RANDOMNESS AND SOLOVAY DEGREES

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Abstract. We consider the behaviour of Schnorr randomness, a randomness notion weaker than Martin-Löf's, for left-r.e. reals under Solovay reducibility. Contrasting with results on Martin-Löf-randomenss, we show that Schnorr randomness is not upward closed in the Solovay degrees. Next, some left-r.e. Schnorr random α is the sum of two left-r.e. reals that are far from random. We also show that the left-r.e. reals of effective dimension $> r$, for some rational r, form a filter in the Solovay degrees.

Keywords. Solovay reducibility, Schnorr randomness and effective dimension.

Mathematical Subject Classification. 03D32 Algorithmic randomness and dimension; 68Q30 Algorithmic information theory.

1. Introduction

Randomness notions. The algorithmic theory of randomness defines randomness notions of reals in $[0, 1]$, or equivalently infinite bit sequences, and studies their properties and interactions with computational complexity. The notion of Martin-Löf (ML) randomness was for a long time considered to be the central one. We review the definition. An open set $U \subseteq [0, 1]$ is *r.e.* if it is a union of computable sequence of open intervals with rational endpoints, that is, $U = \bigcup_{i\in\mathbb{N}}(p_i, q_i)$ where $\langle p_i\rangle$ and $\langle q_i\rangle$ are computable sequence of rationals. A *ML-test* is a sequence $\langle U_n\rangle$ of uniformly r.e. open sets with $\mu(U_n) \leq 2^{-n}$ where μ is the Lebesgue measure. A real x is *ML-random* if $x \notin \bigcap_n U_n$ for every ML-test $\langle U_n\rangle$.

A suite of alternative notions has been introduced by modifying this definition, both stronger and weaker notions than ML-randomness. See e.g. [13, Ch. 3]. Many of these notions have shown their importance in particular for the interaction of randomness and analysis [14, 2]. One of them is the following weakening of ML-randomness, which will be of importance in the present paper. We say that a real x is *Schnorr random* if it passes all ML-tests $\langle U_n\rangle$ such that $\mu(U_n)$ is computable uniformly in n.

Solovay reducibility. Our central notion is Solovay reducibility $\leq_S$. Informally, for left-r.e. reals α, β, $\alpha \leq_S \beta$ expresses that α is no harder to approximate from the left than β. A formal definition will be given in the beginning of Section 2. This implies $\alpha \leq_T \beta$, so β can compute α. Further, $\alpha \leq_S \beta$ also implies that β is in a particular sense no less random than α. Then, Solovay reducibility is a measure of both computational complexity and algorithmic randomness. Background is provided e.g. in [5, Chapter 9], and also in [13, Section 3.2].

Main results. For left-r.e. reals, ML-randomness interacts closely with Solovay reducibility: a left-r.e. real is ML-random iff it is Solovay complete (Calude et al.

[3] and Kučera-Slaman [9]). As discussed shortly, the complete Solovay degree is join irreducible.

The supremum of two left-r.e. reals in the degree structure induced by Solovay reducibility is given by their arithmetic sum. We are guided by the following two facts that restate some of the results above in terms of the sum. Let α, β be left-r.e. reals.

(1) If α is ML-random, then $\alpha + \beta$ is ML-random.
(2) If $\alpha + \beta$ is ML-random, then at least one of α and β is ML-random.

A simple direct proof of the first fact can be found in [13, Theorem 3.2.27]. For the second, see Demuth [4], and as a more recent (and more readable) reference Downey, Hirschfeldt, and Nies [7].

Our first goal is to show that both statements fail for Schnorr randomness (Corollary 3.2 and Theorem 4.1). Thereafter, we prove that in contrast, left-r.e. weakly s-random reals behave like ML-random reals (Theorem 5.6). The reals that have effective packing dimension at most s behave like non-ML-random reals (Proposition 5.7).

