extension g. The idea is now to use g to "guide" the natural construction of a p-cohesive set.

Let e_0 be an index of ω and inductively define

$$A_{e_{n+1}} = \begin{cases} A_{e_n} \cap \overline{A_n}, & \text{if } g(e_n, n) = 0 \\ & \text{(and thus } |A_{e_n} \cap A_n| \leq |A_{e_n} \cap \overline{A_n}|), \\ A_{e_n} \cap A_n, & \text{if } g(e_n, n) = 1 \\ & \text{(and thus } |A_{e_n} \cap \overline{A_n}| \leq |A_{e_n} \cap A_n|). \end{cases}$$

So if A_{e_n} is infinite, then $A_{e_{n+1}}$ is infinite. By induction, the sets A_{e_n} are all infinite and the indices e_n may be computed effectively from a g-oracle. Since g has degree at most $\mathbf{c}'$, the numbers e_n have a uniformly $\mathbf{c}$-recursive approximation $e_{n,x}$.

We now use the standard "diagonal intersection" technique to find an infinite set X with $X \subseteq^* A_{e_n}$ for all n. Let

$$x_0 = 0, \; x_{n+1} = \; \min\{x > x_n : (\forall m \leq n)[x \in A_{e_{m,x}}]\}.$$

We must show that x_n is defined for each n. Assume inductively that x_n is defined. Note that $A_{e_n} \supseteq A_{e_m}$ for all $m \leq n$. Hence all sufficiently large elements x of the infinite set A_{e_n} satisfy the defining condition for x_{n+1}, i.e. $x > x_n$ & $(\forall m \leq n)[x \in A_{e_{m,x}}]$. Thus x_{n+1} is defined. Let $X = \{x_n : n \in \omega\}$. Obviously X is infinite and $\deg(X) \leq \mathbf{c}$. Furthermore, $X \subseteq^* A_{e_m}$ for all m, since $x_{n+1} \in A_{e_{m,x_{n+1}}}$ for all $n \geq m$ and $e_{m,x_{n+1}}$ converges to e_m as n goes to infinity.

Consider any primitive recursive set, say A_n. By construction, $A_{e_{n+1}}$ is a subset of A_n or of its complement. This implies either $X \subseteq^* A_n$ or $X \subseteq^* \overline{A_n}$. Hence X is p-cohesive and so there is a p-cohesive set of degree at most $\mathbf{c}$. Since every infinite subset of a p-cohesive set is p-cohesive and there is an arithmetical p-cohesive set, the p-cohesive degrees are closed upward by [5, Theorem 1] and there is also a p-cohesive set of degree $\mathbf{c}$. ∎

We now obtain a sufficient condition for a degree to be r-cohesive and also characterize the jumps of the r-cohesive degrees. If $\mathbf{c}$ is a degree, we say that the recursive sets are $\mathbf{c}$–*subuniform* if there is a sequence of sets $A_0, A_1, \ldots$ such that $\{\langle i, j\rangle : j \in A_i\}$ is of degree at most $\mathbf{c}$ and every recursive set is equal to some A_i. It is shown in [4, Proposition 4] that, for any degree $\mathbf{a}$, the recursive sets are $\mathbf{a}$-subuniform iff either $\mathbf{a}$ is high or there is a complete extension of Peano arithmetic of degree at most $\mathbf{a}$. Hence (see [6, Proposition 2.2]), the recursive sets are $\mathbf{a}$-subuniform iff either $\mathbf{a}$ is high or there is a $\{0, 1\}$–valued DNR function of degree $\mathbf{a}$.

Theorem 2.2 **(i)** *Let* $\mathbf{c}$ *be any degree such that* $\mathbf{c}' \in \mathbf{C}$ *and the recursive sets are* $\mathbf{c}$*–subuniform. Then* $\mathbf{c}$ *is r-cohesive.*
(ii) *For any degree* $\mathbf{b}$*,* $\mathbf{b}$ *is in* $\mathbf{C}$ *iff* $\mathbf{b}$ *is the jump of an r-cohesive degree.*

Proof: (i) Let $\mathbf{c}$ be a degree as in the hypothesis of (i). Let $A_0, A_1, \ldots$ be sets which are uniformly of degree $\leq \mathbf{c}$ and include all recursive sets. Note also that we may assume without loss of generality that the A_i's are effectively closed under intersection and complementation, by closing them off if necessary. Now simply repeat the proof of Theorem 2.1, using these new A_i's in place of the primitive recursive sets, and replacing p-cohesiveness by r-cohesiveness, to obtain an r-cohesive set of degree $\mathbf{c}$.

(ii) Since every r-cohesive degree is p-cohesive, it follows from Theorem 2.1 that the jump of every r-cohesive degree is in $\mathbf{C}$.

For the converse, assume that $\mathbf{b} \in \mathbf{C}$. Let $\mathbf{a}$ be a low degree such that the recursive sets are $\mathbf{a}$–subuniform. The existence of such a degree [4, Proposition 3] follows at once from the Low Basis Theorem. Note that $\mathbf{b} \geq \mathbf{0}' = \mathbf{a}'$. By the Friedberg Completeness Criterion [15, page 98] relativized to $\mathbf{a}$, there is a degree $\mathbf{c} \geq \mathbf{a}$ such that $\mathbf{c}' = \mathbf{b}$. The degree $\mathbf{c}$ is r-cohesive by part (i) above, so $\mathbf{b}$ is the jump of an r-cohesive degree as desired. ∎

In the foregoing constructions of p-cohesive and r-cohesive sets, it was crucial that the primitive recursive sets and the recursive sets were closed under complementation. Thus, it is not clear how to use these methods directly to construct a cohesive set which is not high. The following surprising result eliminates this obstacle and thus is key for our paper.

