Partial Learning of Recursively Enumerable Languages

Ziyuan Gao[b], Frank Stephan[a], Sandra Zilles[b]

[a]Department of Mathematics and Department of Computer Science, National University of Singapore, Singapore 119076

[b]Department of Computer Science, University of Regina, Regina, SK, Canada S4S 0A2

Abstract

This paper studies several typical learning criteria in the model of partial learning of r.e. sets in the recursion-theoretic framework of inductive inference. Its main contribution is a complete picture of how the criteria of confidence, consistency and conservativeness in partial learning of r.e. sets separate, also in relation to basic criteria of learning in the limit. Thus this paper constitutes a substantial extension to prior work on partial learning. Further highlights of this work are very insightful characterisations of some of the inference criteria studied, leading to interesting consequences about the structural properties of the collection of classes learnable under these criteria. In particular a class is consistently partially learnable iff it is a subclass of a uniformly recursive family.

1. Introduction

Identification in the limit from positive examples, as introduced by Gold [13], models learning as a process in which a learner is presented an infinite sequence of data items belonging to a target, say a language L or a graph of a function; here, in the abstract setting of recursion theory, languages are just considered to be recursively enumerable (r.e.) subsets of the set $\mathbb{N}$ of natural numbers and functions are always required to be recursive mappings from $\mathbb{N}$ to $\mathbb{N}$. The learner processes the data one by one, making a conjecture about the target L in every step. Successful learning of the target L requires the learner, on any infinite input sequence containing all and only the data items contained in L (called a text for L), to return a sequence of hypotheses that stabilises on a single correct hypothesis describing L. In most variations of the model, Gödel numbers are used as hypotheses.

This model is rather restrictive: for example, there is no learner that identifies every regular language in the limit; more generally, no class of sets containing an infinite set L and all its finite subsets is identifiable in the limit [13]. Intuitively, for every potential learner M there is a valid data sequence for L that forces M to conjecture finite subsets of L infinitely often, thus changing its mind infinitely often and failing to stabilise on one conjecture.

To overcome this difficulty, Osherson, Stob and Weinstein introduced the model of partial learning [22], in which the sequence of hypotheses is no longer required to stabilise. Instead, a learner M is considered successful for a target L, if, on any text for L, M returns a sequence of conjectures that contains only one hypothesis infinitely often; this hypothesis must describe L. Osherson, Stob and Weinstein proved that this relaxation of Gold's model allows for the identification of the class of all r.e. sets.

Recently, the model of partial learning has been studied in depth, namely in combination with typical learning criteria that restrict the behaviour of learners. An intuitive such criterion is consistency as introduced by Bārzdiņš [3], which requires the learner to always return conjectures for sets that contain all the examples presented in the text. The consistency requirement imposes a strong restriction on learners both in the context of learning in the limit [3] and in the context of partial learning [16]. Gao and Stephan [10] therefore introduced the notion of essential consistency, which allows the learner to be inconsistent finitely many times. For learning recursive functions, they proved this model to be less restrictive than consistent partial learning, and even more powerful than behaviourally correct learning, a version of learning in the limit in which only a semantic, not a syntactic convergence of the hypothesis sequence is required.

A criterion often considered together with or in contrast to consistency is that of conservativeness: a conservative learner identifying a class of sets in the limit is allowed to change a hypothesis on a valid text segment for a target set L only if that hypothesis is inconsistent with the text segment [1]. This definition does not transfer to the case of partial learning, where a correct hypothesis can be "suspended" infinitely often. Gao, Jain and Stephan [11] adapted the model of conservativeness to partial learning, by requiring that a conservative partial learner

(i) outputs exactly one correct hypothesis and that infinitely often,

(ii) outputs all incorrect hypohteses only finitely often and

(iii) does not output any overgeneralisations of the target language as intermediate hypotheses.

Another criterion that was previously adapted from the limit-learning case to the partial learning case is that of confidence. In the classical setting, a confident learner will produce a sequence of hypotheses that stabilises, even if the input text is not for a set in the target class. In the context of partial learning, the natural adaptation studied by Gao, Stephan, Wu and Yamamoto [9] was to require that the learner outputs exactly one hypothesis infinitely often, even on texts for sets outside the target class.

The main contribution of this paper is a complete study of how the criteria of confidence, consistency and conservativeness in partial learning of r.e. sets separate, also in relation to basic criteria of learning in the limit. In particular, it is determined for any pair of criteria of interest whether or not there are classes learnable under one but not under the other criterion. This goes far beyond the results from previous work on partial learning, which focussed either on learning recursive functions or on only one of the special criteria addressed here.

Many of the results in the setting of set-learning are similar to those in the case of learning recursive functions, but there are also some differences.

Interestingly, most of the separations proved in this paper are already witnessed by uniformly recursive families of sets, which means that a hierarchy of inference criteria similar to that in Figure 1 applies to such families.

Further highlights of the present paper are characterisations of the collection of all classes of r.e. sets that are confidently partially learnable and of the collection of all classes of r.e. sets that are consistently partially learnable. The characterisation of confident partial learnability has the interesting consequence that the union of two confidently partially learnable classes is again confidently partially learnable. A further result is that a class of r.e. sets is consistently partially learnable if and only if it is contained in a uniformly recursive family. It will also be shown that a conservative partial learner can always be made prudent [22] (that is, can be constrained to outputting only hypotheses describing sets it can identify, on any input) or reliable [20] (that is, can be constrained never to output a wrong hypothesis infinitely often, on any input text).

The hierarchy diagram in Figure 1 summarises most of the results of this paper. The inference criteria are abbreviated; *BC* is for behaviourally correct learning [5], *Ex* for learning in the limit, *Part* for partial learning, *Conf* for confidence, *Consv* for conservativeness, *ClsCons* for class-consistency (where the learner is required to be consistent only on valid texts for potential targets), *Cons* for global consistency (where the learner is required to be consistent on all input texts), *EssClsCons* and *EssCons* for the "essential" versions of the two consistency criteria, *Prud* for prudence, *Rel* for reliability, and *ClsPresv* for the class-preserving version of a criterion, requiring the learner to return only hypotheses that represent sets from the target class, on any input. An arrow from criterion A to criterion B means that the collection of classes learnable under model A is contained in that learnable under model B. Each arrow is labelled with the Corollary/Example/Remark/Theorem number(s) that proves (prove) the relationship represented by the arrow. If there is no path from A to B, then the collection of classes learnable under model A is not contained in that learnable under model B.

This article is an extension of a conference paper [12] at the conference Algorithmic Learning Theory 2013 containing full proofs for the results announced at the conference plus some additional theorems, mainly in Section 6.

2. Preliminaries

The notation and terminology from recursion theory adopted in this paper follows in general the book of Rogers [24]. Background on inductive inference can be found in [15]. The symbol $\mathbb{N}$ denotes the set of natural numbers, $\{0, 1, 2, \ldots\}$. Let $\varphi_0, \varphi_1, \varphi_2, \ldots$ denote a fixed acceptable numbering [24] of all partial-recursive functions over $\mathbb{N}$. Given a set S, $\overline{S}$ denotes the complement of S and S^* denotes the set of all finite sequences in S. One defines the e-th r.e. set W_e as $dom(\varphi_e)$ and the e-th canonical finite set by choosing D_e such that $\sum_{x \in D_e} 2^x = e$. This paper fixes a one-one padding function pad with $W_{pad(e,d)} = W_e$ for all e, d. Furthermore, $\langle x, y \rangle$ denotes Cantor's pairing function, given by $\langle x, y \rangle = \frac{1}{2}(x + y)(x + y + 1) + y$. A triple $\langle x, y, z \rangle$ denotes $\langle \langle x, y \rangle, z \rangle$. The notation $\eta(x)\downarrow$ means that $\eta(x)$ is

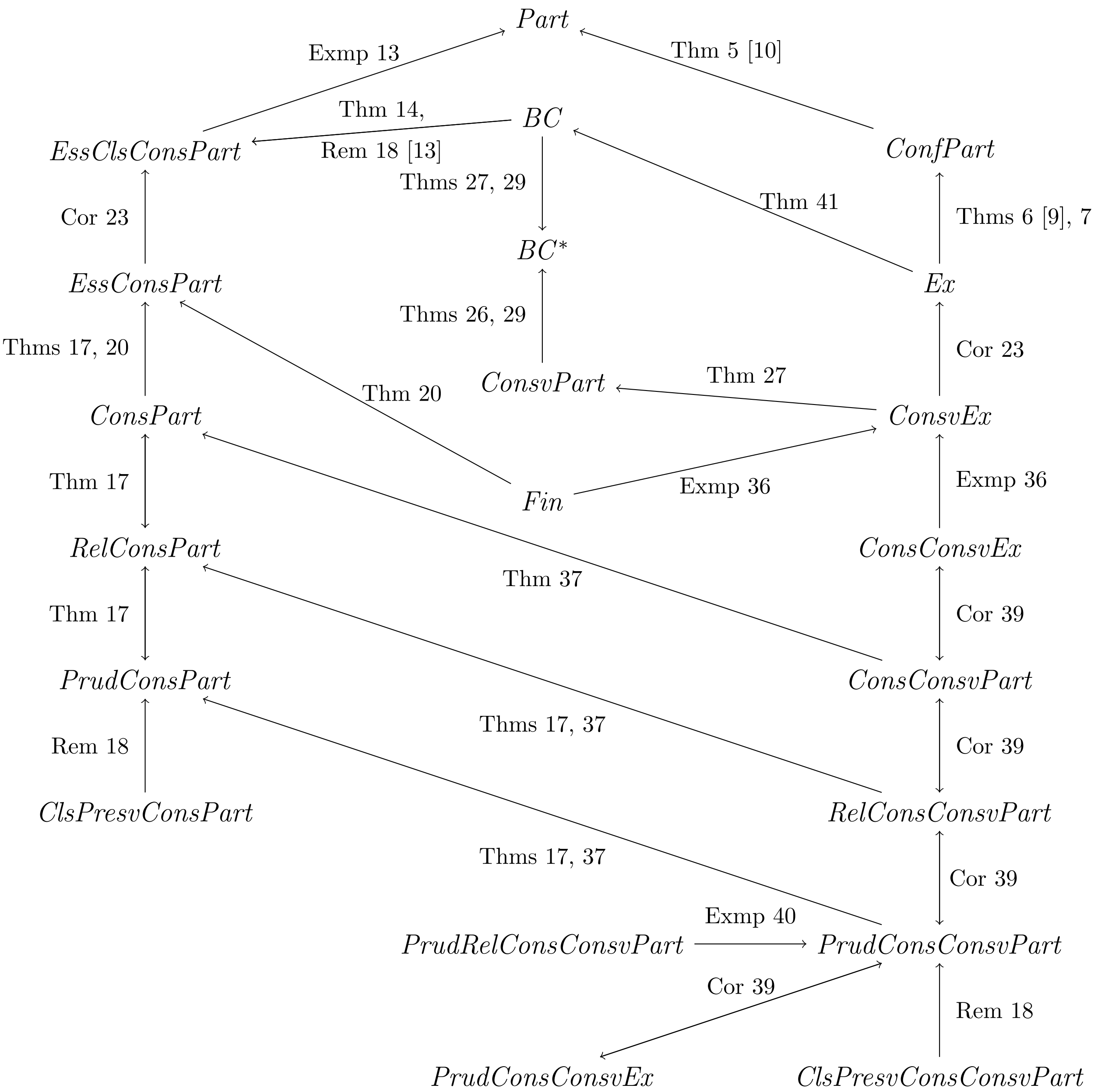

Figure 1: Learning hierarchy

defined, and $\eta(x)\uparrow$ means that $\eta(x)$ is undefined. The notation $\varphi_{e,s}(x)\downarrow$ means that $\varphi_e(x)$ is defined within s steps, that is, the computation of $\varphi_e(x)$ halts within s steps.

Turing reducibility is denoted by $\leq_T$; $A \leq_T B$ holds if A can be computed via a machine that uses B as an oracle; that is, it can give information on whether or not x belongs to B. $A \equiv_T B$ means that $A \leq_T B$ and $B \leq_T A$ both hold, and $\{A : A \equiv_T B\}$ is called the Turing degree of B. For any partial-recursive function g, $graph(g) = \{\langle x, y\rangle : g(x)\downarrow = y\}$. The symbol $\mathbb{K}$ denotes the diagonal halting problem $\{e : \varphi_e(e)\downarrow\}$. Furthermore, $\mathbb{K}_s$ is a recursive approximation to $\mathbb{K}$; without loss of generality, $\mathbb{K}_s \subseteq \{0, 1, \ldots, s\} \cap \mathbb{K}$ and the set $\{\langle x, s\rangle : x \in \mathbb{K}_s\}$ is primitive recursive.

For any two sets A and B, $A \oplus B = \{2x : x \in A\} \cup \{2y + 1 : y \in B\}$. Analogously, $A \oplus B \oplus C = \{3x : x \in A\} \cup \{3y + 1 : y \in B\} \cup \{3z + 2 : z \in C\}$. $A \subseteq^* B$ means that all but finitely many elements of A are in B.

For any $\sigma, \tau \in (\mathbb{N} \cup \{\#\})^*, \sigma \preceq \tau$ if and only if $\sigma = \tau$ or τ is an extension of σ, $\sigma \prec \tau$ if and only if σ is a proper prefix of τ, and $\sigma(n)$ denotes the element in the nth position of σ, starting from $n = 0$. The concatenation of two strings σ and τ shall be denoted by $\sigma \circ \tau$; for convenience, and whenever there is no possibility of confusion, this is occasionally denoted by $\sigma\tau$. Let $\sigma[n]$ denote the sequence $\sigma(0) \circ \sigma(1) \circ \ldots \circ \sigma(n-1)$. The length of σ is denoted by $|\sigma|$.