Preliminaries. Our notation in the algorithmic theory of randomness is standard as in Nies [13] or Downey and Hirschfeldt [5] (except that we write "r.e." instead of "c.e."). Unless otherwise stated, reals in this paper are in $[0, 1]$. A real α is called *left-r.e.* if there exists a computable non-decreasing sequence $\langle a_n \rangle_{n \in \mathbb{N}}$ of rationals such that $\lim_n a_n = \alpha$. We sometimes identify a real α with its binary expansion $X \in 2^\omega$ and the corresponding set $X = \{n \in \mathbb{N} : X(n) = 1\}$. By $X \upharpoonright n$ we denote the initial segment of X of length n. We write $f(n) \leq^+ g(n)$ to mean $f(n) \leq g(n) + O(1)$, that is, there exists a constant c such that $f(n) \leq g(n) + c$ for all n.

2. Solovay reducibility

2.1. **Definitions and basic facts.**

The following was first defined by Solovay in unpublished work.

Definition 2.1 (Solovay [15]). For left-r.e. reals α and β, we say that α is *Solovay reducible* to β, denoted by $\alpha \leq_S \beta$, if there exist a constant c and a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ such that if $q \in \mathbb{Q}$ and $q < \beta$, then $f(q) \downarrow < \alpha$ and $\alpha - f(q) < c(\beta - q)$.

Solovay's used the terminology "α is dominated by β". Recall that this definition expresses that α is less complex than β by asking that β is harder to approximate from below. For, let α_s, β_s be computable nondecreasing sequences of rational converging to α, β respectively. Then $\alpha \leq_S \beta$ if and only if there are a constant $c \in \mathbb{N}$ and a computable function g such that $\alpha - \alpha_{g(n)} < c(\beta - \beta_n)$ for all n. For a proof, see again [5, Chapter 9], and also [13, Section 3.2].

There is a useful algebraic characterization of Solovay reducibility: $\alpha \leq_S \beta$ means that β can be obtained from α by literally "adding" information, in the sense of adding a left-r.e. real γ. To make this work we also need to scale β. (For the precise version of the result used here see [13, 3.2.29].)

Proposition 2.2 (Downey, Hirschfeldt, and Nies [7]). *For left-r.e. reals α, β we have $\alpha \leq_S \beta$ if and only if there are $d \in \mathbb{N}$ and a left-r.e. real γ such that $\alpha + \gamma = 2^d \beta$.*

2.2. **Connections to K-reducibility.** For any reals α, β one writes

$$\alpha \leq_K \beta \text{ if } \forall n \, K(\alpha \upharpoonright n) \leq^+ K(\beta \upharpoonright n),$$

where K denotes prefix-free Kolmogorov complexity. The Levin-Schnorr theorem says that X is ML-random if and only if $K(X \upharpoonright n) \geq^+ n$. Then $\alpha \leq_K \beta$ says that β is no less random than α.

For left-r.e. reals α, β, if $\alpha \leq_S \beta$ then $\alpha \leq_K \beta$ [15]. This fact also follows from Proposition 2.2. We will now obtain an implication in the converse direction under a hypothesis stronger that $\alpha \leq_K \beta$. We write $\alpha \ll_K \beta$ if $\lim_n (K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$.

Proposition 2.3. *Let α, β be left-r.e. reals. If $\alpha \ll_K \beta$ then $\alpha <_S \beta$.*

Notice that in this proposition we can replace $K(\sigma)$ by $C(\sigma)$, $K(\sigma|n)$, or $C(\sigma|n)$ where $n = |\sigma|$. The proof remains almost the same.

Downey, Hirschfeldt, Laforte [6] showed cl-reducibility version, that is $\ll_K$ implies $\leq_{cl}$. We also note that $\emptyset \ll_K \alpha$ implies that α is wtt-complete [5, Theorem 9.12.2].

Proof. First note that $\beta \leq_S \alpha$ would imply $\beta \leq_K \alpha$, contrary to $\alpha \ll_K \beta$. Hence it suffices to show $\alpha \leq_S \beta$. If β is a rational, $\alpha \ll_K \beta$ does not hold. So β is not a rational. If α is a rational, then $\alpha \leq_S \beta$ holds trivially. Hence, we can assume that α is not a rational.