Theorem 2.3 *If* A *is r-cohesive and not high, then* A *is cohesive.*

Proof: Let A be r-cohesive and W_e be an r.e. set such that $W_e \cap A$ is infinite. Let $f(x) = \min\{s : (\exists a \geq x)[a \in W_{e,s} \cap A]\}$. Clearly f is total and A-recursive. Since A is not high, there is an increasing recursive function h not dominated by f. If $f(x) < h(x)$, then there is some $a \geq x$, $a \in A \cap W_{e,h(a)}$ since $f(x) < h(x) \leq h(a)$. Hence the set $R = \{a : a \in W_{e,h(a)}\}$ is a recursive subset of W_e which has an infinite intersection with A. Therefore, $A \subseteq^* R$ and so $A \subseteq^* W_e$. Thus A is cohesive. ∎

Since any cohesive set is r-cohesive and since all high degrees are cohesive [4, Corollary 2], their degrees are equal:

Corollary 2.4 *The Turing degrees of the r-cohesive sets and the cohesive sets are the same.*

Using the relativized Low Basis Theorem [7] there is a degree $\mathbf{b} \in \mathbf{C}$ such that $\mathbf{b}' = \mathbf{0}''$. Thus there is some r-cohesive set X having degree $\mathbf{a}$ with $\mathbf{a}' \leq \mathbf{b}$ and $\mathbf{a}'' = \mathbf{0}''$. This set X is not high and therefore is cohesive. So the following theorem solves the open problem of [5] and [15, page 215]. Cooper's result [1] that any cohesive set below $\emptyset'$ is high does not generalize.

Theorem 2.5 *There is a cohesive set X such that $X'' \equiv_T \emptyset''$.*

Cohesiveness is not the only property of a non-high r-cohesive set A. Another one is that A is effectively immune, *i.e.* A is infinite and there is a recursive function h such that for all e: $W_e \subseteq A \Rightarrow |W_e| < h(e)$.

Theorem 2.6 *If A is r-cohesive but not high, then A is disjoint from some effectively simple set. In particular, A is effectively immune.*

Proof: We may assume that A is cohesive by Theorem 2.3. Let p_A denote the principal function of the set A. Since A is not high, there is an increasing recursive function f such that $f(n) > p_A(2n)$ for infinitely many n and $f(n) \geq 2n$ for all n. By an easy modification of the construction of Post's simple set, there is a simple set B such that p_B majorizes f:

Let ψ be a partial recursive function such that for all e, if W_e has at least one element $\geq f(e)$, then $\psi(e)$ is such an element, and otherwise $\psi(e) \uparrow$. Let B be the range of ψ.

Then, for all e, B has at most e elements $< f(e)$ (*i.e.* the values of ψ on arguments $< e$), so it follows that p_B majorizes f and that $\overline{B}$ is infinite (since $f(e) \geq 2e$). Also, f witnesses that $\overline{B}$ is effectively immune. Since A is cohesive, it must be almost contained in B or almost disjoint from B. If A is almost contained in B, there must exist k such that $p_A(n+k) \geq p_B(n)$ for all n, and so $p_A(2n) \geq f(n)$ for all sufficiently large n, in contradiction to the choice of f. It follows that A is almost disjoint from B, *i.e.* there is a finite set D such that A is disjoint from $B - D$. It is easily seen that $B - D$ is effectively simple since B is, so A is disjoint from an effectively simple set. ▮

Recall that $\mathbf{Q}$ is the class of degrees $\mathbf{a}$ which are not high and satisfy $\mathbf{a}' \in \mathbf{C}$. To characterize the cohesive degrees it suffices to characterize the cohesive degrees in $\mathbf{Q}$, since outside of $\mathbf{Q}$ the cohesive degrees coincide with the high degrees by Theorem 2.2 and [5, Corollary 2]. The following corollary shows that within $\mathbf{Q}$ the cohesive degrees are trapped between two closely related and well-studied classes of degrees involving DNR functions.

Corollary 2.7 *Suppose that* $\mathbf{a} \in \mathbf{Q}$.
(i). *If there is a* $\{0,1\}$*–valued DNR function of degree* $\mathbf{a}$*, then* $\mathbf{a}$ *is cohesive.*
(ii). *If* $\mathbf{a}$ *is cohesive, then there is a DNR function of degree* $\mathbf{a}$.

Proof: (i) By Theorem 2.2 it suffices to show that if h is a $\{0,1\}$–valued DNR function, then the recursive sets are $\deg(h)$-subuniform. This is well known, but we include a proof here for the convenience of the reader. Let g be a recursive function such that $\varphi_{g(e,x)}(u) = \varphi_e(x)$ for all e, x, and u. Then if we let $A_e = \{x : h(g(e,x)) = 0\}$, it is easily seen that the sets $A_0, A_1, \ldots$ are uniformly recursive in h and include all recursive sets.

(ii) By Theorem 2.6 it suffices to show that if A is effectively immune, then there is a DNR function $h \leq_T A$. It is well known that there is a function $g \leq_T A$ such that $(\forall e)[W_{g(e)} \neq W_e]$, i.e. let $W_{g(e)}$ consist of the first $p(e)+1$ elements of A, where p witnesses that A is effectively immune. By [8, Lemma 4.1] there is a DNR function $h \leq_T g$. ∎

Corollary 2.7 gives a necessary condition and a related sufficient condition for a degree to be cohesive, but it does not characterize the cohesive degrees. It is known that the degrees of DNR functions differ from the degrees of $\{0,1\}$–valued DNR functions. This may be shown by observing that almost every degree contains a DNR function but no $\{0,1\}$–valued DNR function (see [10, Corollary 2], [8, Lemma 4.1] and [7, Theorem 5.3]) or one may show from [10, Theorem 9] that there is such a degree below $\mathbf{0}''$. The following extension of the latter result is also due to Kučera and shows that these classes of degrees differ even on $\mathbf{Q}$. It can be proved by the methods of [10] and [11], but since it has not appeared explicitly in the literature we include an outline of the proof.

Proposition 2.8 *[Kučera, unpublished]. For any degree* $\mathbf{d} \geq \mathbf{0}'$ *there is a degree* $\mathbf{a}$ *such that* $\mathbf{a}$ *contains a DNR function but no* $\{0,1\}$*–valued DNR function and* $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{d}$.

