Randomness and Universal Machines

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Abstract. The present work investigates several questions from a recent survey of Miller and Nies related to Chaitin's Ω numbers and their dependence on the underlying universal machine. It is shown that there are universal machines for which Ω_U is just $\sum_x 2^{1-H(x)}$. For such a universal machine there exists a co-r.e. set X such that $\Omega_U[X] = \sum_{p:U(p)\downarrow\in X} 2^{-|p|}$ is neither left-r.e. nor Martin-Löf random. Furthermore, one of the open problems of Miller and Nies is answered completely by showing that there is a sequence U_n of universal machines such that the truth-table degrees of the Ω_{U_n} form an antichain. Finally it is shown that the members of hyperimmune-free Turing degree of a given Π^0_1-class are not low for Ω unless this class contains a recursive set.

1 Introduction

Chaitin [6, 7] started to investigate the halting probability of prefix-free Turing machines M, that is, of machines which never halt on programs p, q where q is an extension of p viewed as a binary string. The halting probability Ω_M is then the probability that a randomly drawn infinite sequence extends a program p of M such that $M(p)$ halts. This is equivalent to the sum

$$\Omega_M = \sum_{p \in \mathrm{dom}(M)} 2^{-|p|}$$

where $|p|$ denotes the lengths of the binary string p. The value of Ω_M can be approximated from below. The word *left-r.e.* is used to denote this property. As a set R represents the number $\sum_{n\in R} 2^{-n-1}$, real numbers between 0 and 1 are identified with the sets representing them. Chaitin was not only interested in the halting probability of machines but in those of universal machines which are defined as follows.

Definition 1. A prefix-free Turing machine $U : \{0,1\}^* \to \{0,1\}^*$ is *universal* if and only if

$$\forall M\ \exists c\ \forall \sigma\ \exists \sigma'\ [U(\sigma') = M(\sigma)\ \wedge |\sigma'| \leq |\sigma| + c].$$

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Furthermore, a prefix-free machine U is called *universal by adjunction* iff for every prefix-free machine V there is a fixed word p such that $U(pq) = V(q)$ for all $q \in \{0,1\}^*$ where pq is the concatenation of p and q and $U(pq) = V(q)$ means that either both sides are defined and equal or both are undefined.

The importance of universal Turing machines is that they permit to define the Kolmogorov complexity in an optimal way, that is, the definitions based on two different machines differ at most by a constant. Given a prefix-free Turing machine M and an element x of its range, let

$$H_M(x) = \min\{|p| : M(p) = x\}$$

be the length of the shortest description of x with respect to M. If M is a universal Turing machine, then H_M is a total function and for every further machine N there is a constant c such that for all x in the range of H_N, $H_N(x) \leq H_M(x) + c$. In this case H_M is referred to as the *prefix-free Kolmogorov complexity based on* M. If there is no need to refer to the underlying universal machine U, one just writes H for H_U and Ω for Ω_U.

Martin-Löf [17] introduced a notion of randomness which became quite accepted in the field and is known as Martin-Löf random. Schnorr [24] found a characterization in terms of Kolmogorov complexity which is here used in place of the original definition:

$$A \text{ is Martin-Löf random } \Leftrightarrow \exists c \forall n\, (H(A(0)\dots A(n)) \geq n - c).$$

Hence Martin-Löf random sets have highly incompressible prefixes.

Chaitin noticed that the halting probability of a universal machine is Martin-Löf random. Further research [5, 15] provided the following equivalence: a left-r.e. set is Martin-Löf random iff it is the halting probability of some universal machine.

The notion of Martin-Löf randomness can easily be relativized to oracles: A is Martin-Löf random relative to B iff there is a constant c such that for all n, $H^B(A(0)\dots A(n)) \geq n-c$. Here H^B is defined as H, but based on an oracle machine which is universal for any oracle B among the prefix-free machines using the same oracle. There is a constant c such that $\forall B \forall x\ (H^B(x) \leq H(x)+c)$, thus if A is Martin-Löf random real relative to B then A is already Martin-Löf random. In case Ω is Martin-Löf random relative to B, B is called *low for* Ω. This definition does not depend on the choice of the universal machine.

Investigations on this topic continues and in a recent survey, Miller and Nies [18] listed a lot of interesting open questions related to the halting probability Ω. The present work addresses some of these questions from the eighth chapter of the survey of Miller and Nies, namely the following three questions.