Let $\langle \alpha_s \rangle_{s \in \mathbb{N}}, \langle \beta_s \rangle_{s \in \mathbb{N}}$ be increasing computable sequences of rationals converging to α, β respectively. Given $n \in \mathbb{N}$, let s_n be the first stage s such that $\alpha_s \upharpoonright n = \alpha \upharpoonright n$. There exists a constant $c \in \mathbb{N}$ such that for each n

$$K(\beta_{s_n} \upharpoonright n) \leq K(\alpha_{s_n} \upharpoonright n) + c = K(\alpha \upharpoonright n) + c.$$

If given $\alpha_{s_n} \upharpoonright n$, we can compute the stage s_n from the approximation $\langle \alpha_s \rangle$, which also computes $\beta_{s_n} \upharpoonright n$, thus the first inequality follows. Since $\lim_n (K(\beta \upharpoonright n) - K(\alpha \upharpoonright n)) = \infty$, we have $K(\beta \upharpoonright n) > K(\alpha \upharpoonright n) + c$ for sufficiently large n. Thus, $\beta_{s_n} \upharpoonright n \neq \beta \upharpoonright n$ for all sufficiently large n. Furthermore, $\beta \upharpoonright n$ is not equal to the lexicographic successor of $\beta_{s_n} \upharpoonright n$, for otherwise $K(\beta \upharpoonright n)$ would be within a constant of $K(\beta_{s_n} \upharpoonright n)$. Hence, $\beta - \beta_{s_n} \geq 2^{-n}$.

Define a partial computable function $f : \mathbb{Q} \to \mathbb{Q}$ by $f(q) = \alpha_s$, where s is the first stage such that $q < \beta_s$. If n is sufficiently large and $s_n \leq s \leq s_{n+1}$, then

$$\alpha - \alpha_s \leq \alpha - \alpha_{s_n} \leq 2^{-n},$$

while

$$\beta - \beta_s \geq \beta - \beta_{s_{n+1}} \geq 2^{-n-1}.$$

Then there is a rational $q^* < \beta$ such that $\alpha - f(q) \leq 2(\beta - q)$ for every rational q with $q^* \leq q < \beta$. Modifying f on the rationals $\leq q^*$ shows that $\alpha \leq_S \beta$. $\square$

2.3. **Connections to ML-randomness.** We restate the two guiding facts from the introduction. They connect Solovay reducibility and ML-randomness.

Proposition 2.4. *Let α, β be left-r.e. reals. If α is ML-random, then $\alpha + \beta$ is ML-random.*

Here $+$ is the real addition. For a direct proof see [13, Theorem 3.2.27].

By Proposition 2.2, for left-r.e. reals α, γ, if α is ML-random and $\alpha \leq_S \gamma$, then γ is ML-random. The implication $\leq_S \Rightarrow \leq_K$ is an extension of this result by the

Levin-Schnorr theorem. This explains that Solovay reducibility is a measure of algorithmic randomness.

Alternatively, recall that a left-r.e. real is ML-random iff it is Solovay complete (Calude et al. [3] and Kučera-Slaman [9]). Proposition 2.4 now follows via Proposition 2.2.

The second fact says that the top degree in the Solovay degrees is join irreducible.

Theorem 2.5 (Demuth [4], Downey, Hirschfeldt, and Nies [7])**.** *Let α, β be left-r.e. reals. If $\alpha + \beta$ is ML-random, then at least one of α and β is ML-random.*

For left-r.e. reals α, β, the degree of $\alpha + \beta$ is the least degree above α, β, thus the degree of $\alpha + \beta$ is the join of α and β. The theorem says that the join of two non-ML-random left-r.e. reals is not ML-random either, and the top degree can not be the join of two lesser degrees.

3. Schnorr randomness is not upward closed in the Solovay degrees

The following will be used to show that for Schnorr randomness, the counterpart of Proposition 2.4 fails.

Theorem 3.1. *Let α be a left-r.e. real such that $\forall n\, K(\alpha \upharpoonright n) < n - f(n)$ for some order function f. There exists a left-r.e. real β such that $\alpha + \beta$ is disjoint from some infinite computable set.*

Proof. Suppose g, h are strictly increasing computable functions such that the range of h is the complement of the range of g. We define $C = A \oplus_g B$ by $C(g(n)) = B(n)$ and $C(h(n)) = A(n)$.