Proof: The proof is very similar to that of the Friedberg jump theorem, but we obtain a set A of the desired degree $\mathbf{a}$ as the intersection of a descending chain $\{T_n\}_{n\in\omega}$ of Π^0_1 classes of positive measure in 2^ω rather than as the union of an ascending chain of strings. In order to ensure that there is a DNR function of degree $\mathbf{a}$ we choose T_0 so that it contains only sets of DNR degree. For example, for T_0 we could use the class $\mathcal{P}_0$ of [10, Theorem 2] and apply [10, Corollary 1] and [8, Lemma 4.1] to show that T_0 contains only sets of DNR degree. The set A must also satisfy the following requirements:

N_e : $\{e\}^A$ is not a (total) $\{0,1\}$–valued DNR function
Q_e : $A'(e)$ is decided (forced) during the construction
P_e : $D(e)$ is appropriately coded into A

(The requirements Q_e and P_e are imprecisely stated but are familiar from the proof of the Friedberg completeness criterion.) We effectively order these requirements as $\{R_n\}_{n\in\omega}$ and choose T_{n+1} so that all of its elements satisfy R_n. To do this, we must show that the above requirements are appropriately "dense". For example we show that given an index of a Π^0_1–class T of positive measure and given e we can $0'$–effectively find an index of a Π^0_1 subclass U of T such that U has positive measure and all elements of U satisfy N_e, and similarly for Q_e in place of N_e. Here the crucial point is that it is a Σ_2 predicate of the index of a Π^0_1 class to say that it has positive measure. For N_e, given a Π^0_1 class T of positive measure, we search $0'$–effectively for x such that the Π^0_1 class consisting of all A in T with $\{e\}^A(x)$ divergent has positive measure. If we find such an x we let $U = \{A \in T : \{e\}^A(x)\uparrow\}$. Simultaneously, we search $0'$–effectively for a clopen set C (determined by finitely many strings) and an x such that $C \cap T$ has positive measure and $\{e\}^A(x)$ is defined and equal to $\varphi_e(e)$ or has value > 1 for all $A \in C$. If we find such a C and x, let $U = C \cap T$. Since almost no set has a $\{0,1\}$–valued DNR function recursive in it [7, Theorem 5.3], at least one of these searches must succeed. The strategy for Q_e is similar but easier. To meet P_e we use a $D \oplus 0'$–oracle. Search for two incompatible strings σ and τ such that the extensions of σ and τ in the given Π^0_1 class T each have positive measure. (Again this is a $0'$–recursive search and must terminate.) Suppose that the code number of σ is less than the code number of τ. Let $U = \{A \in T : \sigma \prec A\}$ if $e \in D$ and otherwise let $U = \{A \in T : \tau \prec A\}$.

One verifies that the construction works as in the proof of the Friedberg completeness criterion. Specifically, the entire construction, including the determination of A', is carried out recursively in $D \oplus 0'$, so $A' \leq_T D \oplus 0'$. The construction is also carried out recursively in $A \oplus 0'$, and from this it is easily seen that $D \leq_T A \oplus 0'$. We omit further details, which are routine.
∎

We have shown that the cohesive degrees coincide with the r-cohesive degrees and that the jumps of the r-cohesive degrees coincide with the jumps of the p-cohesive degrees. On this basis one might hope that the r-cohesive degrees coincide with the p-cohesive degrees. The next result will show that this is not the case. We later obtain a generalization of part (i) using a more complicated proof in Theorem 3.11.

Theorem 2.9 **(i)** *Every cohesive 1-generic degree is high.*
(ii) *There is a cohesive degree* $\mathbf{b}$ *and a noncohesive degree* $\mathbf{a}$ *with* $\mathbf{a}' = \mathbf{b}'$. *Hence, the cohesive degrees cannot be characterized purely in terms of their jumps.*
(iii) *There is a p-cohesive degree which is not r-cohesive.*

Proof: (i) By Theorem 2.6, it suffices to show that no effectively immune set A is Turing reducible to a 1-generic set B. This is well known (see [11, p. 219]). Theorem 4.2 provides a full proof of (i).

(ii) Let $\mathbf{c}$ be any degree in $\mathbf{C}$ with $\mathbf{c} \not\geq \mathbf{0}''$. Since $\mathbf{c} \geq \mathbf{0}'$, the usual proof of the Friedberg completeness criterion shows that there is a 1-generic degree $\mathbf{a}$ with $\mathbf{a}' = \mathbf{c}$. The degree $\mathbf{a}$ is not cohesive by Theorem 2.9 . Also, by Theorem 2.2, there is an r-cohesive degree $\mathbf{b}$ such that $\mathbf{b}' = \mathbf{c}$, and $\mathbf{b}$ is cohesive by Corollary 2.4.

(iii) The degree $\mathbf{a}$ given in the proof of (ii) is not r-cohesive, but it is p-cohesive by Theorem 2.1. ▮

3 Degrees of other immunity notions

In this section we study the degrees of sets with various hyperimmunity-type properties. Such properties are similar to cohesiveness-type properties except that they are defined in terms of the nonexistence of splittings of the set into infinitely many effectively given pieces instead of into two infinite pieces. We show that p-cohesiveness implies hyperimmunity but that there are few other implications between p-cohesiveness and standard hyperimmunity-type properties. We obtain results for the degrees of strongly hyperimmune and strongly hyperhyperimmune sets which parallel those for r-cohesive and cohesive degrees respectively. These will allow us to show that the strongly hyperimmune degrees differ from the strongly hyperhyperimmune degrees but that the corresponding classes of jumps coincide. We also show that the jumps of the cohesive degrees are not the same as the jumps of the strongly hyperhyperimmune degrees.

First we review the definitions of these properties. A sequence of sets $S_0, S_1, \ldots$ is said to *meet* a set A if $(\forall i)[S_i \cap A \neq \emptyset \;\&\; (\forall j \neq i)[S_i \cap S_j = \emptyset]]$. A is *hyperimmune* if A is infinite and there is no sequence of finite sets given effectively by canonical indices which meets A. The notion of *hyperhyperimmune (hhi)* is defined similarly except that r.e. indices are used in place of canonical indices. The notion of *strongly hyperhyperimmune (shhi)* is also defined similarly except that r.e. indices are again used and the restriction that the sets be finite is dropped. The notion of *strongly hyperimmune (shi)* is also defined similarly except that the sets S_i are required to be uniformly recursive (but not necessarily finite) and also $\bigcup_i S_i$ is required to be recursive. Note that, in the definitions of "shhi" and "shi", it would be equivalent to require that $S_i \cap A$ be infinite instead of merely nonempty, since we could define a new sequence T_i with $T_i = \bigcup_j S_{<i,j>}$. In some sense, strong hyperimmunity is analogous to r-cohesiveness and strong hyperhyperimmunity is analogous to cohesiveness. This is clear from the relevant definitions, but will be made more precise in the results of this section.