- Question 8.1: Given a nonrecursive A which is low for Ω. Does A then have hyperimmune Turing degree?
- Question 8.9: Are there universal machines U, V such that $\Omega_U \not\equiv_{tt} \Omega_V$?
- Question 8.10: Given a machine U which is universal by adjunction, is there a co-r.e. set X such that $\sum_{p:U(p)\in X} 2^{-|p|}$ is not Martin-Löf random? Can such an X be taken to be many-one complete?

The present work answers Question 8.9 and obtains some results on the way to settle Questions 8.1 and 8.10. For Question 8.1, it is shown that for every Π^0_1 class without recursive members, every member which is low for Ω is also hyperimmune. For Question 8.10, it is shown that there is some universal machine U for which there is such an X, but this U is not universal by adjunction. Furthermore, it is open whether there is a Π^0_1-complete X with the same property.

For convenience, strings in $\{0,1\}^*$ are identified with natural numbers. More precisely, the string $b_0 b_1 \ldots b_{n-1}$ is identified with $2^n - 1 + \sum_{m<n} 2^m \cdot b_m$. For infinite sequences, $A \in \{0,1\}^\infty$ stands for both, for the set $\{n : A(n) = 1\}$ and for the real number $\sum_{n=0,1,\ldots} A(n) \cdot 2^{-n-1}$ and the relation $A < B$ is transferred from numbers to sets with the additional convention, that for the two representations of numbers of the from $n \cdot 2^{-m}$ the one ending with $011111\ldots$ is below the one ending with $100000\ldots$ so that $<$ becomes a linear ordering on sets. As it is a convention to write strings of the same length in alphabetical order from the left to the right like 000, 001, 010, 011, …, 111, one can say that numbers which are approximated from below have – uniformly in the length n – the prefix $A(0)\ldots A(n-1)$ of the binary sequence of their digits recursively approximated from the left; this explains the term "left-r.e." to denote them. Furthermore, A is called recursively enumerable or just r.e. iff it is recursively enumerable as a set. This definition is important since many authors call all left-r.e. reals just "r.e." which produces a conflict between the notations used for real numbers and the ones used for the sets representing them. Note that there are further synonyms for "left-r.e." like "nearly computable" and "left-computable". More information on recursion theory and algorithmic randomness can be found in the standard textbooks [8, 16, 22, 25, 27].

2 On the Probability that U outputs an element of X

Becher and Grigorieff [1] proposed to study the question how the probability to halt with a value in a given set looks like. The formal definition of this halting probability is the following.

Definition 2. For given X, let

$$\Omega_U[X] = \sum_{p:U(p)\downarrow\in X} 2^{-|p|}$$

denote the probability for U to halts and outputs an element of X.

Given an infinite r.e. set X and a universal machine U, the probability $\Omega_U[X]$ that U halts with the output being an element of X is a left-r.e. Martin-Löf random number. This can easily be seen as follows: There is a partial-recursive one-one function f from the domain X onto all natural numbers with a total and recursive inverse g. Now the machine V given as

$$V(p) = \begin{cases} f(U(p)) & \text{if } U(p)\downarrow\in X; \\ \uparrow & \text{otherwise.} \end{cases}$$

is universal: There is a constant c such that $H_U(g(x)) \leq H_U(x) + c$ for all x. The shortest program p with $U(p) = g(x)$ satisfies then $V(p) = x$. Thus $H_V(x) = H_U(g(x)) \leq H_U(x) + c$ for all x. So V is a universal machine and $U(p)\downarrow\in X$ iff $V(p)\downarrow$. Therefore $\Omega_U[X] = \Omega_V$. The number $\Omega_U[X]$ is left-r.e. and Martin-Löf random since Ω_V is.

Miller and Nies [18] note that $\Omega_U[F]$ is Martin-Löf random for any finite set F whenever U is universal by adjunction. But the following example shows that the randomness of $\Omega_U[F]$ depends in general on the universal machine.

Proposition 3. *There is a universal Turing machine U such that $\Omega_U[\{x\}] = 2^{1-H(x)}$ for all $x \in \{0,1\}^*$ where H is the Kolmogorov complexity based on U.*

Proof. Let $\tilde{U}$ be a universal Turing machine and let $\tilde{H}$ be the Kolmogorov complexity for machine $\tilde{U}$. The new universal machine U is constructed as follows.