Let α be as in the statement. If g is a sufficiently fast growing computable function, then $\gamma = \Omega \oplus_g \emptyset$ satisfies $K(\gamma \upharpoonright n) \geq^+ n - \frac{f(n)}{2}$. Note that γ is left-r.e. Since $\alpha \ll_K \gamma$, by Proposition 2.3 we have $\alpha \leq_S \gamma$. By Prop. 2.2 there exist a natural number d and a left-r.e. real β such that $2^d\gamma = \alpha + \beta$. Since γ is disjoint from an infinite computable set, so is $\alpha + \beta$. □

Partial computable randomness is a notion in between Schnorr and ML-randomness. For the definition see [13, Section 7.4].

Corollary 3.2. *The left-r.e. Schnorr random reals are not upward closed in the Solovay degrees.*

Proof. By a result of Merkle [12] (also see [13, Remark 7.4.17]) there exists a left-r.e. Schnorr random real α such that that $K(\alpha \upharpoonright n) = O(\log n)$. Now take $\alpha + \beta \geq_S \alpha$ as above. Since $\alpha + \beta$ is disjoint from some infinite computable set, this can not be Schnorr random. □

Remark 3.3. The constructed α can be even partial computably random as in [13, Remark 7.4.17]. Furthermore, $\alpha + \beta$ is not even Kurtz random because of its disjointness from an infinite computable set. The definitions of partial computable randomness and Kurtz randomness can be found in [13] and [5].

4. Left-r.e. Schnorr random reals can be split into non-Schnorr random reals

We show that the counterpart of Theorem 2.5 for Schnorr randomness does not hold either. Theorem 4.1 below gives a sufficient condition on α so that α is the join of two far-from-random reals. The condition is given by somewhat slow growing initial segment complexity of a left-r.e. real α. The condition is weak in the sense that α can still be Schnorr random using [13, Remark 7.4.17] as in the proof of Corollary 3.2.

We show that α can be written as a sum of left-r.e. reals which are far from random in various senses. One way to express this is saying that their effective Hausdorff dimensions are 0.

We recall the definition here. By [10], for a real $X \in 2^\omega$, the *effective Hausdorff dimension* $\dim(X)$ can be characterized by

$$\dim(X) = \liminf_n \frac{K(X \upharpoonright n)}{n}. \tag{1}$$

By [1] the *effective packing dimension* $\mathrm{Dim}(X)$ can be characterized by

$$\mathrm{Dim}(X) = \limsup_n \frac{K(X \upharpoonright n)}{n}. \tag{2}$$

Let $\mathbf{a}$ be a Solovay degree. Recall that Solovay reducibility implies K-reducibility. For $\alpha, \beta \in \mathbf{a}$ we have $\alpha \equiv_K \beta$ and hence $\dim(\alpha) = \dim(\beta)$. So we may well-define $\dim(\mathbf{a}) = \dim(\alpha)$ for some real $\alpha \in \mathbf{a}$. A survey of effective Hausdorff dimension is given in [11].

Theorem 4.1. *Let α be a left-r.e. real such that $C(\alpha \upharpoonright n) \leq n - g(n)$ for all n, where g is a computable function such that $\sum_n 2^{-g(n)}$ is finite and a computable real. There exist left-r.e. reals β, γ such that $\alpha = \beta + \gamma$, $\dim(\beta) = \dim(\gamma) = 0$ and both β, γ are not Borel normal and disjoint from infinite computable sets.*

Notice that $\dim(\alpha)$ can be 1 in which case α is more random than β and γ.

Proof. Let $\langle \alpha_s \rangle$ be an increasing computable sequence of dyadic rationals converging to α. We impose additional properties to $\langle \alpha_s \rangle$. For each $k \in \mathbb{N}$, we can compute s such that $C_s(\alpha_s \upharpoonright n) < n - g(n)$ for all $n \leq k$. Since α_s is a dyadic rational, $\alpha_s \cdot 2^j$ is a natural number for some j. By taking a subsequence of such s and considering an increasing computable function h that maps k to j, we can assume that $\alpha_s \cdot 2^{h(s)}$ is a natural number and $C(\alpha_s \upharpoonright n) < n - g(n)$ for all s and all $n \leq h(s)$.