The learnability notions investigated in the present paper are built on the main learning paradigms from positive data – explanatory learning, behaviourally correct learning, and partial learning. Explanatory learning, or "learning in the limit", was introduced by Gold [13] to model the process of language acquisition. This model was later generalised by Bārzdiņš [3] and Case [5]; in their proposed model, known as behaviourally correct learning, the learner is required to almost always output a correct hypothesis of the input set, although it is permitted to output syntactically different hypotheses. Osherson, Stob and Weinstein [22] then relaxed the criterion of behaviourally correct learnability to partial learnability, according to which the learner must output exactly one correct index of the set infinitely often and output any other conjecture only finitely often. In addition, one can specify various constraints on the learner; the next definition imposes a restriction on the hypothesis space of the learner. As was shown by Lange and Zeugmann [19], restricting the hypothesis space of the learner to the target class reduces the power of learning in the limit. Thus, it may be natural to ask if the same constraint on the hypothesis space reduces the strengths of various partial learning criteria.

Definition 1 (Zeugmann, Lange and Kapur [26]). *A learner M is said to* class-preservingly *(ClsPresv) learn $\mathcal{C}$ if it learns $\mathcal{C}$ with respect to a hypothesis space $\{H_0, H_1, H_2, \ldots\}$, where $H_0, H_1, H_2, \ldots$ are r.e. sets, such that $\mathcal{C} = \{H_0, H_1, H_2, \ldots\}$.*

Most of the results in this paper deal with *class-comprising* learning [26], according to which a class $\mathcal{C}$ of r.e. sets must be learnt using a hypothesis space comprising $\mathcal{C}$. Throughout this paper, successful learning with respect to a class $\mathcal{C}$ will generally refer to class-comprising learning unless specified otherwise.

Let $\mathcal{C}$ be a class of r.e. sets. Throughout this paper, the mode of data presentation is that of a *text*, by which is meant an infinite sequence of natural numbers and the # symbol. Formally, a *text* T_L for some L in $\mathcal{C}$ is a map $T_L : \mathbb{N} \to \mathbb{N} \cup \{\#\}$ such that $L = range(T_L)$; here, $T_L[n]$ denotes the sequence $T_L(0) \circ T_L(1) \circ \ldots \circ T_L(n-1)$ and the range of a text T, denoted $range(T)$, is the set of numbers occurring in T. Analogously, for a finite sequence σ, $range(\sigma)$ is the set of numbers occurring in σ. A text, in other words, is a presentation of positive data from the target set. In the following definitions, M is a recursive function mapping $(\mathbb{N} \cup \{\#\})^*$ into $\mathbb{N} \cup \{?\}$; the ? symbol permits M to abstain from conjecturing at any stage. M may also be equipped with an oracle. In this case, a learner that has access to oracle A is an A-recursive function mapping $(\mathbb{N} \cup \{\#\})^*$ into $\mathbb{N} \cup \{?\}$.

Definition 2. (I) *[22] M* partially *(Part) learns $\mathcal{C}$ if, for every L in $\mathcal{C}$ and each text T_L for L, there is exactly one index e such that $M(T_L[k]) = e$ for infinitely many k; furthermore, if M outputs e infinitely often on T_L, then $L = W_e$.*

(II) *[13] M* explanatorily *(Ex) learns $\mathcal{C}$ if, for every L in $\mathcal{C}$ and each text T_L for L, there is a number n for which $L = W_{M(T_L[n])}$ and, for any $j \geq n$, $M(T_L[j]) = M(T_L[n])$.*

(III) *[5] M* behaviourally correctly *(BC) learns $\mathcal{C}$ if, for every L in $\mathcal{C}$ and each text T_L for L, there is a number n for which $L = W_{M(T_L[j])}$ whenever $j \geq n$.*

(IV) *[22] M is* prudent *if it learns the class $\{W_{M(\sigma)} : \sigma \in (\mathbb{N} \cup \{\#\})^*, M(\sigma) \neq ?\}$. In other words, M learns every set it conjectures.*

(V) *[20] M is* reliable *in case for every $L \subseteq \mathbb{N}$, every text T_L for L, and every $e \in \mathbb{N}$, if M outputs e on T_L infinitely often then $W_e = L$. In other words, M never outputs any one incorrect index infinitely often.*

Prudence, reliability or class-preservingness can be enforced with any learning criterion; such a requirement will be denoted by prefixing the given criterion with *Prud*, *Rel* or *ClsPresv*, accordingly.

As a point of departure, the following theorem establishes that the learning criterion of partial learning is quite powerful.

Theorem 3 (Osherson, Stob and Weinstein [22]). *The class of all r.e. sets is partially learnable.*

Before the proof, please note the following general remarks: First, to construct a partial learner N, it is quite often enough to define on each text segment $T[s]$ a finite set $F_s = \{e_1, \ldots, e_l\}$ of indices and specify how many times N must output each index in F_s. One can maintain a sequence I of indices such that if, on the segment $T[s]$, the current approximation I_s of I is equal to $\langle a_1, \ldots, a_k \rangle$ and $F_s = \{e_1, \ldots, e_l\}$ is nonempty, then I_{s+1} is defined to be $\langle a_1, \ldots, a_k, e_1, \ldots, e_1, \ldots, e_l, \ldots, e_l \rangle$, where each e_i is added to I_s as many times as N must output it (as specified at the current stage). On $T[s]$, if $I_{s+1} = \langle a_1, \ldots, a_{l'} \rangle$, then N determines the least i such that a_i has not already been output the number of times it occurs in $\langle a_1, \ldots, a_i \rangle$. N then outputs a_i. If no such i exists, then N outputs an index that is different from all its prior conjectures. Subsequent constructions of partial learners (with the exception of constructions in the following proof) will frequently omit mention of I and F_s, and simply specify how often the learner must output an index.

Proof. To partially learn the class of all r.e. sets, one first constructs a learner M that outputs a hypothesis e at least n times iff there is a stage $t > n$ such that for each $m < n$, m appears in the first t data items iff m is enumerated into W_e within t time steps. More precisely, M can work as follows. Set $I_0 = \emptyset$. On input $T[s+1]$, let F_s consist of all indices $e < s$ for which there is a maximum number $e_h > 0$ such that for some n with $e_h < n \leq s+1$ and for each $m < e_h$, $m \in W_{e,n}$ iff $m \in range(T[n])$. I_{s+1} is defined as an extension of I_s so that for each $e \in F_s$, e occurs e_h times in I_{s+1}. If I_{s+1} is empty, M outputs an index that is different from all its prior conjectures. If I_{s+1} is nonempty and $I_{s+1} = \langle a_1, \ldots, a_l \rangle$, then M determines the least i (if it exists) such that a_i has not already been output as many times as it occurs in $\langle a_1, \ldots, a_i \rangle$, and it outputs a_i. If no such i exists, then M outputs an index that is different from all its prior conjectures.

Let $I = \lim_{s \to \infty} I_s$. If a hypothesis is correct, then it will occur in I infinitely often and therefore be output infinitely often. If a hypothesis e is incorrect, then it will occur in I_s only finitely often: either there is some $x \in W_e$ such that x never occurs in the input and $x \in W_{e,n}$ for sufficiently large n, or there is some $y \notin W_e$ that occurs in $range(T[n])$ for all sufficiently large n. Thus N will output e only finitely often.

Note that M outputs all correct indices infinitely often. One can use a padding technique to isolate a single correct hypothesis based on the outputs of M. Let $d_1, d_2, d_3, \ldots$ be the sequence of conjectures output by M. A new learner N is defined so that for each d_i, N outputs $pad(d_i, m_i)$, where m_i is the number of indices $j < i$ such that $d_j < d_i$. Now if M outputs e infinitely often and the total number of i such that $d_i < e$ is some finite number k, then N outputs $pad(e, k)$ infinitely often and any other index only finitely often. Thus N is a partial learner of all r.e. sets. This padding technique will be employed in many subsequent proofs. ▮

3. Confident Partial Learning

Gao, Stephan, Wu and Yamamoto [9] introduced the notion of confident partial learning, by naturally generalising the constraint that the learner must, with respect to the convergence criterion considered, single out a hypothesis on every possible text for every possible, even non-r.e., set.

Definition 4 (Gao, Stephan, Wu and Yamamoto [9]). *M is said to* confidently partially *(ConfPart) learn $\mathcal{C}$ if it partially learns $\mathcal{C}$ from text and outputs on every infinite sequence (including sequences that are not texts for any member of $\mathcal{C}$) exactly one index infinitely often.*

Confidence is a proper restriction on a partial learner in the sense that the class of all r.e. sets is no longer partially learnable if the learner is required to be confident. This is witnessed even by a class that can be learned behaviourally correctly, the corresponding result for recursive functions can be carried over by considering the graphs.

Theorem 5 (Gao and Stephan [10]). *There is a class of recursive sets that is BC learnable, but not ConfPart learnable.*

Furthermore, one can show that Gold's class containing one infinite set and all its finite subsets [13] is confidently partially learnable but not behaviourally correctly learnable. Thus, while confident partial learnability is not strong enough to identify the whole class of r.e. sets, it is still more powerful than the learning model introduced by Gold [13].

Theorem 6 (Gao, Stephan, Wu and Yamamoto [9]). *There is a uniformly recursive family of sets that is ConfPart learnable, but not BC learnable.*

By contrast, every class that is explanatorily learnable is also confidently partially learnable. This holds true, even when the *Ex*-learner is allowed to converge to an index of a set that disagrees with the target set on at most one number, which is Case and Smith's criterion Ex^1 of learning with at most one anomaly [6].

Theorem 7. *Every Ex^1 learnable class of r.e. sets is confidently partially learnable.*

Proof. Let $\mathcal{C}$ be a class of r.e. sets that is Ex^1 learnable via a recursive learner M. Without loss of generality, it may be assumed that for any nonempty $\sigma \in \mathbb{N}^*$ and any y, $M(\sigma) \neq M(\sigma \circ y)$ implies that $M(\sigma \circ y \circ \tau) > M(\sigma)$ for all $\tau \in \mathbb{N}^*$. Let *pad'* be a one-one recursive function such that for all e, x, $W_{pad'(e,x,0)} = W_e - \{x\}$, $W_{pad'(e,x,1)} = W_e \cup \{x\}$, and $W_{pad'(e,0,2)} = W_{pad'(e,0,3)} = W_e$. On text $T = a_0 \circ a_1 \circ a_2 \circ \ldots$, define a new learner N as follows.

- N outputs $pad'(e, x, 0)$ n times if M outputs e n times and there exists a stage $s > n$ such that $x \in W_{e,s}$ and $x \notin \{a_0, a_1, \ldots, a_s\}$;
- N outputs $pad'(e, x, 1)$ n times if M outputs e n times and there exists a stage $s > n$ such that $x \notin W_{e,s}$ and $x \in \{a_0, a_1, \ldots, a_s\}$;
- N outputs $pad'(e, 0, 2)$ n times if M outputs e n times and there exists a stage $s > n$ such that for all $y < n$, $y \in W_{e,s}$ iff $y \in \{a_0, a_1, \ldots, a_s\}$;
- N outputs $pad'(0, 0, 3)$ n times if M makes at least n mind changes.

The assumption on M made in the second line of the proof implies that if M diverges on T, then M on T outputs each index only finitely often. Thus if M diverges on T, then N outputs $pad'(0, 0, 3)$ infinitely often and every other index only finitely often. If M converges to e and e is an incorrect index, and x is the least number such that either $x \in W_e - range(T)$ or $x \in range(T) - W_e$, then N outputs $pad'(e, x, 0)$ infinitely often if $x \in W_e$ and $x \notin range(T)$; N outputs $pad'(e, x, 1)$ infinitely often if $x \notin W_e$ and $x \in range(T)$. In each of the preceding cases, N outputs every other index only finitely often. If M converges to e and e is a correct index, then N outputs $pad'(e, 0, 2)$ infinitely often and every other index finitely often. This completes the case distinction and establishes that N is a confident partial learner of $\mathcal{C}$. ∎

The following characterisation can be brought over from function learning to set learning.

Theorem 8 (Gao and Stephan [10]). *A class $\mathcal{C}$ of r.e. sets is ConfPart learnable if and only if there is a recursive learner M such that*

- *M outputs on each text exactly one index infinitely often;*

- *if T is a text for a set L in $\mathcal{C}$, and d is the index output infinitely often by M on T, then there is some $e \leq d$ with $W_e = L$.*

A useful consequence of Theorem 8 is the closure of confidently partially learnable classes under finite unions - a property that fails to hold for either explanatorily learnable classes or behaviourally correctly learnable classes [13].

Corollary 9. *If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two classes of r.e. sets, both of which are ConfPart learnable, then their union $\mathcal{C}_1 \cup \mathcal{C}_2$ is also ConfPart learnable.*

4. Essentially Consistent Partial Learning

Consistency [3, 4, 25] is a quite natural condition which postulates that every conjecture should at least enumerate all the data seen so far. Consistency is known to be restrictive, for both explanatory and partial learning. Nonetheless, it will be shown later (Theorem 17) that consistent partial learning is far more powerful than its analogue for explanatory learning, as it allows one to identify any subclass of a uniformly recursive family.

Considering that the whole class of r.e. sets is much richer than the subfamily of recursive sets, a question that suggests itself is whether some weaker variant of consistency may allow even nonrecursive sets to be partially identified. In the present section, consistency is weakened to essentially global consistent and essentially class consistent learning, which generalise the learnability condition without making the criterion so strong that it permits to learn the class RE of all r.e. sets. Note that essentially class consistent learning is a restriction only for partial learning, while it would be automatically implied by the criteria of explanatory and behaviourally correct learning.