Let $p_0, p_1, \ldots$ be a recursive one-one enumeration of the domain of $\tilde{U}$. Given any m, n, let, in stage n, $U(p_n 0^m 1) = \tilde{U}(p_n)$ iff n is the first number k such that $\tilde{U}(p_k) = \tilde{U}(p_n)$ and $|p_k| \leq |p_n| + m$, that is, there is no q of the same length as $p_n 0^m 1$ such that $U(q) = \tilde{U}(p_n)$ defined in some stage before stage n. Furthermore, let U be undefined on all other inputs including those of the form $p_n 0^m 1$ which did not qualify to take the value $\tilde{U}(p_n)$.

Now, for each x and each m there is exactly one program $p_{x,m}$ of length $\tilde{H}(x) + m + 1$ such that $U(p_{x,m}) = x$ and no program of length up to $\tilde{H}(x)$ generates x. Thus one has for every $x \in \{0,1\}^*$ and all sets X the following three equalities:

$$\begin{aligned} H(x) &= \tilde{H}(x) + 1; \\ \Omega_U[\{x\}] &= \sum_{p : U(p)\downarrow = x} 2^{-|p|} = \sum_{n > \tilde{H}(x)} 2^{-n} = 2^{-\tilde{H}(x)} = 2^{1-H(x)}; \\ \Omega_U[X] &= \sum_{x \in X} 2^{1-H(x)}. \end{aligned}$$

The proof is completed by noting that U is prefix-free and that the first of these three equalities guarantees that U is universal. ▮

Notice that by the observation of Miller and Nies [18], the constructed U cannot be universal by adjunction since $\Omega_U[\{x\}]$ is rational.

The following proposition shows the existence of an infinite co-r.e. set X such that for any two elements $x, y \in X$ with $x < y$, the Kolmogorov complexity of y and beyond is guaranteed to be much larger than the one of the strings x and the elements smaller than x. The basic idea of the construction is to check in every stage for every current elements x, y with $x < y$ whether the strings beyond y are much more complicated than x in the way specified below and to enumerate y into the complement of X whenever it turns out that this is not the case. Note that the construction of X does not make any requirements on U and works for every universal machine.

Proposition 4. *There is an infinite co-r.e. set X such that for no $x, y \in X$ with $x < y$ there are v, w such that $y \leq w$, $H(w) \leq v$ and $H(v) \leq x$.*

Proof. One constructs the complement Y of X as follows. Let Y_s denote all the elements enumerated into Y before stage s, so $Y_0 = \emptyset$. Let H_s denote a recursive approximation of H from above.

> At stage s, a number $y \in \{0, 1, \ldots, s\} - Y_s$ is enumerated into Y iff there are $x, v, w \leq s$ with $x < y$, $x \notin Y_s$, $y \leq w$, $H_s(w) \leq v$ and $H_s(v) \leq x$.

It is easy to see that the so constructed enumeration is recursive and thus X is a co-r.e. set. Furthermore, if $x, y \in X$ and $x < y$ there cannot be any v, w such that $y \leq w$, $H(w) \leq v$ and $H(v) \leq x$ since there is a state s where $H_s(v) \leq x$ and $H_s(w) \leq v$ and then y would be enumerated into Y at that stage s latest.

It remains to show that X is infinite. So assume by way of contradiction that X is finite and let a_0 be an upper bound of all elements of X. Then the following maxima and minima exist:

$$a_1 = \max\{u : H(u) \leq a_0\};$$

$$a_2 = \max\{u : H(u) \leq a_1\};$$
$$s = \min\{t : \{0, 1, \ldots, a_2\} \subseteq X \cup Y_t\};$$
$$y = \min\{z : z \notin X \cup Y_s\}.$$

By assumption y is enumerated into Y at some stage $t \geq s$ and so there are $x, v, w \leq t$ witnessing this fact in the sense that $x \notin Y_t$, $x < y$, $H_t(v) \leq x$, $H_t(w) \leq v$ and $y \leq w$. Since $H_t(v) \geq H(v)$ and $H_t(w) \geq H(w)$ one has $H(v) \leq a_0$, $v \leq a_1$, $H(w) \leq v \leq a_1$, $w \leq a_2$ and $y \leq a_2$ in contradiction to $\{0, 1, \ldots, a_2\} \subseteq X \cup Y_s \subseteq X \cup Y_t$ and $y \notin X \cup Y_t$. From this contradiction one can conclude that X is infinite. ∎

Using the above propositions, one obtains a partial result for Question 8.10 of Miller and Nies [18]. But this is not an answer of this question since Question 8.10 considers only machines which are universal by adjunction. Such machines are more difficult to handle.