Let a_n be the number of s such that the binary representation of $\alpha_{s+1} - \alpha_s$ has a bit 1 at position n. The numbers $\{a_n\}$ are uniformly computably approximable from below and

$$\alpha = \sum_n a_n \cdot 2^{-n}.$$

Notice that $a_n \leq 2^{n-g(n)}$ for each n. For s such that $h(s) < n$, $\alpha_s - \alpha_{s-1}$ does not have a bit 1 at position n. For s such that $h(s) \geq n$, we have $C(\alpha_s \upharpoonright n) < n - g(n)$ and there are at most $2^{n-g(n)}$ many different strings $\alpha_s \upharpoonright n$.

Now we split the natural numbers into intervals $I_0, I_1, \ldots$ such that $p(\max(I_n)) > (\min(I_n))^2$ for every n where $p : \mathbb{N} \to \mathbb{N}$ is a computable function such that $\sum_{m \geq n} 2^{-g(m)} \leq 2^{-p(n)}$. For these intervals, the following inequalities hold:

$$\sum_{m > \max(I_n)} a_m \cdot 2^{-m} \leq \sum_{m > \max(I_n)} 2^{-g(m)} \leq 2^{-p(\max(I_n))} \leq 2^{-(\min(I_n))^2}.$$

We define left-r.e. reals β and γ by

$$\beta = \sum_m \{a_m \cdot 2^{-m} \ : \ \exists n (m \in I_{2n})\}, \ \gamma = \sum_m \{a_m \cdot 2^{-m} \ : \ \exists n (m \in I_{2n+1})\}.$$

For β, the bits at the positions from $\min(I_n)$ to $(\min(I_n))^2$ are 0 whenever n is odd. This is because, for every n, the terms $a_m \cdot 2^{-m}$ with $m < \min(I_n)$ affect only digits with positions below $\min(I_n)$ and the terms $a_m \cdot 2^{-m}$ with $m > \max(I_n)$ only contribute to a number bounded by $2^{-p(\max(I_n))}$ which has non-zero digits only at positions larger than $(\min(I_n))^2$. By a similar reason, for γ, the bits at the positions from $\min(I_n)$ to $(\min(I_n))^2$ are 0 whenever n is even.

This fact implies $\dim(\beta) = \dim(\gamma) = 0$. To compute $(\min(I_n))^2$ bits of β for odd n, we only need the information of $\min(I_n)$ bits of β and of n. Thus,

$$\frac{K(\beta \upharpoonright (\min(I_n))^2)}{(\min(I_n))^2} \leq \frac{K(\beta \upharpoonright \min(I_n)) + K(n) + O(1)}{(\min(I_n))^2} \to 0.$$

Furthermore, β and γ are not Borel normal, as the limit superior of the frequences of subwords of the form 0^k goes to 1 for each k. Finally, β is disjoint from the infinite computable set $\{\min(I_{2n+1}) : n \in \mathbb{N}\}$, and γ from $\{\min(I_{2n}) : n \in \mathbb{N}\}$. □

5. Filters and Ideals related to effective dimension

In this final section, we study the relationship between effective Hausdorff dimension and Solovay degrees. Recall that a nonempty subset F of a partially ordered set $(P, \leq)$ is a *filter* if

(1) for every $x \in F$, $x \leq y$ implies $y \in F$,
(2) for every $x, y \in F$, there exists $z \in F$ such that $z \leq x$ and $z \leq y$.

A filter is called *principal* if $F = \{x : x \geq y\}$ for some $y \in P$.

The ML-random left-r.e. reals are Solovay complete, and therefore induce the trivial filter consisting of the largest element. To treat filters related to dimension, we review the following.

Theorem 5.1. *For each rational $r \in (0,1)$, the set of Solovay degrees $\mathbf{a}$ such that $\dim(\mathbf{a}) > r$ is a filter in the Solovay degrees.*

For a proof, we use a generalization of Chaitin's Omega. Let $s \in (0, 1]$. Tadaki [16, Theorems 3.1, 3.2] defined

$$\Omega^s = \sum_{U(\sigma)\downarrow} 2^{-\frac{|\sigma|}{s}}$$

and showed that $s = \dim(\Omega^s) = \mathrm{Dim}(\Omega^s)$. The former equality was independently shown by Mayordomo [10, Corollary 3.3].