Let us call a class $\mathbf{K}$ of degrees *domination closed* if whenever $\mathbf{a} \in \mathbf{K}$ and every $\mathbf{a}$-recursive function is dominated by a $\mathbf{b}$-recursive function, then $\mathbf{b} \in \mathbf{K}$. For example, the high degrees and the hyperimmune degrees are each domination closed. It is perhaps less expected that this holds also for the p-cohesive degrees:

Theorem 3.1 *The class of p-cohesive degrees is domination closed.*

Proof: The proof actually shows the following stronger result: Let $A = \{a_0 < a_1 < \ldots\}$ be p-cohesive and $B = \{b_0 < b_1 < \ldots\}$ be any set with $(\forall i)[a_i \leq b_i]$. Then the degree of B is p-cohesive.

Note that $a_n \in B_n = \{n, n+1, \ldots, b_n\}$ for any n. Let γ be a $\{0,1\}$–valued function partial recursive in $0'$. It suffices to show that there is a $\{0,1\}$–valued function $f \leq_T B'$ such that $f(e) \neq \gamma(e)$ for all e. (Apply Theorem 2.1 to $1-f$.) Let g be a $\{0,1\}$–valued primitive recursive function with $\gamma(e) = \lim_s g(e,s)$ for all e. Let S be the set of all pairs (e,y) such that there are only finitely many n with $g(e,s) = y$ for all $s \in B_n$. Then S is r.e. in X' and for each e there exists y with $(e,y) \in S$. [To prove the latter statement, assume otherwise and fix e such that, for all y, $(e,y) \notin S$. Then, $(\lambda s)g(e,s)$ assumes each value $0,1$ infinitely often on A and hence witnesses that A is not p-cohesive.] Now let $f(e) = y$, where (e,y) is the first pair with first component e found in a fixed X'-recursive enumeration of S. Clearly $f(e) \neq \gamma(e)$. ▮

Corollary 3.2 *Every p-cohesive set is hyperimmune.*

Proof: If A is not hyperimmune and A is p-cohesive, we may take the set B in the proof of the previous theorem to be recursive. Then, by that proof, $\mathbf{0}' \in \mathbf{C}$, which is clearly impossible. ▮

In contrast to Theorem 3.1, the class of cohesive degrees is not domination closed (as defined before the statement of Theorem 3.1): Let $\mathbf{b}$ be a degree which is p-cohesive but not cohesive. Then there is a degree $\mathbf{a} \geq \mathbf{b}$ hyperimmune-free relative to $\mathbf{b}$ such that every partial $\{0,1\}$-valued $\mathbf{b}$-recursive function has a total $\mathbf{a}$-recursive extension [7], and so the recursive sets are $\mathbf{a}$-subuniform. Thus $\mathbf{a}$ is cohesive by Theorem 2.2 and Theorem 2.3. Since $\mathbf{a}$ is hyperimmune-free relative to $\mathbf{b}$, any $\mathbf{a}$-recursive function is dominated by a $\mathbf{b}$-recursive function. Since $\mathbf{a}$ is cohesive and $\mathbf{b}$ is not, this implies that the cohesive degrees are not domination closed.

Theorem 3.3 *There is a hyperhyperimmune degree which is not p-cohesive. In fact, every 2-generic degree has this property.*

Proof: Let A be any 2-generic set. Then A is 1-generic relative to $\mathbf{0}'$ and hence is hyperimmune relative to $\mathbf{0}'$. It follows easily that A is hyperhyperimmune, since any sequence of finite r.e. sets given effectively by r.e. indices can be given effectively relative to $\mathbf{0}'$ by canonical indices.

The proof of Theorem 2.9 shows that no 1-generic degree bounds any DNR degree. Relativizing this to $\mathbf{0}'$ shows that if $\mathbf{a}$ is 2-generic, then $\mathbf{a} \cup \mathbf{0}'$ does not bound any degree in $\mathbf{C}$. Since $\mathbf{a}$ is 1-generic, we have $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}'$. Thus $\mathbf{a}'$ does not bound any degree in $\mathbf{C}$, and it follows that $\mathbf{a}$ is not p-cohesive. ▮

We now begin to study the shi degrees. The following easy and probably well-known characterization of shi sets will be useful:

Lemma 3.4 *For any set A the following are equivalent:*
(i) *A is strongly hyperimmune (shi).*
(ii) *A is infinite and there is no recursive function f such that $f(A) = \omega$.*
(iii) *A is infinite and there is no recursive function g such that $g(s) \leq s$ and $g^{-1}(n) \cap A$ is infinite for all $s, n \in \omega$.*

Proof: To show the contrapositive of (i) $\rightarrow$ (ii), let f be as in (ii), and let $S_n = f^{-1}(n)$ to show that (i) fails. The implication (ii) $\rightarrow$ (iii) is immediate. To show the contrapositive of (iii) $\rightarrow$ (i), let S_n witness that A is not shi with $S_n \cap A$ infinite for all n. Let $g(n)$ be defined to be the unique $i \leq n$ with $n \in S_i$, if such an i exists, and otherwise let $g(n) = 0$. Clearly g witnesses the falsity of (iii). ▮

As in section 2, it is useful to consider a primitive recursive analogue of strong hyperimmunity. We have not checked whether the primitive recursive analogues of (i) - (iii) in Lemma 3.4 are equivalent, but this is not important for our purposes. The primitive recursive analogue of (iii) is the most convenient here. Call an infinite set A *p-small* if there is no primitive recursive function g as in (iii) of Lemma 3.4. (We prefer to avoid the ugly abbreviation "p-shi".) Let $\mathbf{C}_\omega$ be the class of all degrees $\mathbf{c} \geq \mathbf{0}'$ such that for any γ partial recursive in $\mathbf{0}'$ there is a function f of degree at most c such that $f(e) \neq \gamma(e)$ for all e. ($\mathbf{C}_\omega$ consists of the degrees $\geq \mathbf{0}'$ which are DNR relative to $\mathbf{0}'$ by the relativization to $\mathbf{0}'$ of [8, Lemma 4.1].) The following theorem is analogous to Theorem 2.1.