Theorem 5. *There is a universal machine U and a co-r.e. set X such that $\Omega_U[X]$ is neither left-r.e. nor Martin-Löf random.*

Proof. Let U, H be as in Proposition 3 and X be as in Proposition 4 with the only exception that all elements x for which there is an $x' \leq x$ with $H(x') \geq \frac{x-1}{2}$ are removed from X. There are only finitely many such x, so this loss is not essential and X remains infinite.

First assume by way of contradiction that $\Omega_U[X]$ is left-r.e. via an approximation $b_0, b_1, \ldots$ and consider any $x \in X$. Let $a = \sum_{y \in X \cap \{0,1,\ldots,x\}} 2^{1-H(y)}$. One can compute from (x, a) numbers s, v_x such that s is the first number with $b_s > a$ and v_x the least number with $b_s > a + 2^{2-v_x}$.

Let y_x be the next element of X after x. Note that $\Omega_U[X] < a + 2^{2-H(y_x)}$ and thus $2^{2-H(y_x)} > 2^{2-v_x}$. It follows that $H(y_x) < v_x$.

By the choice of X and by $x \in X$, one has $H(x') < x/2$ for all $x' \leq x$. Thus one can compute a from a description of x and of the first $x/2$ bits of its binary representation. So $H(v_x) < x/2 + 2H(x) < x$ for all sufficiently large x. Using $H(v_x) < x$ and taking $w_x = y_x$, the numbers v_x and w_x witness that y_x is eventually enumerated into the complement of X according to the definition of X in Proposition 4. This contradiction to the choice of y_x gives that $\Omega_U[X]$ cannot be left-r.e. and this completes the proof.

Second it is shown that $\Omega_U[X]$ is not random. Note that if $x < y$ and $x, y \in X$ then $x < H(H(y))$ and thus $H(x) < x < H(y)$; the first relation $H(x) < x$ holds since all x with $H(x) \geq \frac{x-1}{2}$ had been removed from X. So $\Omega_U[X] = \sum_{x \in X} 2^{1-H(x)}$ satisfies that all ones in its binary representation correspond to some term $2^{1-H(x)}$ and that between two ones there is at least one zero, namely, the one corresponding to 2^{1-x}. Thus one knows that after every one in the binary representation comes a 0 and so $\Omega_U[X]$ is not Martin-Löf random. ∎

There might be an alternative approach to prove this result. If one succeeds to construct U, X such that $\Omega_U[X]$ is neither left-r.e. nor right-r.e., then $\Omega_U[X]$ is not Martin-Löf random: As $\Omega_U[X]$ is the difference of the two left-r.e. reals Ω_U and $\Omega_U[\overline{X}]$, this follows from a result of Rettinger and Zheng [23].

Becher, Figueira, Gregorieff and Miller [2] show that for every universal machine U and for each sufficiently small but positive recursive real number R there is a K-recursive set X such that $\Omega_U[X] = R$. If one can choose the universal machine freely then one can even get that the corresponding X is a co-r.e. set.

Recall that A is H-trivial iff there is a c such that $\forall n \; H(A(0) \ldots A(n)) \leq H(n) + c$. Hence, the prefix-free Kolmogorov complexity of an H-trivial real is as low as possible. Every H-trivial real is K-recursive and the class of H-trivial reals are closed under $\oplus$ and contain a nonrecursive r.e. set [9].