Lemma 5.2. *Let $r \in (0,1)$ be a real. Suppose α, β are left-r.e. reals such that $\dim(\alpha) > r$ and $\dim(\beta) > r$. There is a left-r.e. real γ such that $\gamma \leq_S \alpha$, $\gamma \leq_S \beta$ and $\dim(\gamma) > r$.*

Notice that r need not be a rational.

Proof. Let p, q be rationals such that $r < p < q$ and $q < \dim(\alpha), \dim(\beta)$. Then, Ω^p is a left-r.e. real and $\dim(\Omega^p) > r$. Furthermore, by Tadaki's results and (1,2) we have $\Omega^p \ll_K \alpha$, and hence $\Omega^p <_S \alpha$ by Proposition 2.3. Similarly, $\Omega^p <_S \beta$. □

Theorem 5.1 is now immediate from the lemma.

Question 5.3. Let $r \in (0, 1]$ be a real. If α, β are left-r.e. reals such that $\dim(\alpha) \geq r$ and $\dim(\beta) \geq r$, does there exist $\gamma \leq_S \alpha, \beta$ with $\dim(\gamma) \geq r$?

Equivalent, is $\{\mathbf{a}\colon \dim(\mathbf{a}) \geq r\}$ a filter? We give a partial answer to it. By the following result it is not a principal filter.

Proposition 5.4. *Let α be a left-r.e. real such that $\dim(\alpha) = r > 0$. There exists a left-r.e. real β such that $\dim(\beta) = r$ and $\beta <_S \alpha$.*

Notice that r need not be computable. We do not use r in the construction of β.

Proof. Let $f(n)$ be the biggest natural number less than $n+\sqrt{n}$. Note that $f : \mathbb{N} \to \mathbb{N}$ is an increasing computable function. Given a lef-r.e. α, let $\beta = \{f(n) \ : \ n \in \alpha\}$. Notice that β is a left-r.e. real with the approximation $\beta_s = \{f(n) \ : \ n \in \alpha_s\}$ where α_s is an increasing computable approximation of α.

Now the following equalities hold:

$$\begin{aligned} r =& \liminf \frac{C(\alpha \restriction n)}{n} = \liminf \frac{C(\beta \restriction f(n))}{n} \\ =& \liminf \frac{C(\beta \restriction f(n))}{f(n)} \cdot \frac{f(n)}{n} = \liminf \frac{C(\beta \restriction n)}{n}. \end{aligned}$$

To describe $\beta \restriction f(n)$ one only needs to describe the bits in the positions $f(k)$ for $k < n$. Since $f(n+1) - f(n) \leq 2$ for every n, $C(\beta \restriction n) = C(\beta \restriction f(k)) + O(1)$ for some k.

Clearly $\beta \leq_S \alpha$. Assume by way of contradiction that the converse also holds. Then there are constants $c, c' \in \mathbb{N}$ such that

$$C(\alpha \restriction f(n)) \leq C(\beta \restriction f(n)) + c' \leq C(\alpha \restriction n) + c.$$

Let $k \in \mathbb{N}$. There exists $a_0 \in \mathbb{N}$ such that $f(n) - n \geq k$ for all $n \geq a_0$. Inductively, we define $a_{n+1} = f(a_n)$. Then, $a_n \geq a_0 + kn$. In contrast,

$$C(\alpha \restriction a_n) \leq C(\alpha \restriction a_{n-1}) + c \leq C(\alpha \restriction a_0) + cn.$$

Hence,

$$\liminf \frac{C(\alpha \restriction a_n)}{a_n} \leq \liminf \frac{C(\alpha \restriction a_0) + cn}{a_0 + kn} \leq \frac{c}{k}.$$

Since k is arbitrary, we have $\dim(\alpha) = 0$, which is a contradiction. $\square$

Next we consider partial randomness.

Definition 5.5 (Tadaki [16], see also [5, Definition 13.5.1])**.** A *test for weak s-ML randomness* is a sequence of uniformly r.e. sets of strings $\langle V_k \rangle$ such that $\sum_{\sigma \in V_k} 2^{-s|\sigma|} \leq 2^{-k}$. A set $X \in 2^\omega$ is *weakly s-ML-random* if $X \notin \bigcap_k [\![V_k]\!]$ for all tests for weak s-ML randomness.