Theorem 3.5 *For any degree* $\mathbf{c}$*,* $\mathbf{c}$ *is p-small iff* $\mathbf{c}' \in \mathbf{C}_\omega$*.*

Proof: (*only if*) Let X be a p-small set of degree $\mathbf{c}$. Let γ be partial recursive in $\mathbf{0}'$. We must find a function $\hat{g} \leq_T X$ with $\hat{g}(e) \neq \gamma(e)$ for all $e \in \mathrm{dom}(\gamma)$. Let g be a primitive recursive approximation to γ, so that $\gamma(n) = \lim_s g(n,s)$. Let S be the set of pairs (e,k) such that $g(e,s) \neq k$ for all sufficiently large $s \in X$. Note that S is r.e. in X'. Since X is p-small, for each e there exists k with $(e,k) \in S$. Let $\hat{g}(e)$ be the first such k found in an X'- recursive enumeration of S. Clearly $g \leq_T X'$ and $\hat{g}(e) \neq \gamma(e)$ for all $e \in \mathrm{dom}(\gamma)$. Hence $\mathbf{c}' \in \mathbf{C}_\omega$.

(*if*) Let $A_0, A_1, \ldots$ be as in Theorem 2.1, and let $f_0, f_1, \ldots$ be a uniformly recursive listing of all primitive recursive functions of one variable. Let $\gamma(n,i) = \lim_{s \in A_i} f_n(s)$. Since $\mathbf{c}' \in \mathbf{C}_\omega$, there is a function g of degree at most $\mathbf{c}'$ such that $g(n,i) \neq \gamma(n,i)$ for all n, i.

We now define inductively a sequence of indices $\{e_n\}_{n \in \omega}$. Let e_0 be such that $A_{e_0} = \omega$. Given e_n, let e_{n+1} be an index e such that $A_e = \{s \in A_{e_n} : f_n(s) \neq g(n, e_n)\}$. The choice of g makes it easy to show by induction on n that A_{e_n} is infinite. Furthermore, the indices e_n may be computed effectively relative to $\mathbf{c}'$ and hence, as in Theorem 2.1, there is an infinite set X of degree at most $\mathbf{c}$ with $X \subseteq^* A_{e_n}$ for all n. For each n there are at most finitely many $s \in X$ with $f_n(s) = g(n, e_n)$, so X is p-small. Hence there is a p-small set of degree $\mathbf{c}$ by [5]. ▮

It follows from Theorem 3.5 and the proof of Theorem 3.3 that no 2-generic degree is p-small and hence that there are hhi degrees which are not p-small.

Theorem 3.6 *The class of p-small degrees is domination closed.*

Proof: The proof is the same as the proof of Theorem 3.1 except that γ and its primitive recursive approximation g are no longer assumed to be $\{0,1\}$–valued, and Theorem 3.5 is applied in place of Theorem 2.1. We omit the details. ▮

Corollary 3.7 *The p-small sets are all hyperimmune.*

The following result is parallel to Theorem 2.2.

Theorem 3.8 **(i)** *Let* $\mathbf{c}$ *be any degree such that* $\mathbf{c}' \in \mathbf{C}_\omega$ *and the recursive sets are* $\mathbf{c}$*–subuniform. Then* $\mathbf{c}$ *is shi.*
(ii) *For any degree* $\mathbf{b}$*,* $\mathbf{b}$ *is in* $\mathbf{C}_\omega$ *iff* $\mathbf{b}$ *is the jump of an shi degree.*

Proof: The proof is parallel to that of Theorem 2.2. The only new point is that in the proof of (i), one must observe that if the recursive sets are $\mathbf{c}$–subuniform, then there exist functions $f_0, f_1, \ldots$ which are uniformly of degree $\leq \mathbf{c}$ and such that every recursive function f with $f(s) \leq s$ for all s appears in the list. For this, let $f_n(i)$ be the least $j \leq s$ such that $\langle i, j\rangle \in A_n$ if such a j exists, and otherwise let $f_n(i) = 0$, where the sets $A_0, A_1, \ldots$ are uniformly of degree $\leq \mathbf{c}$ and include all recursive sets. We omit the remaining details. ▮

There is a similarity between the relations

$$\frac{\text{r-cohesive}}{\text{cohesive}} \qquad \text{and} \qquad \frac{\text{strongly hyperimmune}}{\text{strongly hyperhyperimmune}}.$$

This similarity is mainly that in both cases the sets of non-high degree having the first property also have the second and thus their degrees coincide. So Theorem 2.3 has an analogue:

Theorem 3.9 *If* A *is a strongly hyperimmune set of non-high degree, then* A *is strongly hyperhyperimmune.*

Proof: Assume that A is not strongly hyperhyperimmune, and so there is a uniformly r.e. family $W_{f(i)}$ of pairwise disjoint r.e. sets each having an infinite intersection with A. Let

$$g(i) = \min\{s : (\forall j \leq i)[\,|W_{f(j),s} \cap A| \geq i]\}.$$

Then g is a total A-recursive function. Further assume that A does not have high degree, so there is an increasing total recursive function h not dominated by g. Now let

$$R_i = \{x \geq i : x \in W_{f(i),h(x)}\}.$$

The sets R_i are uniformly recursive disjoint sets, and their union R is also recursive, since $x \in R$ is equivalent to $x \in R_i$ for some $i \leq x$.

Since g does not dominate h, for each i there is some $j \geq i$ with $g(j) \leq h(j)$. Now let $x = \max(A \cap W_{f(i),h(j)})$. By definition of g, $|A \cap W_{f(i),h(j)}| \geq j$ and so $x \geq j$. Since $h(x) \geq h(j)$, $x \in W_{f(j),h(x)}$ and so $|R_i \cap A| \geq 1$. Therefore A is not strongly hyperimmune. The theorem follows by contraposition. ▮

Corollary 3.10 **(i)** *The shi degrees are the same as the shhi degrees.*
(ii) *There is a degree which is shhi but not cohesive. In fact, there is an shhi degree which does not have the same jump as any p-cohesive degree.*

Proof: Part (i) is immediate from the theorem and the fact that every high degree is shhi.

For part (ii), it suffices to prove the second statement. For this, by Theorems 2.1, 3.8, and 3.9, it suffices to show that $\mathbf{C} \neq \mathbf{C}_\omega$. This is the relativization to $\mathbf{0}'$ of the known result that there is a degree which contains a DNR function but no $\{0,1\}$–valued DNR function (see the discussion before Proposition 2.8.) In fact, by relativizing Proposition 2.8 to $\mathbf{0}'$ it follows that every degree $\geq \mathbf{0}''$ is the jump of a degree in $\mathbf{C}_\omega - \mathbf{C}$. ▮

We do not know whether the analogue of Theorem 2.6 holds for shi sets, i.e. we do not know whether every non-high shi set is effectively immune (or of DNR degree). Nonetheless, we obtain the analogue of Theorem 2.9 by a more involved direct construction. Its proof uses methods from the theory of inductive inference (see Theorems 4.18 and 5.8 of [2]).