Proposition 6. *There is a universal machine U and an integer m such that for every H-trivial real R between 0 and 2^{-m} there is a co-r.e. set X with $R = \Omega_U[X]$.*

Proof. Given a universal machine $\tilde{U}$ which outputs within s steps only numbers smaller than 2^s, one can construct a new universal machine $\tilde{V}$ with the following property: If $\tilde{U}$ outputs on input p a number x after s steps then $\tilde{V}(p00) = x$ and $\tilde{V}(p1^k0) = 2^s \cdot 3^k$ for $k > 0$. Note that $\tilde{V}$ is also prefix-free. In particular the complexity $\tilde{H}$ based on $\tilde{V}$ has an approximation $\tilde{H}_s$ such that for almost all n there is an x such that $\tilde{H}_x(x) = \tilde{H}(x) = n$. By the way, this x is the largest number with $\tilde{H}(x) = n$. If one now constructs U from $\tilde{V}$ instead from $\tilde{U}$ as in Proposition 3, this property is preserved to U and the complexity H based on U: There is an approximation H_s such that the numbers

$$x_n = \max\{z : z = 0 \vee H(z) \leq n\}$$

satisfy $H_{x_n}(x_n) = H(x_n)$ for all n. Furthermore, $H(x_n) = n$ for almost all n; one now chooses the constant m for the proposition such that $m \geq 2$ and $\forall n \geq m\,(H(x_n) = n)$.

Let R be an H-trivial real with $0 < R < 2^{-m}$. The aim is now to build a co-r.e. set X such that

$$R = \sum_{r \in R} 2^{-1-r} = \sum_{x \in X} 2^{1-H(x)} = \Omega_U[X]$$

where this goal is by choosing $X \subseteq \{x_m, x_{m+1}, \ldots\}$ such that

$$x_n \in X \Leftrightarrow n - 2 \in R.$$

The further construction makes use of the fact that there is an r.e. H-trivial set $Q \geq_T R$ [20, Theorem 7.4]. This fact guarantees that R has a recursive approximation $R_0, R_1, \ldots$ such that the function c_R defined as

$$c_R(n) = \max\{s : s = 1 \vee \exists m < n\,(R_{s-1}(m) \neq R(m))\}$$

can be computed relative to Q. For all n let

$$y_n = \max\{z \leq c_R(n) : z = 0 \vee H_z(z) = n\}$$

Note that the sequence $y_0, y_1, \ldots$ can be computed relative to Q. Since Q is H-trivial and therefore H^Q differs from H only by a constant, one has that $H^Q(y_n)$ and $H(n)$ also differ at most by a constant. Thus, for almost all n, $H^Q(y_n) < H^Q(x_n)$ and $y_n < x_n$. For these n it holds that $R_{x_n}(n-2) = R(n-2)$. Without loss of generality one can assume this property for all $n \geq m$ since a finite modification of the approximation $R_0, R_1, \ldots$ would enforce it. After ensuring this property, one defines the co-r.e. set

$$X = \{x : H_x(x) \geq m \wedge R_x(H_x(x) - 2) = 1 \wedge \forall y > x\, \forall t\,(H_t(y) > H_x(x))\}.$$

Now the connection between X and R is verified. On one hand, consider any $x \in X$ and let $n = H(x)$. Then the condition $\forall y > x\, \forall t\,(H_t(y) > H_x(x))$ enforces that $H(y) > H(x)$ for all $y > x$ and thus $x = x_n$. Since $H_{x_n}(x_n) = H(x_n)$, one furthermore has that $n \geq m$. So $X \subseteq \{x_m, x_{m+1}, \ldots\}$. On the other hand, if $n \geq m$, then x_n satisfies $H_{x_n}(x_n) = H(x_n) = n$ and

$R_{x_n}(H_{x_n}(x_n)-2) = R(n-2)$. So one has for $n \geq m$ that $x_n \in X \Leftrightarrow n-2 \in R$. Since by choice of R no number below m is in R, the equivalence $x_n \in X \Leftrightarrow n-2 \in R$ holds for all n and one has the following equalities:

$$\begin{aligned}\Omega_U[X] &= \sum_{x \in X} R_x(H_x(x)-2)2^{1-H(x)} \\ &= \sum_{x \in X} R(H(x)-2)2^{1-H(x)} \\ &= \sum_{n=m}^{\infty} R(n-2)2^{1-n} = R.\end{aligned}$$

This completes the proof. ∎

3 Halting Probability and Truth-Table Reducibility

In Question 8.9, Miller and Nies [18] asked whether there are two different universal machines such that the corresponding Ω numbers are not tt-equivalent. One can show that the tt-degrees of the Ω numbers contain even an infinite antichain. Note that the resulting universal machines are universal by adjunction whenever the starting machine U is universal by adjunction; thus Question 8.9 is answered completely by the below theorem.