Definition 10 (Angluin [1]; Bārzdiņš [3]; Gao and Stephan [10]). *Let $\mathcal{C}$ be a class of r.e. sets and M be a recursive learner.*

(i) *M is said to* essentially globally consistently partially *(EssConsPart) learn $\mathcal{C}$ if it partially learns $\mathcal{C}$, and for each set L and every text T for L, $range(T[n]) \subseteq W_{M(T[n])}$ holds for cofinitely many n.*

(ii) *M is said to* essentially class consistently partially *(EssClsConsPart) learn $\mathcal{C}$ if it partially learns $\mathcal{C}$, and for each set L in $\mathcal{C}$ and every text T for L, $range(T[n]) \subseteq W_{M(T[n])}$ holds for cofinitely many n.*

(iii) *M is* consistent *(Cons) if for all $\sigma \in (\mathbb{N} \cup \{\#\})^*$, $range(\sigma) \subseteq W_{M(\sigma)}$.*

(iv) *For any text T, M is* consistent on T *if $range(T[n]) \subseteq W_{M(T[n])}$ for all $n > 0$.*

(v) *M is said to be* class consistent *(ClsCons) with respect to $\mathcal{C}$ if it is consistent on each text for every L in $\mathcal{C}$.*

(vi) *M is said to* consistently partially *(ConsPart) learn $\mathcal{C}$ if it partially learns $\mathcal{C}$ from text and is consistent.*

(VII) *M is said to* class consistently partially *(ClsConsPart) learn* $\mathcal{C}$ *if it partially learns* $\mathcal{C}$ *from text and is class consistent.*

Example 11. *The class of r.e. sets* $\mathcal{C} = \{\mathbb{K} \cup D : D \text{ is finite}\} \cup \{\mathbb{N}\}$ *is EssClsConsPart learnable relative to an oracle* A *only if* $\mathbb{K} \leq_T A$.

Proof. Let A be any oracle such that there is an A-recursive essentially class consistent partial learner M of the class $\mathcal{C}$. Then, as $\mathbb{N} \in \mathcal{C}$ and M is essentially class consistent, there must exist a $\sigma \in \mathbb{N}^*$ such that for all $\tau \in \mathbb{N}^*$, $range(\sigma \circ \tau) \subseteq W_{M(\sigma \circ \tau)}$. Fixing any such σ, one can build a text T for $\mathbb{K} \cup range(\sigma)$ as follows.

1. For all $x < |\sigma|$, $T(x) = \sigma(x)$.

2. At stage s, let a_s be the last position on which T has been defined up to the present stage. Let $b_s = \min((\mathbb{K} \cup range(\sigma)) - range(T[a_s + 1]))$ and $F_s = \{M(T[k]) : k \leq a_s + 1 \wedge M(T[k]) \neq ? \wedge W_{M(T[k])} = \mathbb{K} \cup range(\sigma)\}$. Search noneffectively for an x_s such that $x_s \in \mathbb{K} \cup range(\sigma)$ and the condition $\{M(T[a_s+1] \circ x_s), M(T[a_s+1] \circ x_s \circ b_s)\} \cap F_s = \emptyset$ holds. If such an x_s is found, set $T(a_s + 1) = x_s$ and $T(a_s + 2) = b_s$.

There must be a stage s at which the search for an x_s fails to terminate successfully. For, by the construction of T, if the stages proceed through infinitely often, then M on every text segment of T outputs either ? or an index different from all of its prior correct conjectures, contradicting the fact that it partially learns $\mathcal{C}$. Thus there is a stage s such that if $x \in \mathbb{K} \cup range(\sigma)$, then $M(T[a_s + 1] \circ x) \in F_s \vee M(T[a_s + 1] \circ x \circ b_s) \in F_s$ holds. The global consistency of M on any text extension of σ gives the condition that $x \notin \mathbb{K} \cup range(\sigma) \Rightarrow M(T[a_s + 1] \circ x) \notin F_s \wedge M(T[a_s + 1] \circ x \circ b_s) \notin F_s$. Noting that $a_s + 1$ and b_s are fixed numbers, while F_s is a fixed finite set and σ is a fixed string, one therefore has the reduction $x \in \mathbb{K} \cup range(\sigma) \Leftrightarrow \{M(T[a_s + 1] \circ x), M(T[a_s + 1] \circ x \circ b_s)\} \cap F_s \neq \emptyset$, which shows that $\mathbb{K} \leq_T A$, as required. ∎

Corollary 12. *The class RE of all r.e. sets is EssClsConsPart learnable relative to an oracle* A *iff* $\mathbb{K} \leq_T A$.

Proof. Example 11 provides a subclass of RE that is essentially class consistently partially learnable relative to an oracle A only if $\mathbb{K} \leq_T A$. Conversely, if $\mathbb{K} \leq_T A$, then there is an A-recursive globally consistent partial learner M of RE: on text $T = a_0 \circ a_1 \circ a_2 \circ \ldots$, let N be an A-recursive learner that determines at every stage s the set F_s of all indices $e \leq s$ such that $\{a_0, a_1, \ldots, a_s\} - \{\#\} \subseteq W_e$. If $F_s \neq \emptyset$ and there exists a least index e in F_s such that

- there is a stage t and a number n with $n < t \leq s$ and for all $x < n$, $x \in \{a_0, a_1, \ldots, a_t\}$ holds iff $x \in W_e$, and

- e has not already been output at least n times, then N outputs e. If no such e exists or if $F_s = \emptyset$, N conjectures $\{a_0, a_1, \ldots, a_s\} - \{\#\}$. The A-recursive globally consistent partial learner M then outputs on input σ the index $pad(N(\sigma), d)$ where d is the number of proper prefixes γ of σ for which $N(\gamma) < N(\sigma)$. M is globally consistent, as it only outputs at each stage an index consistent with the current data. Furthermore, N outputs an index infinitely often only if it is correct, and M translates the minimum index that N outputs infinitely often into a padded index. M outputs the padded index infinitely often, while every other padded index is output only finitely often. Hence M must globally consistently partially learn RE. ▮

Example 13. *The class $\mathcal{C}$ in Example 11 is not EssClsConsPart learnable with finitely many queries to any oracle.*

Proof. Assume by way of contradiction that $\mathcal{C}$ is essentially class consistently partially learnable with finitely many queries to some oracle A via a recursive learner M. As M essentially class consistently partially learns $\mathbb{N}$, there is a $\sigma \in \mathbb{N}^*$ such that for all $\tau \in \mathbb{N}^*$, $range(\sigma \circ \tau) \subseteq W_{M(\sigma\circ\tau)}$. Fix such a σ. There must be a $\tau \in \mathbb{N}^*$ so that for all $\eta \in \mathbb{N}^*$, $M(\sigma \circ \tau \circ \eta)$ is A-computable without queries to A. For, if it is assumed otherwise, then one can inductively build a text for $\mathbb{N}$ as follows: set $\sigma_0 = \sigma$, and for all k, $\sigma_{k+1} = \sigma_k \circ \min(\{x : x \notin range(\sigma_k)\}) \circ \theta_{k+1}$, where θ_{k+1} is a string for which $M(\sigma_{k+1})$ is A-computable only with at least one query to A; such a θ_{k+1} exists by hypothesis. Then $T = \lim_{k\to\infty} \sigma_k$ is a text for $\mathbb{N}$ on which M must necessarily make a query to A at infinitely many stages, contradicting the assumption that M can essentially class consistently partially learn $\mathcal{C}$ using only finitely many queries to A. Thus there is indeed a $\tau \in \mathbb{N}^*$ for which the computation of $M(\sigma \circ \tau \circ \eta)$ does not involve any queries to A for any $\eta \in \mathbb{N}^*$. This in turn implies that the class $\mathcal{C}_0 = \{\mathbb{K} \cup range(\sigma \circ \tau)\}$ is consistently partially learnable by an unrelativised recursive learner: let M' be a learner such that on any input η, M' conjectures $W_{M(\sigma\circ\tau\circ\eta)}$. The consistency of M' follows from the consistency of M on input $\sigma \circ \tau \circ \eta$ for any $\eta \in \mathbb{N}^*$. M' also partially learns $\mathcal{C}_0$ since M partially learns $\mathbb{K} \cup range(\sigma \circ \tau)$, which is a set in $\mathcal{C}$. By [11, Theorem 24], however, no nonrecursive set can be consistently partially learnt, and so $\mathcal{C}$ cannot be essentially class consistently partially learnt with finitely many queries to any oracle. ▮

The next theorem makes a connection between partial learning and learning with a strict convergence criterion.

Theorem 14. *Every BC learnable class of r.e. sets is EssClsConsPart learnable.*

Proof. Let $\mathcal{C}$ be a class of r.e. sets that is behaviourally correctly learnable via a recursive learner M. On text $T = a_0 \circ a_1 \circ a_2 \circ \ldots$, define a learner N as follows. First, let $e_0, e_1, e_2, \ldots$ be a one-one enumeration of the set $\{M(a_0 \circ \ldots \circ a_i) : i \in \mathbb{N}\}$. N outputs for each i the conjecture e_i at least n times iff there is a stage $s > n$ such that $W_{e_i,s} \cap \{0, 1, \ldots, n\} = \{a_0, a_1, \ldots, a_s\} \cap \{0, 1, \ldots, n\}$. Since M is a BC learner of $\mathcal{C}$, it outputs on a text for any $L \in \mathcal{C}$ only finitely many incorrect conjectures, and so there is a stage s after which N only outputs indices of L; furthermore, it conjectures every correct index output by M infinitely often. Let $d_0, d_1, d_2, \ldots$ be the sequence of conjectures of N on some text T'. One can define a learner N' that outputs on T' the index $pad(d_i, m_i)$ for each conjecture d_i of N on T', where $m_i = |\{k < i : d_k < d_i\}|$. By construction, if d_m is the minimum correct index among all of N's conjectures, there is a unique number k such that N' outputs $pad(d_m, k)$ infinitely often, while every other index is output only finitely often, and thus N' essentially class consistently partially learns $\mathcal{C}$. ∎

In contrast to confident partial learnability, essentially class consistent partial learning is not closed under finite unions.

Corollary 15. *Essentially class consistent partial learning is not closed under finite unions; that is, there are classes of r.e. sets $\mathcal{L}_1$ and $\mathcal{L}_2$, each of which is EssClsConsPart learnable, such that $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ is not EssClsConsPart learnable.*

Proof. The class $\mathcal{L} = \{\mathbb{K} \cup D : D \text{ is finite}\}$ is behaviourally correctly learnable, and so by Theorem 14, it is also essentially class consistently partially learnable. The class $\{\mathbb{N}\}$ is also essentially class consistently partially learnable. However, the union $\mathcal{L} \cup \{\mathbb{N}\}$ is not essentially class consistently partially learnable, as Example 11 has shown. ∎

In prior work [10], it was shown that essentially globally consistent partial learning of recursive functions is closed under finite unions. Theorem 16 establishes the analogue of this result for the case of learning r.e. sets. Corollary 15 may thus be interpreted as follows: by relaxing the criterion of global consistency to class consistency, some regularity properties of essentially globally consistent partial learning may be lost.

Theorem 16. *Essentially globally consistent partial learning is closed under finite unions; that is, if $\mathcal{L}_1$ and $\mathcal{L}_2$ are both EssConsPart learnable, then $\mathcal{L}_1 \cup \mathcal{L}_2$ is EssConsPart learnable.*

Proof. Assume that M_1 and M_2 are two *EssConsPart*-learners. Now make M_3 from M_1 as follows: If M_1 on input σ conjectures e then count the number of times M_3 has conjectured e on prefixes $\tau \prec \sigma$; let m be this number. If now for all $x < m$ it holds that $range(\sigma)(x) = W_{e,|\sigma|}(x)$ then let $M_3(\sigma) = e$ else let $M_3(\sigma)$ be an index d of range(σ) which in addition satisfies $d \geq |\sigma|$, such an index can be found by the padding lemma.

It is easy to see that M_3 is recursive. Furthermore, if M_3 outputs an index e infinitely often then W_e is equal to the set to be learnt. On the other hand, if M_1 outputs an index e infinitely often and W_e equals the set to be learnt then one can show by induction that e is output infinitely often; if M_3 outputs e at least m times and σ is a sufficiently long prefix of the text with $M_1(\sigma) = e$ then $range(\sigma)(x) = W_{e,|\sigma|}(x)$ for all $x < m$ and therefore M_3 will also output e for at least the $(m+1)$-st time. Furthermore, if $M_3(\sigma) \neq M_1(\sigma)$ then $M_3(\sigma)$ is consistent. Hence it follows that M_3 is an *EssConsPart*-learner for the class of sets learnt by M_1. One can make a similar learner M_4 out of M_2.

Now $M_5(\sigma) = \min\{M_3(\sigma), M_4(\sigma)\}$ is a further learner; as M_3 and M_4 are consistent for almost all prefixes of a given text, so is M_5. Furthermore, the least index e output on a given text infinitely often by either M_3 or M_4 is also output infinitely often by M_5. Hence M_5 outputs on every set learnt by either learner at least one index infinitely often and every infinitely often output index is correct. Following the usual padding construction [10], one can modify M_5 to a further learner M_6 which is also essentially partially consistent and if M_5 outputs at least one index infinitely often, then M_6 outputs a padded version of the least such index infinitely often. Hence M_6 is an *EssConsPart*-learner that learns every set learnt either by M_1 or by M_2. ∎

As shown in [11, Theorem 24], every consistently partially learnable class of r.e. sets is contained in a uniformly recursive family of sets. The following theorem establishes a strong converse of this result, showing that every subclass of a uniformly recursive family may even be prudently consistently partially learnt and reliably partially learnt. This provides a complete characterisation of all consistently partially learnable classes of sets.