Theorem 3.11 *Every degree which is both 1-generic and strongly hyperimmune is also high.*

Proof: Let $B = \varphi_e^A$ be strongly hyperimmune and let A be non-high. The idea of the proof is to construct a set $V \subseteq \{0,1\}^*$ witnessing that A is not 1-generic. Now let

$$B^\tau = \{x : \varphi_e^\tau(x) = 1\}.$$

Thus $B = \bigcup_{\tau \preceq A} B^\tau$. Since B is strongly hyperimmune, there is no recursive function $f : \omega \to \omega$ such that $f(B) = \omega$. The idea is to construct f such that using that $f(B) \neq \omega$ one can construct an r.e. set V of strings witnessing that A is not 1-generic. Since there is no effective way to control A, the construction uses some way to diagonalize against all possible oracles simultaneously. From a given finite function σ (viewed as a potential initial segment of f), a function ψ constructs within a certain time t an extension $\psi(\sigma, t)$ of σ which shall disprove as many oracles as possible. ψ tries to use all oracles up to the given length $s = |\sigma|$. A finite set $T(\sigma, t)$ is used to keep track of which oracles of this length are not already diagonalized. Formally, the functions are defined inductively:

$$\psi(\sigma, 0) = \sigma, \quad T(\sigma, 0) = \{\tau \in \{0,1\}^* : |\tau| = s\}.$$

If for given t there are some $\tau, \tau', x_0, x_1, \ldots, x_s$ ($s = |\sigma|$) such that

(1) $|\psi(\sigma, t)| < x_0 < x_1 < \ldots < x_s < t$
(2) $x_0, x_1, \ldots, x_s \in B^{\tau'}$
(3) $\tau \preceq \tau'$,
(4) $\tau \in T(\sigma, t)$,
(5) $|\tau'| \leq t$,

then take the first such $(\tau, \tau', x_0, x_1, \ldots, x_s)$ and let

$$\begin{aligned} \psi(\sigma, t+1)(x) &= \begin{cases} \psi(\sigma, t)(x) & \text{if } x < |\psi(\sigma, t)|, \\ 0 & \text{if } |\psi(\sigma, t)| \leq x \leq x_0, \\ i+1 & \text{if } 0 \leq i < s \text{ and } x_i < x \leq x_{i+1}, \\ \uparrow & \text{otherwise } (x > x_s); \end{cases} \\ T(\sigma, t+1) &= T(\sigma, t) - \{\tau\} \end{aligned}$$

and otherwise let ψ, T be unchanged:

$$\psi(\sigma, t+1) = \psi(\sigma, t), \quad T(\sigma, t+1) = T(\sigma, t).$$

So $\{\psi(\sigma, t)\}_t$ is an increasing sequence of extensions of σ. Note that for all $x \geq |\sigma|$ either $\psi(\sigma, t)(x) \leq |\sigma|$ or $\psi(\sigma, t)$ is undefined. Furthermore for each σ, there are only finitely many t such that $\psi(\sigma, t+1) \neq \psi(\sigma, t)$. Let t now be so large that $\psi(\sigma, t)$ has taken its final value $\psi(\sigma)$. Furthermore assume for a contradiction that $\tau = A \upharpoonright |\sigma| \in T(\sigma, t)$. Since B is infinite, there is an extension $\tau' \succ \tau$ such that $B^{\tau'}$ has $|\sigma| + 1$ elements beyond the domain of $\psi(\sigma)$. Therefore at some sufficiently large stage, this extension is found and, contrary to our choice of t, $\psi(\sigma, t') \neq \psi(\sigma, t)$ for some $t' > t$. This shows that $A \upharpoonright |\sigma| \notin T(\sigma, t)$.

Thus $\hat{g}(\sigma) = (\mu t)[A \upharpoonright |\sigma| \notin T(\sigma, t)]$ is an A-recursive function and so is $g(s) = \max\{\hat{g}(\sigma) : \sigma \in \{0, \ldots, q\}^q \;\&\; q \leq s\}$. Since A is not high, there is an increasing recursive function h not dominated by g.

Let $\sigma_0 = 0$, $\sigma_{x+1} = \psi(\sigma_x, h(t)) \cdot 0$ for the first t such that $t > |\psi(\sigma_x, h(t))|$. The union of these strings σ_x is a recursive function f. Since B is strongly hyperimmune, there is some $n \notin f(B)$. Fix such an n. Now let

$$V = \{\tau : (\exists x \in B^\tau)\,[n = f(x)]\,\}.$$

V is an r.e. set. Furthermore if $\tau \in V$, then $\tau \not\preceq A$, since otherwise $n \in f(B)$. Now, to complete the proof it suffices to show that V does not "strongly avoid" A and that therefore A is not 1-generic. For this we must show that there are arbitrarily long strings $\tau \preceq A$ which have an extension $\tau' \in V$.

We know that there are infinitely many y such that $h(y) > g(y)$. Consider any such y and let k be the unique number such that $|\sigma_k| \leq y < |\sigma_{k+1}|$. We may assume that y is sufficiently large that $|\sigma_k| \geq n$. By construction,

$$\sigma_{k+1} \succ \psi(\sigma_k, h(|\sigma_{k+1}|)) \succeq \psi(\sigma_k, h(y)) \succeq \psi(\sigma_k, g(y)) \succeq \psi(\sigma_k, \hat{g}(\sigma_k)).$$

(In verifying the final inequality above, note that $g(y) \geq \hat{g}(\sigma_k)$ because $y \geq |\sigma_k| \geq \sigma_k(u)$ for all $u < |\sigma_k|$.) By the definition of $\hat{g}(\sigma_k)$ and the construction, $T(\sigma_k, \hat{g}(\sigma_k)) = T(\sigma_k, \hat{g}(\sigma_k) - 1) - \{\tau\}$, where $\tau = A \upharpoonright |\sigma_k|$. By the conditions (1-5) with $s = |\sigma_k| \geq n$ there are $\tau' \succ \tau$ and $x_n \in B^{\tau'}$ such that $f(x_n) = \psi(\sigma_k, \hat{g}(\sigma_k))(x_n) = n$. Thus $\tau' \in V$. Furthermore, $|\tau| = |\sigma_k| \geq k$, and k may be made arbitrarily large by choosing y large, so V witnesses that A is not 1-generic. ▮

Corollary 3.12 *There are degrees which are p-cohesive but not strongly hyperimmune.*

Proof: Let $\mathbf{a}$ be a non-high 1-generic degree with jump in $\mathbf{C}$. Then $\mathbf{a}$ is p-cohesive by Theorem 2.1 but is not strongly hyperimmune by Theorem 3.11. ▮

4 Further Results

In this section we obtain various further results on the degrees of sets with cohesiveness and hyperimmunity properties.