Theorem 7. *Given a universal machine U, one can construct a whole sequence $U_1, U_2, \ldots$ of universal machines and an r.e. real X such that the Ω numbers $\Omega_{U_1}, \Omega_{U_2}, \ldots$ given as $\Omega_{U_m} = 2^{-1} \cdot \Omega_U + 2^{-m}X$ form an antichain.*

Proof. Let X be creative subset of the odd natural numbers and define U_m such that

$$U_m(ap) = \begin{cases} U(p) & \text{if } a = 0 \text{ and } U(p)\downarrow; \\ 0 & \text{if } ap = 1^{m+n-1}0 \text{ for an } n \in X; \\ \uparrow & \text{otherwise.} \end{cases}$$

Then one easily sees that $\Omega_{U_m} = 2^{-1} \cdot \Omega_U + 2^{-m}X$ and that U_m is prefix-free. Furthermore, all programs of U are translated into U_m by placing a 0 in front, hence U_m is universal. Finally it is easy to see that $\Omega_{U_i} - \Omega_{U_j} = (2^{-i} - 2^{-j})X$ for all i, j. Assume that $i \neq j$. Then $X = (\Omega_{U_i} - \Omega_{U_j})/(2^{-i} - 2^{-j})$ and X is truth-table reducible to Ω_{U_j} whenever $\Omega_{U_i} \leq_{tt} \Omega_{U_j}$ and to Ω_{U_i} whenever $\Omega_{U_j} \leq_{tt} \Omega_{U_i}$. But since X is creative and not truth-table reducible to a Martin-Löf random set [3, 4], it cannot happen that Ω_{U_i} and Ω_{U_j} are tt-comparable and $\Omega_{U_1}, \Omega_{U_2}, \ldots$ is an infinite antichain for truth-table reducibility. ∎

A related question is whether Ω numbers of incomparable tt-degrees form a minimal pair. This is still unknown, but at least one knows that they do not have to.

Theorem 8. *There are Ω numbers which are incomparable but do not form a minimal pair with respect to truth-table reducibility.*

Proof. Recall that a set Z is identified with the real $\sum_{n \in Z} 2^{-1-n}$ and by the same way any real is identified with a set. Furthermore, let $\Omega_{U,s}$ be an approximation of Ω_U from below.

Taking any universal machine U, there are a creative set X, a recursive set Y, an enumeration of X with convergence modulus c_X and an enumeration from the left of Ω_U with convergence modulus c_{Ω_U} such that

- $\Omega_U + Y < 1$ and $X \subseteq Y$;
- For all $x, y \in Y$ with $x < y$ there is a $z \notin \Omega_U$ with $x < z < y$;
- For all n, $c_X(n) \leq c_{\Omega_U}(n)$.

There is a prefix-free machine V with $\Omega_V = \Omega_U + X$; note that Ω_V is Martin-Löf random since it is the sum of two left-r.e. reals with one of them being Martin-Löf random. Assume that $x \in Y$. Then the digits of Ω_V up to the one corresponding to x depend only on the $X \cap \{0, 1, \ldots, x\}$ but not on $X \cap \{x+1, x+2, \ldots\}$ since the least member of that set is at least y and there is a 0 in Ω_U between x and y. Taking an enumeration of V which satisfies

$$\Omega_{V,s} = \Omega_{U,s} + X_s$$

one has that $c_{\Omega_V}(n) = c_{\Omega_U}(n)$ for all n. So these two functions are identical. The graph

$$\text{graph}(c_{\Omega_U}) = \{(n, c_{\Omega_U}(n))\}$$

is not recursive but graph(c_{Ω_U}) is tt-reducible to both, Ω_U and Ω_V: one can easily check whether for a pair (n, t) the value t is equal to the minimal s such that $\Omega_{U,s}$ and Ω_U coincide on the first n bits. On the other hand, Ω_U and Ω_V are tt-incomparable by the methods given above and so they are an example of tt-incomparable Ω numbers which do not form a minimal pair for tt-reducibility. ∎

Call a tt-reduction M order-preserving iff $M(X) \leq M(Y)$ for all reals X, Y with $X \leq Y$. The next result shows that for order-preserving tt-reducibility, all Ω numbers are either equivalent or incomparable; thus together with the previous result one has that their degrees form an infinite antichain.