Theorem 17. *The following statements are equivalent for a class $\mathcal{C}$ of r.e. sets:*

(I) *$\mathcal{C}$ is a subclass of a uniformly recursive family;*

(II) *$\mathcal{C}$ is ConsPart learnable;*

(III) *$\mathcal{C}$ is RelConsPart learnable;*

(IV) *$\mathcal{C}$ is PrudConsPart learnable;*

(V) *$\mathcal{C}$ is PrudConsPart learnable using a uniformly recursive hypothesis space.*

Proof. The following two chains of implications will be established: (I)⇒(V)⇒(IV)⇒(II)⇒(I) and (I)⇒(III)⇒(II). The implications (V)⇒(VI), (IV)⇒(II) and (III)⇒(II) follow directly from the definitions of the learning criteria. First it is shown that any subclass of a uniformly recursive family is reliably *ConsPart* learnable. For this purpose, fix a uniformly indexed family $\mathcal{S} = \{L_1, L_2, L_3, \ldots\}$; without loss of generality, it may be assumed that $\mathcal{S}$ is one-one and contains all the cofinite sets. Define a learner M as follows. For a text T, M on $T(0)T(1)\ldots T(n)$ finds the least pair $\langle i, j\rangle$ such that $j \leq n$, $|\{m : m < n \wedge M(T(0)T(1)\ldots T(m)) = i\}| = j$ and

$$L_i \cap \{0, 1, \ldots, j\} \subseteq \{T(0), T(1), \ldots, T(n)\} \subseteq L_i \cup \{\#\}.$$

Having identified the pair $\langle i, j\rangle$, M conjectures L_i. Now if $range(T) \neq L_i$, then either $L_i \cap \{0, 1, \ldots, j\} \not\subseteq range(T)$ for some j or $range(T) \not\subseteq L_i$. Hence there is a least j for which the pair $\langle i, j\rangle$ will never be a candidate at sufficiently large stages; this implies, in particular, that M will not conjecture L_i at least j times, and so it is a reliable learner. Furthermore, if $L_i = range(T)$, then for every j, M will identify the pair $\langle i, j\rangle$ at some stage, since it holds that for all $\langle i', j'\rangle < \langle i, j\rangle$, $\langle i', j'\rangle$ can only qualify at most once. Consequently, M conjectures L_i infinitely often, and so it reliably partially learns $\mathcal{S}$.

Next, the implication from the first statement to the last is shown. Let the class be contained in the class-comprising hypothesis space $L_0, L_1, \ldots$ which is also uniformly recursive and in addition one-one. Furthermore, the hypothesis space contains all cofinite sets (in order to always have sufficiently many hypotheses to choose from). So the learner will below conjecture L_i when needed.

Given any text T, on input $T(0)T(1)\ldots T(n)$, the learner finds the least pair $\langle i,j\rangle$ such that the L_i has been conjectured j times before on inputs $T(0)T(1)\ldots T(m)$ with $m<n$ and $j\le n$ and

$$L_i \cap \{0,1,\ldots,j\} \subseteq \{T(0),T(1),\ldots,T(n)\} \subseteq L_i \cup \{\#\}.$$

Note that no wrong set is conjectured infinitely often: if $j \in L_i - range(T)$ or $T(j) \notin L_i \cup \{\#\}$ then the pair $\langle i,j\rangle$ will never qualify and therefore L_i will be conjectured at most j times. Furthermore, if L_i is the set to be learnt then each pair $\langle i,j\rangle$ will qualify from that point onwards where L_i has been conjectured j times and where all the members of $L_i \cap \{0,1,\ldots,j\}$ have been observed in the input; as there are only finitely many smaller pairs which will be dealt with in only finitely many steps, the learner will eventually address the pair $\langle i,j\rangle$ and conjecture L_i again. It is easy to see that the learner is consistent. Furthermore, for every n there is a cofinite set such that its members below n are exactly those which appear in $\{T(0),T(1),\ldots,T(n)\}$ and therefore, for every input $T(0)T(1)\ldots T(n)$ there is a pair $\langle i,j\rangle$ which qualifies so that the learner is total. The implication from the second to the first statement has been established in prior work [11, Theorem 24], and this completes the proof. ▮

Remark 18. *Theorem 17 shows that the family of behaviourally correctly learnable classes is properly contained in the family of essentially class consistently partially learnable classes: by the equivalence of (I) and (II) in Theorem 17, the uniformly recursive class $\{\mathbb{N}\} \cup \{D :$ D is finite$\}$ is essentially class consistently partially learnable, but it is not behaviourally correctly learnable [13]. Further, class-preserving consistent partial learnability is strictly subsumed under prudent consistent partial learnability. For example, one can take the class $\mathcal{F}_0$ of all finite sets which do not contain 0. This class is uniformly recursive as well as prudently consistently partially learnable, but it is not class-preservingly consistently partially learnable since any consistent partial learner must conjecture a hypothesis containing 0 on any input with 0 in its range; such a hypothesis thus cannot be a member of $\mathcal{F}_0$. The next example further refines the distinction between these two learning criteria, exhibiting a class of r.e. sets that is prudently consistently partially learnable but not class-preservingly class consistently partially learnable.*

Example 19. *Let a class $\mathcal{C}$ contain the set of all pairs of natural numbers plus, for each x, the following set: $L_x = \{y : \forall z < y\,[\langle x,z\rangle \in L_x]$ and the x-th machine M_x outputs on the sequence $\langle x,0\rangle\,\langle x,1\rangle \ldots \langle x,y\rangle$ either ? or an index W_e such that W_e contains some pair $\langle x',y'\rangle$ with $x' \neq x\}$. Then the class $\mathcal{C}$ is PrudConsPart learnable but not ClsPresvClsConsPartlearnable.*

Proof. First, the *PrudConsPart*-learnability follows from the fact that each set L_x is either a finite set or a set of the form $\{x\} \times \mathbb{N} = \{\langle x,y\rangle : y \in \mathbb{N}\}$ and hence the class is a subset of the uniformly recursive family consisting of the set of all pairs, the sets $\{x\} \times \mathbb{N}$ and the finite sets. Furthermore, the set of all pairs is in the class and therefore it is always possible to form a consistent hypothesis.

Second, the class is uniformly r.e., as the definitions given for the L_x are uniform enumeration procedures (which might get stuck as a finite set if M_x is partial) and just one set is added to this family.

Third, the class does not have a class-preserving consistent partial learner. Assume by contradiction that M_x would be such a learner. Then there are two cases.

If L_x is infinite then M_x outputs on the text $\langle x, 0\rangle \langle x, 1\rangle \ldots$ for L_x on every input of the form $\langle x, 0\rangle \langle x, 1\rangle \ldots \langle x, y\rangle$ either ? or an index which enumerates some pair outside L_x and therefore M_x does not partially learn this set.

If L_x is finite then there is a first $\langle x, y\rangle$ not in this set and M_x outputs on input $\langle x, 0\rangle \langle x, 1\rangle \ldots \langle x, y\rangle$ some index e. Now W_e contains all the data observed by consistency and is therefore different from L_x. Furthermore, W_e does not contain any pair $\langle x', y'\rangle$ with $x' \neq y'$ as otherwise $\langle x, y\rangle$ would be enumerated into L_x. Hence the conjecture e represents a set not in the class to be learnt and the learner M_x is not using a class-preserving hypothesis space. ∎

One implication of Theorem 17 is that every consistently partially learnable class contains only recursive sets. This characterisation, however, does not extend to the notion of essentially consistent partial learnability, as the following example demonstrates.

Theorem 20. *The class $\mathcal{C} = \{\mathbb{K}\} \cup \{D : D$ is finite$\}$ is PrudEssConsPart learnable but not ClsConsPart learnable.*

Proof. The learner works on a text T as follows:

- It outputs a default index for $\mathbb{K}$ at least n times iff there have already been observed more than n elements in the text T and the first n elements observed are enumerated into $\mathbb{K}$;
- if the default index of $\mathbb{K}$ is not the current output by the previous case then the learner outputs a default index of a set just consisting of the finitely many elements observed so far.

If the learner learns a set $L \not\subseteq \mathbb{K}$ then the learner outputs the hypothesis for $\mathbb{K}$ only finitely often and outputs consistent hypotheses all other times.

If the learner learns a set $L \subseteq \mathbb{K}$ then the learner always outputs a consistent hypothesis, either $\mathbb{K}$ or the range of the data seen so far.

Furthermore, it is easy to see that the learner partially learns $\mathbb{K}$ as well as every finite set. The class cannot be *ClsConsPart* learnt, as it contains for every $\sigma \in (\mathbb{N} \cup \{\#\})^*$ some member L with $range(\sigma) \subseteq L$, and so a *ClsConsPart*-learner would already be a consistent learner; such a learner, however, cannot exist by Theorem 17 and the nonrecursiveness of $\mathbb{K}$. ∎

It was shown in earlier work [9] that the class of all cofinite sets is not confidently partially learnable. As this class is uniformly recursive, it follows from Theorem 17 that it is

PrudConsPart learnable. Thus the criterion of *PrudConsPart* learnability does not imply confident partial learnability in general.

Corollary 21. *The class of all cofinite sets is ConsPart learnable but not ConfPart learnable.*

The next theorem completes the proof that confident partial learnability and consistent partial learnability have incomparable learning strengths.

Theorem 22. *There is a ConfPart learnable class consisting of recursive sets that is not EssClsConsPart learnable.*

Proof. Let $M_0, M_1, M_2, \ldots$ be an enumeration of all partial-recursive learners. For each $\sigma \in \mathbb{N}^*$ and $i \in \mathbb{N}$, let $A_{\langle \sigma,i \rangle}$ denote the set $\{\langle \sigma, i, k \rangle : k \in \mathbb{N}\}$, and define an r.e. set $L_{\langle \sigma,i \rangle}$ in stages as follows. The construction proceeds by trying to build a text for $L_{\langle \sigma,i \rangle}$ on which M_i either never outputs any index infinitely often, or is inconsistent at infinitely many stages. $\tau_0 = \sigma$ is an initial approximation to this text; at stage $s+1$, one defines a further approximation τ_{s+1} based on the outputs of M_i on some potential extensions of τ_s. For bookkeeping, define approximations $B_0, B_1, B_2, \ldots$ to an auxiliary r.e. set B; B records the numbers that must not be added into $L_{\langle \sigma,i \rangle}$ in order to maintain the inconsistency of M_i on some earlier constructed text segment.

1. Let $L_{\langle \sigma,i \rangle,0} = range(\sigma), \tau_0 = \sigma$, and $B_0 = \emptyset$.
2. At stage $s+1$, search for either (I) the first $w \in A_{\langle \sigma,i \rangle} \cap \{w : w > \max(B_s \cup L_{\langle \sigma,i \rangle,s})\}$ such that $M_i(\tau_s \circ w) \downarrow \notin \{M_i(\gamma) : \gamma \preceq \tau_s\}$, or (II) the first pair x, y with $x \neq y$, $\{x, y\} \subseteq A_{\langle \sigma,i \rangle} \cap \{w : w > \max(B_s \cup L_{\langle \sigma,i \rangle,s})\}$, so that for some e,
   - $M_i(\tau_s \circ x) \downarrow = M_i(\tau_s \circ y) \downarrow = e$;
   - $x \in W_e \vee y \in W_e$ holds.

   In case (I), let $L_{\langle \sigma,i \rangle,s+1} = L_{\langle \sigma,i \rangle,s} \cup \{w\}$, $\tau_{s+1} = \tau_s \circ w$, and $B_{s+1} = B_s$. In case (II), let z be the first element in $\{x, y\}$ that W_e enumerates, and z' be the other member of $\{x, y\}$. Then set $L_{\langle \sigma,i \rangle,s+1} = L_{\langle \sigma,i \rangle,s} \cup \{z'\}, \tau_{s+1} = \tau_s \circ z'$, and $B_{s+1} = B_s \cup \{z\}$.

Let $L_{\langle \sigma,i \rangle} = \bigcup_{s \in \mathbb{N}} L_{\langle \sigma,i \rangle,s}$, and define $\mathcal{C}_1 = \{L_{\langle \sigma,i \rangle} : \sigma \in \mathbb{N}^* \wedge i \in \mathbb{N}\}$, $\mathcal{C} = \mathcal{C}_1 \cup \{\mathbb{N}\}$. Then $\mathcal{C}$ is confidently partially learnable. The class $\mathcal{C}_1$ is confidently partially learnable: the subclass of all $L_{\langle \sigma,i \rangle}$ which are infinite may be explanatorily learnt via a learner which, on a text T, converges to an index for $L_{\langle \sigma,i \rangle}$ if $range(T) \subseteq^* A_{\langle \sigma,i \rangle}$, and outputs a default index infinitely often otherwise; the subclass of all $L_{\langle \sigma,i \rangle}$ which are finite may also be explanatorily learnt by a learner which, on a given text segment $T[n]$, outputs a canonical index for $range(T[n])$. Therefore each of these two subclasses of $\mathcal{C}_1$ is confidently partially learnable, and so the union $\mathcal{C}_1 \cup \{\mathbb{N}\}$ is confidently partially learnable as well.

Next, assume for a contradiction that M_n essentially class consistently partially learns the class $\mathcal{C}$. Since M_n must also essentially class consistently partially learn $\mathbb{N}$, there must exist some $\sigma \in \mathbb{N}^*$ such that for all $\tau \in \mathbb{N}^*$, $range(\sigma \circ \tau) \subseteq W_{M_n(\sigma\circ\tau)}$. Fix such a σ. By the construction of $L_{\langle\sigma,n\rangle}$, there is a text T for $L_{\langle\sigma,n\rangle}$ such that on every text segment of T, M_n either outputs a conjecture different from all of its previous ones, or it outputs an index e such that for some y, $y \in W_e - L_{\langle\sigma,n\rangle}$, that is, the index is incorrect. Consequently, M_n cannot be an essentially class consistent partial learner of $\mathcal{C}$. ∎

The next corollary shows that in Theorem 14, class consistency cannot be replaced with global consistency.