Call a cohesive set A *standard cohesive* if $A \subseteq^* B$ for every simple set B. (Use of the forcing method described in the introduction often produces a standard cohesive set since the infinite r.e. sets used in the forcing conditions have infinite intersection with every simple set.) By Theorem 2.6, if A is standard cohesive, then p_A must dominate every recursive function and therefore be of high degree. On the other hand, JOCKUSCH constructed in [3, Theorem 4.1] a standard cohesive set below any given high degree. Since the standard cohesive sets are closed under inclusion and there is an arithmetical standard cohesive set, their degrees are closed upward by [5, Theorem 1] and so every high degree contains a standard cohesive set. Thus we have obtained the following result.

Theorem 4.1 *A degree is high iff it contains a standard cohesive set.*

There are even high degrees in which *all* cohesive sets are standard cohesive:

Theorem 4.2 *If a cohesive set A has 1-generic degree, then A is standard cohesive.*

Proof: Let $A = \{e\}^B$ where B is 1-generic. Further, let S be a simple set. The r.e. set $A_\sigma = \{x : (\exists \tau \succeq \sigma)[\{e\}^\tau(x) = 1]\}$ is infinite for any $\sigma \preceq B$, since $A \subseteq A_\sigma$. Thus A_σ intersects S and so B does not strongly avoid $W = \{\tau : (\exists x \in S)[\{e\}^\tau(x) = 1]\}$. Since B is 1-generic, there is some $\sigma \preceq B$ with $\sigma \in W$, so A intersects S. Since by the same argument A also intersects $S - D$ for any finite set D, $A \cap S$ is infinite. Therefore $A \subseteq^* S$ and A is standard cohesive. ∎

So this theorem gives an alternative proof of Theorem 2.9 (i): Any cohesive set of 1-generic degree is standard cohesive and therefore high by Theorem 4.1.

COOPER [1] showed that any hyperhyperimmune set, and therefore any cohesive set, of degree below $\mathbf{0}'$ has high degree. This result does not generalize to all degrees by Theorem 2.5, but there is still an improvement: "degree below $\mathbf{0}'$" can be replaced by "degree k-r.e. in $\mathbf{0}'$". Note that hhi sets recursive in $0'$ are shhi by [1].

Theorem 4.3 *If A is p-small and of degree k-r.e. in $\mathbf{0}'$ for some k, then A has high degree.*

Proof: Let A be a p-small set whose degree is k-r.e. relative to $\mathbf{0}'$. There are degrees $\mathbf{a_0}, \mathbf{a_1}, \ldots, \mathbf{a_k}$ such that $\mathbf{a_{i+1}}$ is r.e. in $\mathbf{a_i} \cup \mathbf{0}'$ and $\mathbf{a_{i+1}} \geq \mathbf{a_i}$. Furthermore $\mathbf{a_0} = \mathbf{0}$ and A has degree $\mathbf{a_k}$. Then A' has degree $\mathbf{a_{k+1}} = \mathbf{a_k'}$, which is r.e. in $\mathbf{a_k} \cup \mathbf{0}'$ and above $\mathbf{a_k}$. Now let $\mathbf{b_i} = \mathbf{a_i} \cup \mathbf{0}'$. These degrees satisfy $\mathbf{0}' = \mathbf{b_0} \leq \mathbf{b_1} \leq \ldots \leq \mathbf{b_{k+1}}$, each $\mathbf{b_{i+1}}$ is r.e. in $\mathbf{b_i}$ and $\mathbf{b_{k+1}}$ is the degree of A'. Such a degree $\mathbf{b_{k+1}}$ is called $k+1$-REA relative to $\mathbf{0}'$. (Note that the degree $\mathbf{b_{k+1}}$ is not required to be below $\mathbf{0}''$.) Since A is p-small, $\mathbf{b_{k+1}} \in \mathbf{C}_\omega$. Now $\mathbf{C}_\omega$ consists of degrees which are DNR relative to $\mathbf{0}'$ and hence of degrees containing functions which are fixed-point free relative to $\mathbf{0}'$.

Theorem 5.1 of [8] states that any k-REA degree having a fixed point free function is above $\mathbf{0}'$. By relativizing this theorem to $\mathbf{0}'$, it follows $\mathbf{b_{k+1}} \cup \mathbf{0}' \geq \mathbf{0}''$. So $\mathbf{b_{k+1}} \geq \mathbf{0}''$ since $\mathbf{b_{k+1}} \geq \mathbf{0}'$. A' has degree above $\mathbf{0}''$ and A has high degree. ▮

Corollary 4.4 *If a $\mathbf{0}'$-recursive function f majorizes p_A for some p-cohesive set A, then f has high Turing degree.*

Proof: Let f be $\mathbf{0}'$-recursive, A be p-cohesive and f majorize p_A. By the proof of Theorem 3.1 there is a p-cohesive set B of degree below f. Then by Theorem 4.3, the set B and therefore also f have high degree. ▮

Theorem 2.3 showed that any non-high r-cohesive set is in fact cohesive. The following results are similar in spirit:

Theorem 4.5 *Let $A \leq_T B$, A of k-r.e. degree and let B be cohesive, but not high. Then either $B \subseteq^* A$ or $B \subseteq^* \overline{A}$.*

Proof: First consider the case that A has r.e. degree. χ_A is the limit of some approximation a with A-recursive modulus m:

$$m(x) = \min\{s : (\forall y \leq x)(\forall t \geq s)[a(y,t) = \chi_A(y)]\}.$$

Further let $B = \{b_0, b_1, \ldots\}$, where $b_0 < b_1 < \ldots$. Now the function $\hat{m}(n) = m(b_n)$ is B-recursive and therefore, there is some increasing recursive function h not dominated by $\hat{m}$. For infinitely many $x = b_n \in B$, $m(x) = m(b_n) \leq h(n) \leq h(x)$ and therefore, $B \not\subseteq^* \{y : (\exists t > h(y))[a(y,t) \neq a(y, h(y))]\}$. So for almost all $y \in B$, $a(y, h(y)) = \chi_A(y)$. Since $a(y, h(y))$ is a recursive function, there is a recursive set R such that $R \cap B =^* A \cap B$. So either $B \subseteq^* R \cap B \subseteq^* A \cap B \subseteq A$ or $B \subseteq^* \overline{R} \cap B \subseteq^* \overline{A} \cap B \subseteq \overline{A}$ follows.