The strings can be replaced with open intervals with dyadic rational endpoints [5, after Definition 13.5.8].

Tadaki [16] showed that X is weakly s-ML-random if and only if $K(X \restriction n) > sn - O(1)$. Hence, for every weakly s-ML-random real X, we have $\dim(X) \geq s$. It is known that the converse fails. This also follows from the construction of Proposition 5.4.

Further, for left-r.e. reals, being weakly s-ML-random is upward closed under $\leq_S$.

Theorem 5.6. *Let $s \in (0, 1] \cap \mathbb{Q}$. The set of left-r.e. weakly-s-random Solovay degrees is a principal filter in Solovay degrees with the degree of Ω^s as the bottom element.*

Here, we consider only left-r.e. reals, so we restrict s to be a rational.

Proof. We follow the argument of Kučera-Slaman theorem [9]. Let α be a left-r.e. weakly s-random real. The goal is to show $\Omega^s \leq_S \alpha$.

We construct a test for weak s-ML-randomness $\langle U_k \rangle_{k\in\mathbb{N}}$. At stage t act as follows. If $\alpha_t \in U_k[t]$, then do nothing. Otherwise let t' be the last stage at which we put anything into U_k (or $t = 0$ if there is no such stage), and put the intervals $(\alpha_t, \alpha_t + 2^{-\frac{k}{s}}(\Omega^s_t - \Omega^s_{t'}))$ into U_k.

Let $t_0 = 0,\ t_1,\ \cdots$ be the stages at which we put numbers into U_k. The total weight is bounded by

$$\sum_i (2^{-\frac{k}{s}}(\Omega^s_{t_{i+1}} - \Omega^s_{t_i}))^s \leq \sum_{U(\sigma)\downarrow} (2^{-\frac{k}{s}})^s (2^{-\frac{|\sigma|}{s}})^s \leq 2^{-k}.$$

Thus, $\langle U_k \rangle$ is a test for weak s-ML-randomness.

Since α is weakly s-random, there exists a constant k such that $\alpha \notin U_k$. Then,

$$\alpha_{t_{i+1}} - \alpha_{t_i} > 2^{-k/s}(\Omega^s_{t_i} - \Omega^s_{t_{i-1}})$$

for each $i > 0$, so $\Omega^s \leq_S \alpha$. □

Now we turn to ideals, the dual notion of filters. A nonempty subset I of a partially ordered set $(P, \leq)$ is an *ideal* if

(1) for every $x \in I$, $y \leq x$ implies $x \in I$,
(2) for every $x, y \in I$, there exists $z \in I$ such that $x \leq z$ and $y \leq z$.

Theorem 2.5 says that the non-ML-random Solovay degrees form an ideal. The following is an easy observation.

Proposition 5.7. *For each $r \in [0,1]$, the set of left-r.e. degrees $\mathbf{a}$ such that $\mathrm{Dim}(\mathbf{a}) < r$ is an ideal of left-r.e. Solovay degrees. We can replace $\mathrm{Dim}(\mathbf{a}) < r$ by $\mathrm{Dim}(\mathbf{a}) \leq r$.*

Proof. Let α, β be left-r.e. reals such that $\mathrm{Dim}(\alpha) < r$ and $\mathrm{Dim}(\beta) < r$. Then, $\gamma = \alpha + \beta$ satisfies $\alpha \leq_S \gamma$ and $\beta \leq_S \gamma$. One can easily check that $K(\gamma \upharpoonright n) =^+ \max(K(\alpha \upharpoonright n), K(\beta \upharpoonright n))$ ([8, Theorem 7.4]). Hence,

$$\mathrm{Dim}(\gamma) = \limsup \frac{K(\gamma \upharpoonright n)}{n} \leq \limsup \frac{\max(K(\alpha \upharpoonright n), K(\beta \upharpoonright n))}{n} < r.$$

The proof in the case that $\mathrm{Dim}(\mathbf{a}) \leq r$ is almost the same. □

The set of left-r.e. K-trivial degrees is an ideal in the Solovay degrees because the sum of two K-trivial reals is again K-trivial [8, Theorem 7.2].

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