Corollary 23. *There is an Ex learnable class of recursive sets that is not EssConsPart learnable.*

Proof. Let $\mathcal{C}_1' = \{L_{\langle\sigma,i\rangle} : \sigma \in \mathbb{N}^* \wedge i \in \mathbb{N} \wedge |L_{\langle\sigma,i\rangle}| = \infty\}$ and $\mathcal{C}_1'' = \{L_{\langle\sigma,i\rangle} : \sigma \in \mathbb{N}^* \wedge i \in \mathbb{N} \wedge |L_{\langle\sigma,i\rangle}| < \infty\}$ where $L_{\langle\sigma,i\rangle}$ is as defined in Theorem 22. As was argued in the proof of Theorem 22, $\mathcal{C}_1'$ is explanatorily learnable. Suppose, however, that $\mathcal{C}_1'$ were essentially globally consistently partially learnable. As $\mathcal{C}_1''$ consists of only finite sets, $\mathcal{C}_1''$ is essentially globally consistently partially learnable, and so by Theorem 16 the class $\mathcal{C} = \mathcal{C}_1' \cup \mathcal{C}_2'' \cup \{\mathbb{N}\}$ is also essentially globally partially learnable, contradicting the conclusion in the proof of Theorem 22. Hence $\mathcal{C}_1'$ is not essentially globally consistently partially learnable. ∎

5. Conservative Partial Learning

Angluin [1] introduced the notion of *conservativeness* in the model of explanatory learning, and she gave sufficient conditions for an indexed family of nonempty recursive sets to be inferable by a conservative learner [1, Theorem 5]. Subsequent studies on conservative learning in the case of uniformly r.e. classes as well as indexed families [18, 26] yielded fairly succinct characterisations of this learning criterion. In prior work [11], the notion of conservativeness was adapted to the model of partial learning; in this modified version of conservative learning, the learner is required to output exactly one correct index of the input set L infinitely often and it cannot conjecture any proper superset of L.

In particular when considered together with consistency, conservativeness turned out to rule out many irregularities although not all which can arise from Pitt's delaying trick [4, 23]. When it comes to partial learning, the combination of consistency and conservative also reduces the learning power and brings the criterion down to conservative explanatory learning, as shown below. The present section aims to shed further light on the nature of conservative partial learning alone as well as on its combination with consistency.

Definition 24 (Gao, Jain and Stephan [11]). *A recursive learner M is said to* conservatively partially *(ConsvPart) learn $\mathcal{C}$ if it partially learns $\mathcal{C}$ and outputs on each text for every L in $\mathcal{C}$ exactly one index e with $L \subseteq W_e$.*

The following example shows that confident partial learnability does not imply conservative partial learnability in general.

Example 25. $\{D : D$ *finite*$\} \cup \{\mathbb{N}\}$ *is ConfPart learnable, but not ConsvPart learnable.*

One can in fact construct an explanatorily learnable class of sets that is not conservatively partially learnable, as the following theorem demonstrates. By Theorem 7, the class given in Example 26 is also confidently partially learnable.

Theorem 26. *There is a uniformly recursive family of sets that is Ex learnable, but not ConsvPart learnable.*

Proof. Define an indexed family that contains for every e

- the set $\{e, e+1, e+2, \ldots\}$ and
- the first set of the form $\{e, e+1, \ldots, e+t\}$ found such that the e-th learner conjectures on the input $e \circ (e+1) \circ (e+2) \circ \ldots \circ (e+t)$ a set containing $e, e+1, \ldots, e+t+1$; if such a t does not exist then no finite set with minimum e is in the class.

It is easy to see that the resulting family can be made recursive and that none of the learners *ConsvPart*-learn this family. Furthermore, an explanatory learner would find in the limit the least element e in the text. In the case that a set of the form $\{e, e+1, \ldots, e+t\}$ is added in the family and the text does not contain any element larger than t then the learner converges to an index of this set, else the learner converges to the index of $\{e, e+1, \ldots\}$ which, without loss of generality, comes first in the indexed family while the index of $\{e, e+1, \ldots, e+t\}$ is the second index with least element e (if any). ∎

For completeness, the next theorem states that conservative partial learnability does not imply confident partial learnability or behaviourally correct learnability in general. Gao, Jain and Stephan [11, Example 9] have proven that the class of graphs of all recursive functions witnesses this separation.

Theorem 27 (Gao, Jain and Stephan [11]). *There is a class of infinite recursive sets that is ConsvPart learnable, but neither ConfPart learnable nor BC learnable.*

By contrast, when weakening the constraints of behaviourally correct learning by allowing finitely many anomalies in each hypothesis, every *ConsvPart* learnable class becomes learnable under the new model.

Definition 28. *A learner M BC^* learns $\mathcal{C}$ iff for every $L \in \mathcal{C}$ and each text T_L for L, there is a number n for which $|(W_{M(T_L[j])} - L) \cup (L - W_{M(T_L[j])})| < \infty$ whenever $j \geq n$.*

Theorem 29. *Every ConsvPart learnable class of r.e. sets is BC^* learnable.*

Proof. Suppose that M *ConsvPart* learns a class $\mathcal{C}$ of r.e. sets. Define a new learner N on a given text T as follows. On input $T(0)T(1)\ldots T(n)$, N outputs an index c such that

$$W_c = \begin{cases} range(T[n+1]) & \text{if } \forall k \leq n+1[range(T[n+1]) \not\subseteq W_{M(T[k])}]; \\ \bigcup_{j \leq n+1 \wedge s \in A_j} W_{M(T[j]),s} & \text{if } \exists k \leq n+1[range(T[n+1]) \subseteq W_{M(T[k])}] \\ & \wedge A_j = \{t : \forall i < j[range(T[n+1]) \subseteq W_{M(T[j]),t} \\ & \wedge range(T[n+1]) \not\subseteq W_{M(T[i]),t}]\}. \end{cases}$$

Suppose that N reads a text T for some $L \in \mathcal{C}$. By the conservativeness of M, there is a least k such that $L = W_{M(T[k])}$, and for all $m < k$, $L \not\subseteq W_{M(T[m])}$. Thus for all sufficiently large $n \geq k$, N on $T[n]$ will conjecture the union of $W_{M(T[k])}$ and a finite set, so that N is a BC^* learner of $\mathcal{C}$, as required. ∎

Remark 30. *The preceding result also implies that the class of recursive functions is BC^* learnable, a fact that was first observed by Harrington and published by Case and Smith [6]. Furthermore, one may note that any class containing an infinite set S and all its finite subsets cannot be BC^* learnt.*

6. Consistent and Conservative Partial Learning

Consistent partial learning has been studied previously mainly in the context of learning recursive functions [16] and it turned out that for the case of learning recursive functions from arbitrary texts, consistent partial learnability is equivalent to consistent explanatory learnability. The next theorem provides an analogue of this result for the case of learning r.e. sets. First, recall Angluin's [1] definition of conservative explanatory learning.

Definition 31 (Angluin [1]). *A recursive learner M is said to* conservatively explanatorily *(ConsvEx) learn $\mathcal{C}$ if it explanatorily learns $\mathcal{C}$ and, for any $\sigma, \tau \in (\mathbb{N} \cup \{\#\})^*$ such that $M(\sigma) \neq M(\sigma\tau)$, it holds that $range(\sigma\tau) \not\subseteq W_{M(\sigma)}$.*

Theorem 32. *A class $\mathcal{C}$ of r.e. sets is ConsConsvPart learnable iff $\mathcal{C}$ is PrudConsConsvEx learnable iff $\mathcal{C}$ is PrudConsConsvPart learnable.*

This result is an immediate consequence of the following two lemmas.

Lemma 33. *If a class $\mathcal{C}$ of r.e. sets is ConsConsvPart learnable, then $\mathcal{C}$ is ConsEx learnable by a learner that does not output any index for a proper superset of a given target set $L \in \mathcal{C}$ on every text for L.*

Proof. The following proof is a modification of a technique used in [16, Theorem 8]. Let M be a recursive *ConsConsvPart* learner of the given class $\mathcal{C}$. Define a partial-recursive function f by setting

$$f(\sigma) = \max(\{s : \exists \tau \in (\{0, 1, \ldots, \max(\{|\sigma|, \max(range(\sigma))\})\} \cup \{\#\})^*[|\tau| \leq |\sigma| \wedge s = \min(\{t : range(\tau) \subseteq W_{M(\tau),t}\})]\})$$

for every $\sigma \in (\mathbb{N} \cup \{\#\})^*$. By the global consistency of M, f is total. Next, denote by E_σ the set $\{M(\tau) : \tau \in (range(\sigma) \cup \{\#\})^* \wedge |\tau| \leq |\sigma|\}$. An explanatory learner N of $\mathcal{C}$ may work as follows. On input σ, N outputs the least $e \in E_\sigma$, if it exists, such that $range(\sigma) \subseteq W_{e,f(\sigma)}$; if no such e is found, N outputs a canonical index for $range(\sigma)$. Obviously, N is a consistent learner.

Fix $L \in \mathcal{C}$, and let T be any text for L. It shall be shown that there is an index d such that for almost all n, $d \in E_{T[n]}$ and $range(T[n]) \subseteq W_{d,f(T[n])}$. Assume by way of a contradiction that no such d exists. Then for every $j \in \bigcup_{i\in\mathbb{N}} E_{T[i]}$, there are infinitely many stages n such that $range(T[n]) \not\subseteq W_{j,f(T[n])}$. Now one can build a sequence T' over $\mathbb{N}$ such that

1. T' is a text for L;
2. there is an index e^* for L such that M on T' outputs e^* exactly once.

The construction of T' proceeds as follows.

- Stage 0. Fix σ such that $range(\sigma) \subseteq L$ and $W_{M(\sigma)} = L$. Denote $M(\sigma)$ by e^*. Set $T'(z) = \sigma(z)$ for all $z < |\sigma|$.
- Stage $2n+1$. Search for some $x \in L$ such that $x > \max(range(T'[2n+|\sigma|]))$ and $x \notin W_{e^*,f(T'[2n+|\sigma|]\circ x)}$. Set $T'(2n+|\sigma|) = x$.
- Stage $2n+2$. Let y be the least number such that $y \le T'(2n+|\sigma|)$ and $y \in L - range(T'[2n+|\sigma|])$. Set $T'(2n+1+|\sigma|) = y$.

Proof of 1.

It follows directly from the construction that $range(T') \subseteq L$. To show that $L \subseteq range(T')$, one first notes that σ always exists since the learner M is a *ConsConsvPart* learner of $\mathcal{C}$, and that $e^* \in \bigcup_{i\in\mathbb{N}} E_{T[i]}$. Furthermore, there is at every stage $2n+1$ an x which satisfies the required conditions of the construction that $x \in L$, $x > \max(range(T'[2n+|\sigma|]))$, and $x \notin W_{e^*,f(T'[2n+|\sigma|]\circ x)}$. For, by assumption there exist infinitely many k such that $range(T[k]) \not\subseteq W_{e^*,f(T[k])}$. One can choose $k > 2n+|\sigma|$ so large that $e^* \in E_{T[k]}$, $range(T'[2n+|\sigma|]) \subseteq range(T[k])$, $\{0,1,\ldots,\max(range(T'[2n+|\sigma|]))\}\cap L \subseteq W_{e^*,f(T[k])}$, and $range(T[k]) \not\subseteq W_{e^*,f(T[k])}$. Hence there is some number $z \in L$ with $z \in range(T[k]) - W_{e^*,f(T[k])}$ and $z > \max(range(T'[2n+|\sigma|]))$. Furthermore, as $range(T'[2n+|\sigma|]) \cup \{z\} \subseteq range(T[k])$ and $k > 2n+|\sigma|$, it follows that $f(T'[2n+|\sigma|]\circ z) \le f(T[k])$ and so $z \notin W_{e^*,f(T'[2n+|\sigma|]\circ z)}$. Moreover, if T' is defined on position $2n+|\sigma|$ at stage $2n+1$ and $T'(2n+|\sigma|) = x$, then T' must be defined on position $2n+1+|\sigma|$ at the subsequent stage, since x satisfies the requirements that $x \le T'(2n+|\sigma|)$ and $x \in L - range(T'[2n+|\sigma|])$. This shows that $T'(i)$ is defined for all i. Assume now that there is a least number $w \in L - range(T')$. As was argued earlier, there is at every stage $2n+1$ some $x \in L$ such that $x > \max(range(T'[2n+|\sigma|]))$ and $T'(2n+|\sigma|) = x$. Thus at some stage $2k+2$ large enough, $T'(2k+|\sigma|) > w$ holds, and so $T'(2k+1+|\sigma|) = w$ by construction. Therefore T' is a complete text for L.

Proof of 2.

For every n, since $T'(2n+|\sigma|) \notin W_{e^*,f(T'[2n+1+|\sigma|])}$ and $T'(2n+1+|\sigma|) \le T'(2n+|\sigma|)$, it must hold that $\{M(T'[2n+1+|\sigma|]), M(T'[2n+2+|\sigma|])\} \cap \{e^*\} = \emptyset$. Consequently, M outputs e^*, an index for L, exactly once on T'.

Property 2 contradicts the assumption that M is a *ConsConsvPart* learner of $\mathcal{C}$. This contradiction implies that there is some $d \in \bigcup_{i\in\mathbb{N}} E_{T[i]}$ such that for almost all n, $d \in E_{T[n]}$ and $range(T[n]) \subseteq W_{d,f(T[n])}$. Thus N on T converges to the least such d; let this number be

d'. As $L = range(T) \subseteq W_{d'}$ by the choice of d', and $L \not\subset W_{d'}$ because M is a *ConsConsvPart* learner and so cannot output a proper superset of L on any string τ with $range(\tau) \subseteq L$, one may conclude that $L = W_{d'}$. Thus N is an explanatory learner for $\mathcal{C}$ which never conjectures a proper superset of any given target set $L \in \mathcal{C}$ on every text for L. ∎

Lemma 34. *If a class $\mathcal{C}$ of r.e. sets is ConsEx learnable by a learner that does not output any index for a proper superset of a given target set $L \in \mathcal{C}$ on every text for L, then $\mathcal{C}$ is PrudConsConsvEx learnable.*

Proof. Let M be a consistent explanatory learner of $\mathcal{C}$ that never outputs any index for a proper superset of a given target set $L \in \mathcal{C}$ on any text for L. First, it is shown how to construct a consistent and conservative learner N of $\mathcal{C}$. Consider a recursive function f such that $W_{f(\sigma)} = \bigcup_{s \in A_\sigma} W_{M(\sigma),s}$, where $A_\sigma = \{t : \forall \tau \in (W_{M(\sigma),t} \cup \{\#\})^*[|\tau| > t \vee M(\sigma) = M(\sigma \circ \tau)]\}$; the programme for $f(\sigma)$ enumerates $W_{M(\sigma)}$ until it identifies a string $\tau \in (W_{M(\sigma)} \cup \{\#\})^*$ such that $M(\sigma) \neq M(\sigma \circ \tau)$, whereupon the enumeration halts. Without loss of generality, assume that $f(\tau) < f(\tau')$ if $|\tau| < |\tau'|$. Denote by $B_{\sigma,s}$ the approximation $\bigcup_{t \leq s \wedge t \in A_\sigma} W_{M(\sigma),t}$. One defines N as follows.