Since the same argument goes through for sets $C \leq_T A$, B is r-cohesive relative to A. Furthermore, by relativizing Theorem 2.3 to A, B is cohesive relative to A. Of course, B still is not high relative to A.

If A has k-r.e. degree, there are degrees $\mathbf{a_0} = \mathbf{0} \leq \mathbf{a_1} \leq \ldots \leq \mathbf{a_k}$ such that each $\mathbf{a_{i+1}}$ is r.e. in $\mathbf{a_i}$ and A has degree $\mathbf{a_k}$, for example by [9, Theorem 1.4]. Inductively, one shows that B is cohesive relative to each $\mathbf{a_i}$. At the end, B is cohesive relative to $\mathbf{a_k}$. So either $B \subseteq^* A$ or $B \subseteq^* \overline{A}$. ▮

We close by showing that our construction of a non-high cohesive degree can be combined with cone avoidance.

Theorem 4.6 *Let $\mathbf{c}_0, \mathbf{c}_1, \ldots$ be a sequence of nonzero degrees. Then there is a cohesive non-high degree $\mathbf{a}$ such that $\mathbf{c}_i \not\leq \mathbf{a}$ for all i.*

Proof: Define inductively a sequence of degrees $\mathbf{d}_0, \mathbf{d}_1, \ldots$ ($\mathbf{d}_i \not\geq \mathbf{0}''$) as follows. Let $\mathbf{d}_0 = \mathbf{0}'$. If $\mathbf{d}_i \cup \mathbf{c_i} \not\geq \mathbf{0}''$, then let $\mathbf{e}_i = \mathbf{c}_i$ else $\mathbf{e}_i = \mathbf{0}'$; in both cases let $\mathbf{d}_{i+1} = \mathbf{d}_i \cup \mathbf{e_i}$. By Spector's Theorem [13, p. 485] the ideal generated by the degrees $\mathbf{d}_i$ has an exact pair $\mathbf{f}$ and $\mathbf{g}$ such that the degrees $\mathbf{e}_i$ are uniformly recursive in both $\mathbf{f}$ and $\mathbf{g}$. Since $\mathbf{0}''$ does not belong to the ideal, one half of the exact pair is not above $\mathbf{0}''$, say $\mathbf{g} \not\geq \mathbf{0}''$.

Posner and Robinson [14, Theorem 3] showed that there is a degree $\mathbf{a}$ such that $\mathbf{a}' = \mathbf{g}$ and $\mathbf{a}' = \mathbf{a} \cup \mathbf{e_i}$ for all i. By [7, Theorem 2.4] relativized to $\mathbf{a}'$ there is a degree in $\mathbf{C^a}$ (the relativization of $\mathbf{C}$ to $\mathbf{a}$) which is hyperimmune-free relative to $\mathbf{a}'$. Hence by Theorem 2.2 and Corollary 2.4 relativized to $\mathbf{a}$ there is a degree $\mathbf{b} \geq \mathbf{a}$ such that $\mathbf{b}$ is cohesive relative to $\mathbf{a}$ and $\mathbf{b}'$ is hyperimmune-free relative to $\mathbf{a}'$. Clearly $\mathbf{b}$ is also cohesive in the unrelativized world.

Since $\mathbf{a}' \cup \mathbf{0}''$ is r.e. in but not recursive in $\mathbf{a}'$, $\mathbf{a}' \cup \mathbf{0}''$ is hyperimmune relative to $\mathbf{a}'$, and hence $\mathbf{a}' \cup \mathbf{0}''$ is also hyperimmune relative to $\mathbf{b}'$. Thus $\mathbf{0}'' \not\leq \mathbf{b}'$ so $\mathbf{b}$ is not high.

Assume now for a contradiction that $\mathbf{c}_i \leq \mathbf{b}$. If $\mathbf{e}_i = \mathbf{c}_i$, then $\mathbf{b} = \mathbf{b} \cup \mathbf{e_i} \geq \mathbf{a} \cup \mathbf{e_i} = \mathbf{a}'$ and $\mathbf{b}' \geq \mathbf{a}'' \geq \mathbf{0}''$. Otherwise, $\mathbf{c}_i \cup \mathbf{d_i} \geq \mathbf{0}''$. Then $\mathbf{b}' \geq \mathbf{c}_i \cup \mathbf{a}' = \mathbf{c_i} \cup \mathbf{g} \geq \mathbf{c_i} \cup \mathbf{d_i} \geq \mathbf{0}''$. Both cases contradict the above assertion that $\mathbf{b}$ is not high, so $\mathbf{c}_i \not\leq \mathbf{b}$. ▮

5 Open Questions

A number of interesting open questions remain. The main problem is still to find an exact characterization of the cohesive degrees. In particular, we do not know whether the cohesive degrees coincide with either the DNR degrees or the degrees of complete extensions of arithmetic on $\mathbf{Q}$, the family of non-high degrees with jump in $\mathbf{C}$. The corresponding question for shhi degrees is also open for $\mathbf{Q}_\omega$, the family of non-high degrees with jump in $\mathbf{C}_\omega$.

A question related to the above is whether there are non-high cohesive degrees which are minimal degrees. (It is still open whether there are minimal degrees which are DNR [6].) It is also not known whether the cohesive degrees are first-order definable in the first-order theory of the partial ordering of the degrees. The corresponding questions for shhi degrees are also open.

Concerning other degree classes, it would be interesting to have a characterization of the hhi degrees or of their jumps. Of course, every degree $\geq \mathbf{0}'$ is the jump of a 1-generic and hence of a hyperimmune degree, but no characterization of the jumps of hyperimmune-free degrees is known. The corresponding definability questions are also open here.

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