Proposition 9. *Let U, V be universal machines. If $\Omega_U \leq_{tt} \Omega_V$ via an order-preserving tt-reduction then $\Omega_U \equiv_{tt} \Omega_V$ and the reverse tt-reduction is also order-preserving.*

Proof. Let M denote the order-preserving tt-reduction from Ω_U to Ω_V and let f be its recursive use. For every length n there are with respect to the ordering $<$ on real numbers a least set $X_n \subseteq \{0, 1, \ldots, f(2n)\}$ and a greatest set $Y_n \subseteq \{0, 1, \ldots, f(2n)\}$ such that both $M(X_n)$ and $M(Y_n)$ coincide with Ω_U on the first $2n$ bits. If n is sufficiently large, then there is no finite set $F \subseteq \{0, 1, \ldots, n\}$ such that $X_n < F \leq Y_n$ as real numbers since otherwise the first $2n$ bits of Ω_U could be computed from the $n+1$ bits coding F and some code for n using $H(n)$ bits in contradiction to Ω_U being random. As a consequence, the first n bits of X_n and Y_n must be the same and both coincide with those of Ω_V. Since only the first $2n$ bits of Ω_U are relevant for these considerations, one can, for almost all n, compute the first n bits of Ω_V from the first $2n$ of Ω_U and patch the remaining finitely many cases from a table. So there is a tt-reduction from Ω_V to Ω_U. It is easy to see that this reverse tt-reduction is also order-preserving. ∎

4 On low for Ω sets

In the following let T be an infinite recursive tree, that is, let $T \subseteq \{0, 1\}^*$ be recursive and have the property that $\sigma \in T$ whenever $\sigma\tau \in T$ for $\sigma, \tau \in \{0, 1\}^*$. A set A is an infinite branch of T iff all nodes of the form $A(0)A(1)\ldots A(n)$ are members of T. For recursive trees the effective analogue of König's Lemma fails and T may fail to have recursive infinite branches. But several results guarantee that some infinite branches of T are near to being recursive: Jockusch and Soare [14] showed that every infinite recursive tree has infinite branches of low degree and infinite branches of hyperimmune-free degree; Downey, Hirschfeldt, Miller and Nies [10] showed that every infinite recursive tree has an infinite branch which is low for Ω. If a tree has only recursive branches, then all of its branches are low for Ω. But one might ask under which conditions all infinite branches of an infinite recursive tree are low for Ω. The next result shows that this cannot happen if all infinite branches are nonrecursive; indeed in that case the infinite branches of hyperimmune-free degree are not low for Ω.

Theorem 10. *If T is an infinite recursive binary tree without infinite recursive branches then every A on T which is low for Ω does also have hyperimmune Turing degree.*

Proof. Assume by way of contradiction that T is an infinite recursive binary tree without infinite recursive branches and A is an infinite branch of T which is low for Ω and has hyperimmune-free Turing degree. Let U be a prefix-free universal oracle Turing machine, that is, for every oracle A and every prefix-free partial-recursive V^A there is a constant c such that $H_U^A(x) \leq H_V^A(x) + c$ for all x in the range of V^A. From now on, let H^A denote H_U^A and let H_s^A be an A-recursive approximation from above to H^A where for the s-th approximation the oracle A is queried only below s.

Since A is low for Ω, there is a constant c such that $H^A(\Omega(0)\ldots\Omega(n)) \geq n-c$ for all n. Let Ω_s be an approximation of Ω from the left. For every n there is a $s \geq n$ such that

$$\forall m \leq n\,(H_s^A(\Omega_s(0)\ldots\Omega_s(m)) \geq m-c)$$

and since A has hyperimmune-free Turing degree, there is a recursive function f such that this s is between n and $f(n)$ for all n. Now one can construct a new recursive binary tree $S \subseteq T$ such that

an infinite branch B of T is also an infinite branch of S iff
$\forall n\,\exists s \in \{n, n+1, \ldots, f(n)\}\,\forall m \leq n\,(H_s^B(\Omega_t(0)\ldots\Omega_s(m)) \geq m-c)$.

The listed condition is a Π_1^0 condition since the first quantifier is unbounded and universal and all other quantifiers have recursively bounded range. Thus there is such a recursive tree S. Furthermore, A is on this tree S and all infinite branches on the tree S are low for Ω.