- Stage 0. Set $\sigma_0 = T(0)$ and $N(T(0)) = f(\sigma_0)$.
- Stage $s+1$; N reads the current input $T[s+2]$. Let x_{s+1} be the least number, if such a number exists, for which $x_{s+1} \in range(T[s+2]) - range(\sigma_s)$; if no such number exists, let $x_{s+1} = \#$. N searches for some $\theta \in (range(T[s+2]) \cup \{\#\})^*$ such that $|\theta| \leq s+1$, $range(\sigma_s \circ \theta) \not\subseteq B_{\sigma_s,|\theta|}$, and $M(\sigma_s \circ \theta) \neq M(\sigma_s)$. If N finds θ, set $\sigma_{s+1} = \sigma_s \circ \theta \circ x_{s+1}$ and $N(T[s+2])$ equal to a canonical index for $range(T[s+2])$. If no such θ exists, set $\sigma_{s+1} = \sigma_s$. For all $\theta' \in (range(T[s+2]) \cup \{\#\})^*$ such that $|\theta'| \leq s+1$ and $M(\sigma_s \circ \theta') = M(\sigma_s)$, determine whether $range(\sigma_s \circ \theta') \subseteq W_{f(\sigma_s)}$; this condition can be checked recursively as M is consistent. Then $range(\sigma_s \circ \theta') \subseteq W_{f(\sigma_s)}$ holds for all such θ' iff $range(T[s+2]) \subseteq W_{f(\sigma_s)}$. If $range(T[s+2]) \subseteq W_{f(\sigma_s)}$ and $range(T[s+1]) \subset range(T[s+2])$, or if $range(T[s+2]) \subseteq W_{f(\sigma_s)}$ and $N(T[s+1]) = f(\sigma_s)$, set $N(T[s+2]) = f(\sigma_s)$. Otherwise, set $N(T[s+2])$ equal to a canonical index for $range(T[s+2])$.

The consistency of N follows directly from the construction. N is also syntactically conservative due to the following reasons.

- If $N(T[s+1]) = f(\sigma_s)$ and $N(T[s+2])$ is a canonical index for $range(T[s+2])$, then $range(T[s+2]) \not\subseteq W_{f(\sigma_s)}$.
- If $N(T[s+1])$ is a canonical index for $range(T[s+1])$ and $N(T[s+2]) = f(\sigma_s)$, then $range(T[s+1]) \subset range(T[s+2])$.
- If $N(T[s+1]) = f(\sigma_s)$ and $N(T[s+2]) = f(\sigma_{s+1})$, then $\sigma_s = \sigma_{s+1}$.

Now it is shown that if T is a text for some $L \in \mathcal{C}$, then N explanatorily learns L on T. Suppose N reads a text T for some L in $\mathcal{C}$. If $\sigma_n \prec \sigma_{n+1}$ holds for infinitely many n, then $\lim_{k\to\infty} \sigma_k$ is a text for L on which M makes infinitely many syntactic mind changes, contradicting the fact that it explanatorily learns L. Hence there is a least l such that $\sigma_k = \sigma_l$ for all $k \geq l$. Firstly, suppose there is some $\eta \in (L \cup \{\#\})^*$ such that $M(\sigma_l\eta) = M(\sigma_l)$ and $\sigma_l\eta$ is a stabilising sequence for M on L, that is, for all $\alpha \in (L \cup \{\#\})^*$, it holds that $M(\sigma_l\eta\alpha) = M(\sigma_l\eta)$. Then, since M explanatorily learns L, one has $W_{M(\sigma_l)} = L$, which also implies, by the choice of l, that $W_{f(\sigma_l)} = W_{M(\sigma_l)}$. Thus N converges on T to either $f(\sigma_l)$, a correct index for L, or, if L is finite, to a canonical index for L. Secondly, suppose that for all $\eta \in (L \cup \{\#\})^*$, there is some $\rho \in (L \cup \{\#\})^*$ for which $M(\sigma_l) \neq M(\sigma_l\eta\rho)$. Consequently, it holds by construction that for all $\eta \in (L \cup \{\#\})^*$, $range(\eta) \subseteq W_{f(\sigma_l)}$, so that $L \subseteq W_{f(\sigma_l)}$. However, since $W_{f(\sigma_l)}$ cannot be a proper superset of L, it follows that $L = W_{f(\sigma_l)}$. Hence N again converges syntactically on T to either the correct index $f(\sigma_l)$ or, if L is finite, to a canonical index for L.

By simulating N on any input text, one can construct a prudent, consistent, as well as conservative learner P of $\mathcal{C}$. First, for any set L, define the canonical text T_L of L as follows: if $x \in L$, then $T_L(x) = x$; else $T_L(x) = \#$.

Given σ, let T_σ be the canonical text for $range(\sigma)$. Now $P(\sigma)$ determines the least prefix η of T_σ such that

$$W_{N(\eta)} \supseteq range(\sigma) \text{ and } W_{N(\eta)} \cap \{0, 1, \ldots, |\eta| - 1\} = range(\eta).$$

If such an η is found then (a) $P(\sigma) = N(\eta)$ else (b) $P(\sigma)$ outputs a canonical index for $range(\sigma)$.

Due to the consistency and conservativeness of N, P is a recursive learner. Further, P is consistent by definition. Now it is argued that P is conservative. Assume that $P(\sigma\tau) \neq P(\sigma)$. By construction, if $range(\sigma\tau) = range(\sigma)$, then $P(\sigma\tau) = P(\sigma)$. So it must hold that $range(\sigma\tau) \supset range(\sigma)$. Thus if case (b) holds on σ, then P's mind change from σ to $\sigma\tau$ is conservative. If case (a) holds on σ and $W_{M(\eta)}$ contains $range(\sigma\tau)$, then every element of $range(\sigma\tau) \cap \{0, 1, \ldots, |\eta| - 1\}$ is already in $range(\eta)$ and so $T_{\sigma\tau}$ extends η. As η is the shortest prefix of T_σ such that $W_{N(\eta)}$ contains $range(\sigma)$, it is also the shortest prefix of $T_{\sigma\tau}$ such that $W_{N(\eta)}$ contains $range(\sigma\tau)$. Thus $N(\sigma\tau) = N(\sigma)$, a contradiction. Therefore P is conservative.

Now assume that N explanatorily learns L from T_L. Then there is a shortest prefix η of T_L such that $W_{N(\eta)} = L$. By conservativeness, $range(\eta) \not\subseteq W_{N(\gamma)}$ for all proper prefixes γ of η. If L is finite and $range(\sigma) = L$ then $P(\sigma)$ is defined by case (a) and P explanatorily learns L. If L is infinite there is a first $\eta \preceq T_L$ on which N conjectures L and whenever $range(\eta) \subseteq range(\sigma)$ for a prefix σ of a text for L, $P(\sigma)$ will be $N(\eta)$. Again, P explanatorily learns L.

As $P(\sigma) = P(\sigma')$ holds for any prefixes σ, σ' such that $range(\sigma) = range(\sigma')$, P explanatorily learns every set for which it outputs a conjecture by case (b). Furthermore, assume now that P outputs a conjecture by case (a) for a set $L = W_{N(\eta)}$ and the corresponding η in the definition. Then $W_{N(\eta)} \cap \{0, 1, \ldots, |\eta| - 1\} = range(\eta)$ and N learns L on the canonical text T_L for L due to conservativness. Again, P explanatorily learns the set conjectured. Thus P is prudent. This establishes that P is a *PrudConsConsvEx* learner of $\mathcal{C}$. ▮

Note that Lemma 34 also holds without consistency requirements; the proof is analogous to that of Lemma 34, in fact a bit simpler.

Observation 35. *If a class $\mathcal{C}$ of r.e. sets is Ex learnable by a learner that does not output any index for a proper superset of a given target set $L \in \mathcal{C}$ on every text for L, then $\mathcal{C}$ is PrudConsvEx learnable.*

Proof. M, f, A_σ, and $B_{\sigma,s}$ are defined as in the proof of Lemma 34.

- Stage 0. Set $\sigma_0 = T(0)$ and $N(T(0)) = f(\sigma_0)$.
- Stage $s+1$; N reads the current input $T[s+2]$. Let x_{s+1} be the least number, if such a number exists, for which $x_{s+1} \in range(T[s+2]) - range(\sigma_s)$; if no such number exists, let $x_{s+1} = \#$. N searches for some $\theta \in (range(T[s+2]) \cup \{\#\})^*$ such that $|\theta| \leq s+1$, $range(\sigma_s \circ \theta) \not\subseteq B_{\sigma_s,|\theta|}$, and $M(\sigma_s \circ \theta) \neq M(\sigma_s)$. If N finds θ, set $\sigma_{s+1} = \sigma_s \circ \theta \circ x_{s+1}$ and $N(T[s+2]) = f(\sigma_{s+1})$. If no such θ exists, set $\sigma_{s+1} = \sigma_s$ and $N(T[s+2]) = f(\sigma_s)$.

Now the following properties of N are established.

1. N is syntactically conservative.
2. If T is a text for some $L \in \mathcal{C}$, then N explanatorily learns L on T.

Proof of 1.
Suppose that $N(T[s+2]) \neq N(T[s+1])$ for some s. Then at stage $s+1$, there is some $\theta \in (range(T[s+2]) \cup \{\#\})^*$ such that $|\theta| \leq s+1$, $range(\sigma_s \circ \theta) \not\subseteq B_{\sigma_s,|\theta|}$, and $M(\sigma_s \circ \theta) \neq M(\sigma_s)$. The syntactic conservativeness of N will follow if one can show that $range(\sigma_s \circ \theta) \not\subseteq W_{f(\sigma_s)}$. Assume by way of a contradiction that there is a least number t such that $range(\sigma_s \circ \theta) \subseteq B_{\sigma_s,t+1}$. By the choice of θ, $t+1 > |\theta|$ must hold. One notes also that $B_{\sigma_s,t+1} \subseteq W_{M(\sigma_s),t+1}$, whence, $range(\sigma_s \circ \theta) \subseteq W_{M(\sigma_s),t+1}$, and since $M(\sigma_s \circ \theta) \neq M(\sigma_s)$ and $|\theta| < t+1$, it follows that $t' \notin A_{\sigma_s}$ for all $t' \geq t+1$. Therefore $W_{f(\sigma_s)} = B_{\sigma_s,t}$, implying by the choice of t that $range(\sigma_s \circ \theta) \not\subseteq W_{f(\sigma_s)}$, contrary to assumption.

Proof of 2.
Suppose N reads a text T for some L in $\mathcal{C}$. There are two possible cases: (I) N makes infinitely many syntactic mind changes on T and (II) there is a least l such that N converges to $f(\sigma_l)$ on T.

Case (I). If this case holds, then $\lim_{s\to\infty} \sigma_s$ is a text T' for L, since the choice of x_{s+1} at every stage $s+1$ ensures that every number that appears in T will eventually be added into $range(T')$. Furthermore, by construction there are infinitely many stages $s+1$ at which $M(\sigma_s \circ \theta) \neq M(\sigma_s)$ for some θ with $\sigma_s \circ \theta \preceq \sigma_{s+1}$; in particular, M fails to converge to an index on T'. This contradicts the fact that M must explanatorily learn L on every text for L.

Case (II). If there is a least number l such that N converges to $f(\sigma_l)$ on T, then for all $\tau \in (L \cup \{\#\})^*$, either $range(\sigma_l \circ \tau) \subseteq B_{\sigma_l,|\tau|}$ or $M(\sigma_l \circ \tau) = M(\sigma_l)$ must hold. Assume for the sake of a contradiction that $M(\sigma_l)$ is not an index for L. As M will never conjecture a proper superset of L on any sequence in $(L \cup \{\#\})^*$, there is some $z \in L - W_{M(\sigma_l)}$. Fixing such a z gives that for all $\tau \in (L \cup \{\#\})^*$, $range(\sigma_l \circ z \circ \tau) \not\subseteq W_{M(\sigma_l)}$, and by virtue of the fact that $W_{f(\sigma_l)} \subseteq W_{M(\sigma_l)}$, one has that $range(\sigma_l \circ z \circ \tau) \not\subseteq B_{\sigma_l,|\tau|+1}$, and so $M(\sigma_l \circ z \circ \tau) = M(\sigma_l)$ for all $\tau \in (L \cup \{\#\})^*$. Consequently, there is a text extension of $\sigma_l \circ z$ for L on which M converges to $M(\sigma_l)$, an incorrect index; this contradicts the assumption that M is an explanatory learner of $\mathcal{C}$. Hence $M(\sigma_l)$ must indeed be an index for L. It remains to show that $W_{f(\sigma_l)} = W_{M(\sigma_l)}$. First, $W_{f(\sigma_l)} \subseteq W_{M(\sigma_l)}$ holds by the definition of f. If $W_{f(\sigma_l)} \subset W_{M(\sigma_l)}$, then there is some $\eta \in (W_{M(\sigma_l)} \cup \{\#\})^*$ such that $M(\sigma_l) \neq M(\sigma_l \circ \eta)$. Owing to the definition of f, $range(\sigma_l \circ \eta) \not\subseteq B_{\sigma_l,|\eta|}$; otherwise, if t'' is the least number with $range(\sigma_l \circ \eta) \subseteq B_{\sigma_l,t''+1}$, then $t''' \notin A_{\sigma_l}$ whenever $t''' \geq t''+1$, that is, $W_{f(\sigma_l)} = B_{\sigma_l,t''}$. Furthermore, $range(\sigma_l \circ \eta) \subseteq L$, implying that there is a sufficiently large stage $l'+1 > \max(\{l, |\eta|\})$ at which $\eta \in (range(T[l'+2]) \cup \{\#\})^*$, $range(\sigma_l \circ \eta) \not\subseteq B_{\sigma_l,|\eta|}$, and $M(\sigma_l \circ \eta) \neq M(\sigma_l)$. As a consequence, $f(\sigma_{l'+1}) \neq f(\sigma_l)$ and so $N(T[l'+2]) \neq f(\sigma_l)$, contrary to the assumption that l is the least number such that N converges syntactically to $f(\sigma_l)$ on T.

Finally, using the fact that every *ConsvEx* learnable class is also *PrudConsvEx* learnable [11, Theorem 10], one can construct a prudent and conservative explanatory learner of $\mathcal{C}$. ∎

The next example shows that the equivalence of consistent conservative partial learning and consistent conservative explanatory learning will no longer hold if consistency is removed from either side of the relation.

Example 36. *The class $\{\mathbb{K}\}$ is finitely learnable but not ConsConsvPart learnable. The class $\mathcal{F}$ of all finite sets is PrudConsConsvPart learnable but not finitely learnable.*

Proof. A class $\mathcal{C}$ of r.e. sets is *ConsPart* learnable only if every set in $\mathcal{C}$ is recursive; hence $\{\mathbb{K}\}$ is not *ConsPart* learnable. $\mathcal{F}$ is *PrudConsConsvPart* learnable via a learner M which outputs a canonical index for $range(\sigma)$ on input σ. On the other hand, if there is a finite learner N of $\mathcal{F}$ which outputs e on some string $\#^k$ of pause symbols, where $W_e = \emptyset$, then N fails to learn $\{0\}$ on the text $\#^k \circ 0^\infty$. Therefore $\mathcal{F}$ is not finitely learnable. ∎

The next result shows that a target class of r.e. sets can be consistently partially identifiable as well as conservatively partially identifiable, but not consistently conservatively partially identifiable, that is, with both constraints imposed at once.

Theorem 37. *There is a class of r.e. sets that is ConsPart learnable and ConsvPart learnable but not ConsConsvPart learnable.*

Proof. Consider the class $\mathcal{C} = \{e\} \oplus \{x : x \leq |W_e|\}$. $\mathcal{C}$ is contained in the uniformly recursive family $\{\{e\} \oplus \{0, 1, \ldots, x\} : e, x \in \mathbb{N}\} \cup \{\{e\} \oplus \mathbb{N} : e \in \mathbb{N}\}$, so that by Theorem 17, $\mathcal{C}$ is consistently partially learnable. A finite learner of $\mathcal{C}$, which is also a *ConsvPart* learner, works as follows: The learner waits and abstains from preliminary conjectures until the first even number $2e$ appears in the input and from then onwards it conjectures $\{e\} \oplus \{x : x \leq |W_e|\}$ without further mind change.

If $\mathcal{C}$ would have a *ConsConsvEx* learner, then one could use the halting problem to find out in the limit that either W_e is finite (this is r.e. in $\mathbb{K}$) or that $\{e\} \oplus \mathbb{N}$ has a stabilising sequence σ (what is the case if W_e is infinite); in the second case, the hypothesis on σ must be $\{e\} \oplus \mathbb{N}$ or a superset thereof by consistency of the learner. Thus one would take this σ and check whether $|W_e| \leq x$ for the largest x with $2x + 1 \in range(\sigma)$; now W_e is finite iff $|W_e| \leq x$ and this is a $\mathbb{K}$-recursive condition, contradicting the fact that $\{e : W_e \text{ is infinite}\}$ cannot be decided relative to $\mathbb{K}$. Thus by Theorem 32, $\mathcal{C}$ is not *ConsConsvPart* learnable. ∎

While reliable explanatory learning is so restrictive that only finite sets can be reliably identified in the limit [21], one may suspect that reliability limits partial learning less severely. It turns out that consistent and conservative limit learners can be made reliable when weakening the limit learning constraint to partial learning.

Theorem 38. *Every ConsConsvEx learnable class of r.e. sets is RelConsConsvPart learnable.*

Proof. Suppose that M is a *ConsConsvEx* learner of a class $\mathcal{C}$ of r.e. sets. Define a new learner N on a given text T as follows. On input $T[n]$, N determines whether or not $W_{M(T[n]),k} = range(T[n])$ for the least $k \geq n$ such that $range(T[n]) \subseteq W_{M(T[n]),k}$; if so, N outputs $M(T[n])$. Otherwise, if $n = 1$ or $T(n-1) = \#$, N outputs a canonical index for $range(T[n])$; if $n > 1$ and $T(n-1) \neq \#$, N determines whether or not

(a) $T(n-1) \in W_{M(T[n-1]),k'} - range(T[n-1])$ for the least $k' \geq n-1$ such that $range(T[n-1]) \subseteq W_{M(T[n-1]),k'}$,

(b) for all $y < T(n-1)$, $W_{M(T[n]),k} \cap \{0, 1, \ldots, y\} = range(T[n]) \cap \{0, 1, \ldots, y\}$,

(c) $M(T[n-1]) = M(T[n])$.

If (a),(b) and (c) are all satisfied, N outputs $M(T[n])$; otherwise, N outputs a canonical index for $range(T[n])$.

The consistency of N follows directly by construction. Suppose N is fed with a text T for some target set $L \in \mathcal{C}$. Suppose that l_0 is the least number such that $W_{M(T[l_0'])} = L$ and $M(T[l_0']) = M(T[l_0])$ for all $l_0' \geq l_0$. As M never outputs indices of proper supersets of L on T, and for every n, $N(T[n])$ is equal to either $M(T[n])$ or a canonical index for $range(T[n])$, N never outputs indices of proper supersets of L on T. If L is finite, then, for the least $l' \geq l_0$ such that $L = range(T[l'])$, $N(T[l'']) = M(T[l_0])$ for all $l'' \geq l'$; for all $l''' < l'$, $N(T[l'''])$ is equal to either $M(T[l'''])$ or to a canonical index for some finite set $range(\tau)$ with $range(\tau) \subset L$. Hence N is conservative in the partial sense. If L is infinite, then, since N either outputs a conjecture of M on T, or conjectures a finite set, N is also conservative (in the partial sense) on T.

Now if there are infinitely many n such that $W_{M(T[n]),k} = range(T[n])$ for the least $k \geq n$ such that $range(T[n]) \subseteq W_{M(T[n]),k}$, then $N(T[p]) = M(T[l_0])$ holds for infinitely many p, so that N outputs a single correct hypothesis for L infinitely often on T. On the other hand, suppose that $range(T[n]) \subset W_{M(T[n]),k}$ holds for almost all n, where, for each n, k is the least number with $k \geq n$ and $range(T[n]) \subseteq W_{M(T[n]),k}$. For each such $n \geq l_0$, there is a least $x \in W_{M(T[n]),k} - range(T[n])$, and also a least $n_0 > n$ such that $T(n_0) = x$. Thus $N(T[n_0 + 1]) = M(T[n_0 + 1])$, implying that $N(T[n']) = M(T[l_0])$ holds for infinitely many n'. Thus N is a *ConsConsvPart* learner of L.

It remains to show that N is reliable. If T' is a text for some set L' such that M on T' outputs some index d infinitely often, then the consistency of M gives that $L \subseteq W_d$. If there is a least number $y \in W_d - L$, then $y \neq T'(n)$ for all n, and therefore $N(T'[n']) \neq d$ whenever n' is sufficiently large. Since N outputs on $T'[n']$ either $M(T'[n'])$ or a canonical index for $range(T'[n'])$, it follows that N never outputs any incorrect hypothesis infinitely often. Hence N is indeed a reliable partial learner. ▮

Corollary 39. *The following statements are equivalent for a class $\mathcal{C}$ of r.e. sets.*

(i) $\mathcal{C}$ *is ConsConsvPart learnable.*

(ii) $\mathcal{C}$ *is ConsConsvEx learnable.*

(iii) $\mathcal{C}$ *is PrudConsConsvPart learnable.*

(iv) $\mathcal{C}$ *is PrudConsConsvEx learnable.*

(v) $\mathcal{C}$ *is RelConsConsvPart learnable.*

Example 40. *The class $\{\mathbb{N}\}$ is PrudConsConsvPart learnable but not PrudRelConsConsvPart learnable.*

Proof. Obviously a learner always outputtting a fixed index for $\mathbb{N}$ is a *PrudConsConsvPart* learner for the class $\{\mathbb{N}\}$. Now assume by way of contradiction that M is a *PrudRelConsConsvPart* learner for this class. There is a locking sequence σ for $\mathbb{N}$. Due to reliability, M can only finitely often output the hypothesis $M(\sigma)$ on the text $\sigma\#\#\ldots$; due to consistency, it must on some input $\sigma\#^t$ conjecture an index different from $M(\sigma)$ for a set L with $range(\sigma) \subseteq L \subseteq \mathbb{N}$. If $L \subset \mathbb{N}$ then the learner is either not prudent or not conservative. If $L = \mathbb{N}$ then one can make a text for $\mathbb{N}$ starting with $\sigma\#^t$ on which M outputs two different indices for $\mathbb{N}$ what contradicts the constraint that a conservative learner outputs only one index for the target language. ▮

The next theorem shows that prudence in partial learning is not equivalent to class-preservingness. Further, in combination with Theorem 26, it shows that conservative partial learnability and explanatory learnability have incomparable learning strengths.

Theorem 41. *There exists a uniformly recursive class of sets that is PrudConsvBC learnable as well as EssConsPrudConsvPart learnable but neither ConsvPart learnable with respect to a class-preserving hypothesis space nor Ex learnable.*

Proof. Let X and Y be two r.e. sets with $Y \subset X$ and $X - Y$ not being r.e. and $0 \notin X$. The class $\mathcal{C}$ contains the following sets:

1. The empty set $\emptyset$ and the full set $\{\langle x, y, z\rangle : x, y, z \in \mathbb{N}\}$;
2. For all x and all $y \notin X$ the set $A_{x,y} = \{\langle x, y, z\rangle : z \in \mathbb{N}\}$;
3. For all x and all $y \notin X$ the set $B_{x,y} = \{\langle x, y, z\rangle : \exists u \geq z\, [u = 0 \vee W_{x,u} \neq W_{x,z-1}]\}$;
4. For all x, z and all distinct $y, y' \in Y$ the set $C_{x,y,y',z} = \{\langle x, y, 0\rangle, \langle x, y', 0\rangle, \langle x, y, 1\rangle, \langle x, y', 1\rangle, \ldots, \langle x, y, z\rangle, \langle x, y', z\rangle\}$.

Note that the superclass $\mathcal{D}$ of the empty set, the full set and all the sets $A_{x,y}, B_{x,y}, C_{x,y,y',z}$ with $x, y, y', z \in \mathbb{N}$ is uniformly recursive; the main difference between $\mathcal{C}$ and $\mathcal{D}$ is that $\mathcal{D}$ has no restrictions on the parameter y considered in the various entries. To see that $\mathcal{D}$ has a uniformly recursive family, one starts with a uniformly recursive enumeration of all sets $\emptyset$, $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$, $A_{x,y}$ and $C_{x,y,y',z}$; then one can for every stage x, y, s add a set uniformly recursive in x, y, s which consists of the elements $\langle x, y, 0\rangle, \langle x, y, 1\rangle, \ldots, \langle x, y, z\rangle$ currently enumerated into $B_{x,y}$ and one defines the set on all other triples as 0 until eventually some further element goes into $B_{x,y}$; in the case that this happens, one makes the set equal to $C_{x,y,y',z}$ for some sufficiently large y' where it had not yet been fixed before whether any $\langle x, y', u\rangle$ is in the set or not. Note that the infinite sets $B_{x,y}$ are also in this enumeration, as they are equal to $A_{x,y}$. This gives that $\mathcal{D}$ can be represented by a uniformly recursive family and thus $\mathcal{D}$ is *PrudConsPart*-learnable.

An *EssConsPrudConsvPart*-learner for the class does the following: If no data has been observed so far then the conjecture is an index for the empty set. If there are x, y, z such that $\emptyset \neq range(\sigma) \subseteq \{\langle x, y, 0\rangle, \langle x, y, 1\rangle, \ldots, \langle x, y, z\rangle\}$ where z is taken as small as possible then let $e(x, y)$ be the default index for $B_{x,y}$ and let σ be the input. Let $\langle x, y, z\rangle$ been the largest datum in $range(\sigma)$ and let u be the largest element enumerated into $B_{x,y}$ with $\max\{z, |\sigma|\}$ steps; note that $u \leq \max\{z, |\sigma|\}$. If $e(x, y)$ has been conjectured so far less than u times or if $z \leq u$ then conjecture $e(x, y)$ else conjecture $f(x, y, u)$ where

$$W_{f(x,y,u)} = \{\langle x, y, s\rangle : W_{x,s} \subseteq W_{x,u} \vee \exists u' \geq \max\{s, 1\}\, [W_{x,u'} \neq W_{x,u'-1}]\}.$$

Note that $W_{f(x,y,u)} = B_{x,y}$ in the case that $B_{x,y}$ contains an element larger than $\langle x, y, u\rangle$ and $W_{f(x,y,u)} = A_{x,y}$ in the case that $B_{x,y}$ contains only elements less or equal to $\langle x, y, u\rangle$. If there are exactly one x and two y, y' such that $range(\sigma)$ contains triples $\langle x, y, u\rangle$ and $\langle x, y', u'\rangle$ for some u, u' then the learner lets z be the maximum of all these u, u' and conjectures a default index for $C_{x,y,y',z}$. In the remaining cases, only the full set contains $range(\sigma)$ and the learner conjectures a default index for the full set.

For a verification of the learner note that the learner is consistent in all cases except for the case when it conjectures $e(x, y)$. But in that case, this only happens when the data observed is a subset of $A_{x,y}$; furthermore, the problem can only arise if the data observed is an infinite subset of $A_{x,y}$ while $B_{x,y}$ is finite. But then the learner will eventually converge to an index of the form $f(x, y, u)$ where u is the maximum value with $\langle x, y, u\rangle \in B_{x,y}$ and therefore the hypothesis will from then onwards be a hypothesis for $A_{x,y}$ which is consistent with all data observed. Furthermore, assume now by way of contradiction that the learner overgeneralises on some σ when learning a set L in the class. This can only happen when $range(\sigma) \subseteq A_{x,y}$. However, then the hypothesis must be of the form $W_{f(x,y,u)} = A_{x,y}$ with u being the maximum element of $B_{x,y}$ while the learner is learning $B_{x,y}$. However, this hypothesis will only be issued when $z > u$ and actually already data outside $B_{x,y}$ have been observed. Hence the given learner is conservative. Furthermore, one has to show by case distinction that the learner partially learns all the members of the class $\mathcal{D}$. In the case that the learner learns the empty set, a finite set $B_{x,y}$, a set $C_{x,y,y',z}$ or the full set then one can easily see from the algorithm that the learner actually converges to an index of this set. In the case that the learner learns an infinite set $B_{x,y}$ then the learner will output the default index for $B_{x,y}$ infinitely often and all indices of the type $f(x, y, u)$ only finitely often; again the learner partially infers the set. In the case that the learner learns a set $A_{x,y}$ while $B_{x,y}$ is finite then the learner converges again to $f(x, y, u)$ syntactically where u is the maximum number with $\langle x, y, u\rangle \in B_{x,y}$; this is then an index for $A_{x,y}$. Thus, the learner learns all sets in $\mathcal{D}$. Furthermore, the learner is prudent as it only conjectures indices for sets in $\mathcal{D}$. These arguments complete that the given learner *PrudEssConsConsvPart*-learns $\mathcal{C}$.

A *PrudConsvBC*-learner differs from the above learner only when dealing with a σ of nonempty range contained in $A_{x,y}$ for some x, y. If $z \leq u$ then the learner conjectures $e(x, y)$ else the learner conjectures $f(x, y, u)$. It is easy to see that the learner is behaviourally correct and that it *PrudConsConsvBC*-learns the given class $\mathcal{C}$; again all members of $\mathcal{D}$ are learnt and all conjectures are for sets in $\mathcal{D}$.

In a class-preserving recursive enumeration of the class, one has that for $y \in X - Y$ the full set is the only set which contains any triple $\langle x, y, z\rangle$. Hence, when starting to enumerate $A_{x,y}$ or $B_{x,y}$ on some data from $A_{x,y}$, the learner has for all $y \in X - Y$ eventually to redirect the hypothesis to the full set. As $X - Y$ is not r.e., this might also happen for some data started for $\langle x, y, z\rangle$ where $y \in Y$ and so a tuple $\langle x', y', z'\rangle$ with $x \neq x'$ might be enumerated which enforces that the full set is the only set containing the data enumerated so far; however, when y goes into Y, the data observed so far might also stem from a set of the form $C_{x,y,y',z}$ with y' from another member of Y and z be sufficiently large, thus the learner overgeneralises and is not conservative.

It is well-known that a class containing, for every x, the sets $A_{x,0}$ and $B_{x,0}$ is not explanatorily learnable. This mainly follows from the fact that otherwise a stabilising sequence found for $A_{x,0}$ would permit to get, in the case that W_x is finite, an upper bound on the last s where $W_{x,s+1} \neq W_{x,s}$ and therefore permit to decide in the limit whether W_x is finite – which is impossible. ∎

7. Conservative Partial Learning of Uniformly Recursive Families

The last results in the present work are on partially conservative learning with respect to uniformly recursive families. In particular, these observations illustrate the connection between partial learning and learning in the limit (in both the syntactic and the semantic sense).

Theorem 42. *If a uniformly recursive class $\mathcal{C}$ of sets is ConsvPart learnable, then $\mathcal{C}$ is BC learnable.*

Proof. Let M be a recursive *ConsvPart* learner of $\mathcal{C}$, and T be a text for any set L_e in $\mathcal{C}$. As M *ConsvPart* learns L_e, there is a number n sufficiently large so that $W_{M(T[n])} = L_e$. Let $H_e = range(T[n])$. Since M is partially conservative, $L_d = L_e$ holds for every L_d in $\mathcal{C}$ with $H_e \subseteq L_d \subseteq L_e$, for otherwise one may build a text for L_d extending $T[n]$ on which M outputs a proper superset of L_d. Hence there is a family of finite tell-tale sets for $\mathcal{C}$. As shown in [2, Section 3.2, Corollary 3], a uniformly recursive class non-effectively satisfying Angluin's tell-tale condition is *BC* learnable, and therefore $\mathcal{C}$ is *BC* learnable. ∎

Theorem 43. *If a uniformly recursive class $\mathcal{C}$ of sets is ConsvPart learnable with respect to a class-preserving hypothesis space, as well as Ex learnable with respect to a class-preserving hypothesis space, then it is ClsPresvEx learnable by a learner that does not output any index for a proper superset of a given target set $L \in \mathcal{C}$ on every text for L.*

Proof. Let M be a recursive *ConsvPart* learner of the given class $\mathcal{C}$ which uses a class-preserving hypothesis space. One may assume that M uses any general class-preserving hypothesis space. As $\mathcal{C}$ is also explanatorily learnable, there is a uniformly r.e. family of finite tell-tale sets for $\mathcal{C}$. Suppose $L_0, L_1, L_2, \ldots$ is a uniformly recursive numbering of $\mathcal{C}$, and that $H_0, H_1, H_2, \ldots$ is the corresponding family of tell-tale sets, that is, for all e, $H_e \subseteq L_e$ and there is no d such that $H_e \subseteq L_d \subset L_e$. One can define a learner N as follows. On input σ, N searches for the least $e \leq |\sigma|$, if such an e exists, with $H_{e,|\sigma|} \subseteq range(\sigma) \subseteq L_e$; if no such e is found, N outputs ?. If e' is the least such number, N then searches for the shortest $\tau \preceq \sigma$ such that $H_{e',|\sigma|} \subseteq W_{M(\tau),|\sigma|} \subseteq L_{e'}$; if no such τ exists, N outputs ?. If τ' is the shortest such prefix found, then N outputs $M(\tau')$.

Suppose N is fed with a text T for the set L in $\mathcal{C}$. Since N only outputs indices conjectured by M on T, and M is a *ConsvPart* learner of $\mathcal{C}$, N never conjectures a proper superset of L. It remains to show that N explanatorily learns L. Suppose that in the numbering $L_0, L_1, L_2, \ldots$, e is the least index for L. There is an n sufficiently large so that for all $k > n$, $H_{e,k} = H_e$ and e is the least index not exceeding k with $H_{e,k} \subseteq range(T[k]) \subseteq L_e$. Furthermore, as M outputs at least one correct index for L, there is a least number l such that $H_e \subseteq W_{M(T[l])} \subseteq L_e$. Thus N will converge to $M(T[l])$ in the limit, and since M only outputs indices of sets in $\mathcal{C}$, it follows that $W_{M(T[l])} = L_e$. Thus N is a class-preserving explanatory learner of $\mathcal{C}$ which never conjectures a proper superset of any target set $L \in \mathcal{C}$ on every text for L. ∎

Observation 35 and Theorem 43 imply the following corollary.

Corollary 44. *If a uniformly recursive class $\mathcal{C}$ of sets is ConsvPart learnable with respect to a class-preserving hypothesis space, as well as Ex learnable, then $\mathcal{C}$ is ConsvEx learnable.*

8. Conclusion

This paper complements earlier work on partial learning of recursive functions [10], and extends earlier results on conservative learning of r.e. sets [11]. It is hoped that in doing so, more light has been shed on the relationship between partial learning and the traditional models of learning, such as learning in the limit and behaviourally correct learning, in which strict convergence to a correct hypothesis is enforced. For example, it was shown that while conservative partial learnability is powerful enough to identify the whole class of recursive functions, the additional constraint of consistency reduces its strength to that of consistent conservative learning in the limit (Theorem 32). Moreover, the characterisation of consistently partially learnable classes of r.e. sets as subclasses of uniformly recursive families (Theorem 17) suggests that many natural classes of recursive sets - even those that do not satisfy the tell-tale condition [1] - can be consistently partially identified. It may be fruitful to explore refined notions of consistent partial learning, and to characterise them in a way similar to Theorem 17.

Acknowledgements. F. Stephan was partially supported by NUS grant R252-000-420-112; S. Zilles was partially supported by the Natural Sciences and Engineering Research Council of Canada (NSERC).

References

[1] Dana Angluin. Inductive inference of formal languages from positive data. *Information and Control* 45(2) (1980): 117–135.

[2] Ganesh Baliga, John Case and Sanjay Jain. The synthesis of language learners. *Information and Computation* 152 (1999): 16–43.

[3] Janis Bārzdiņš. Two theorems on the limiting synthesis of functions. In *Theory of Algorithms and Programs, vol. 1*, pages 82–88. Latvian State University, 1974. In Russian.

[4] John Case and Timo Kötzing. Difficulties in forcing fairness of polynomial time inductive inference. *Algorithmic Learning Theory*, Twentieth International Conference, ALT 2009, Porto, Portugal, October 3-5, 2009. Proceedings. Springer LNAI 5809 (2009): 263–277.

[5] John Case and Chris Lynes. Machine inductive inference and language identification. *Proceedings of the Ninth International Colloquium on Automata, Languages and Programming*, Springer LNCS 140 (1982): 107–115.

[6] John Case and Carl Smith. Comparison of identification criteria for machine inductive inference. *Theoretical Computer Science* 25 (1983): 193–220.

[7] Mark Fulk. A study of inductive inference machines. PhD thesis, SUNY at Buffalo, 1985.

[8] Mark Fulk. Prudence and other conditions on formal language learning. *Information and Computation* 85 (1990): 1–11.

[9] Ziyuan Gao, Frank Stephan, Guohua Wu and Akihiro Yamamoto. Learning families of closed sets in matroids. *Computation, Physics and Beyond; International Workshop on Theoretical Computer Science*, WTCS 2012, Springer LNCS 7160 (2012): 120–139.

[10] Ziyuan Gao and Frank Stephan. Confident and consistent partial learning of recursive functions. *Algorithmic Learning Theory,* Twenty-third International Conference, ALT 2012, Lyon, France, October 2012, Proceedings. Springer LNAI 7568 (2012): 51–65.

[11] Ziyuan Gao, Sanjay Jain and Frank Stephan. On conservative learning of recursively enumerable languages. *Ninth Conference on Computability in Europe*, CiE 2013, Milan, Italy, July 1–5, 2013. Springer LNCS 7921 (2013): 181–190.

[12] Ziyuan Gao, Frank Stephan and Sandra Zilles. Partial learning of recursively enumerable languages. *Algorithmic Learning Theory,* Twenty-fourth International Conference, ALT 2013, Singapore, October 2013, Proceedings. Springer LNAI 8139 (2013): 113–127.

[13] E. Mark Gold. Language identification in the limit. *Information and Control* 10 (1967): 447–474.

[14] Sanjay Jain and Arun Sharma. Prudence in vacillatory language identification. *Algorithmic Learning Theory,* Third International Conference, ALT 1992, Tokyo, Japan, October 1992, Proceedings. Springer LNAI 743 (1993): 159–168.

[15] Sanjay Jain, Daniel Osherson, James S. Royer and Arun Sharma. 1999. *Systems that learn: an introduction to learning theory.* MIT Press, Cambridge, Massachusetts.

[16] Sanjay Jain and Frank Stephan. Consistent partial identification. *Proceedings of the Twentysecond Annual Conference on Computational Learning Theory*, COLT 2009, pages 135–145.

[17] Sanjay Jain, Frank Stephan and Nan Ye. Prescribed learning of r.e. classes. *Theoretical Computer Science* 410(19) (2009): 1796–1806.

[18] Dick de Jongh and Makoto Kanazawa. Angluin's theorem for indexed families of r.e. sets and applications. *Proceedings of the Ninth Annual Conference on Computational Learning Theory*, COLT 1996, pages 193–204, ACM Press, 1996.

[19] Steffen Lange and Thomas Zeugmann. Language learning in dependence on the space of hypotheses. *Proceedings of the Sixth Annual ACM Conference on Computational Learning Theory*, COLT 1993, pages 127–136, ACM Press, 1993.

[20] Eliana Minicozzi. Some natural properties of strong-identification in inductive inference. Theoretical Computer Science, 2:345–360, 1976.

[21] Daniel N. Osherson, Michael Stob and Scott Weinstein. Learning strategies. *Information and Control* 53 (1982): 32–51.

[22] Daniel N. Osherson, Michael Stob and Scott Weinstein. 1986. *Systems that learn: an introduction to learning theory for cognitive and computer scientists*. Cambridge, Massachusetts.: MIT Press.

[23] Leonard Pitt. Inductive inference, DFAs, and computational complexity. *Analogical and Inductive Inference, Proceedings of the Second International Workshop*, AII 1989. Springer LNAI 397 (1989): 18–44.

[24] Hartley Rogers, Jr. 1987. *Theory of recursive functions and effective computability.* Cambridge, Massachusetts: MIT Press.

[25] Rolf Wiehagen and Thomas Zeugmann. Learning and consistency. *Algorithmic Learning for Knowledge-Based Systems*, GOSLER Final Report, Springer LNAI 961 (1995): 1–24.

[26] Thomas Zeugmann, Steffen Lange and Shyam Kapur. Characterizations of monotonic and dual monotonic language learning. *Information and Computation* 120(2) (1995): 155–173.