Since $S \subseteq T$, S does not have any infinite recursive branch. It follows that no infinite branch of S is isolated, indeed through every node of S go either no or uncountably many infinite branches. Using the oracle K one can decide for any node which of these two cases applies. If a set B is not H-trivial then there is for every n a number u_n such that $H^B(u_n)+n < H(u_n)$. Since there are only countably many sets which are H-trivial, one can construct a sequence $\sigma_0, \sigma_1, \ldots$ of nodes of S with the following properties:

- σ_n is above all nodes σ_m with $m < n$;
- there are infinitely many nodes in S above σ_n;
- if B is an infinite branch through σ_n then there is a number u_n such that $H^B_{|\sigma_n|}(u_n) + n < H(u_n)$.

The sequence $\sigma_0, \sigma_1, \ldots$ defines a K-recursive branch D of S. Furthermore, $H^D(u_n)+n < H(u_n)$ for each n since D and the B above both extend σ_n and the approximation $H^B_{|\sigma_n|}(u_n)$ evaluates B only at places which actually belong to the domain of the string σ_n. It follows that D is low for Ω, K-recursive and not H-trivial. But this contradicts a result of Hirschfeldt, Nies and Stephan [12] which says that such a set D does not exist. ∎

Nies, Stephan and Terwijn [21] showed that every set which is Martin-Löf random and low for Ω is already Martin-Löf random relative to K and thus does not have hyperimmune-free degree. Since not only the Martin-Löf random sets but also all sets of diagonally nonrecursive and hyperimmune-free degree are on a recursive tree without recursive infinite branches, one obtains as a corollary the following stronger result. Recall that a set A has diagonally nonrecursive degree iff there is a total function $f \leq_T A$ such that $f(x) \neq \varphi_x(x)$ whenever $\varphi_x(x)$ is defined.

Corollary 11. *If a set has diagonally nonrecursive degree and is low for Ω then it has also hyperimmune degree.*

Although Question 8.1 is not completely answered, one can use Theorem 10 to get a weaker but related result. Here a Turing degree is hyperimmune relative to K if it can compute a function which is not dominated by any K-recursive function.

Theorem 12. *Let A be a nonrecursive set such that the Turing-degree of A is hyperimmune-free unrelativized and the Turing-degree of $A \oplus K$ is hyperimmune-free relative to K. Then A is on a recursive binary tree without infinite recursive branches; in particular, A is not low for Ω.*

Proof. Assume that A is a nonrecursive set such that the Turing-degree of A is hyperimmune-free unrelativized and the Turing-degree of $A \oplus K$ is hyperimmune-free relative to K. Since A is not recursive, the function

$$F(e) = \min\{x : A(x) \neq \varphi_e(x) \vee \varphi_e(x) \uparrow\}$$

is total. F can be computed relative to $A \oplus K$. There is a K-recursive function majorizing F which is recursively approximated by a function G_s. Let G be the maximum of the G_s so that

$$G(e) = \max\{G_s(e) : s = 0, 1, \ldots\} \geq F(e)$$

holds for all e. Now also G majorizes F. One can compute relative to A for every e, s a number t such that $\Psi^A(e, s, t)$ is true where the formula Ψ is given as follows.

$\Psi^A(e, s, t)$ is true iff one of the following three conditions holds:
- $\exists u \leq t\,(G_{s+u}(e) > \max\{G_0(e), G_1(e), \ldots, G_s(e)\})$;
- $\exists r \leq G_s(e)\,(\varphi_{s+t}(r) \uparrow)$;
- $\exists r \leq G_s(e)\,(\varphi_{s+t}(r) \downarrow \neq A(r))$.

Since A has hyperimmune-free Turing degree there is a recursive function Q giving an upper bound on the t computed from e, s relative to A. Now one can define an infinite recursive binary tree T such that B is an infinite branch of T iff

$$\forall e \forall s \exists t \leq Q(e, s)\,(\Psi^B(e, s, t)).$$

It is clear that A is on the tree T. If B is recursive then there is an index e such that φ_e is the characteristic function of B. Let s be so large that there is a $u \leq s$ with $G_u(e) = G(e)$ and $\varphi_{e,s}(x)$ being defined for all $x \leq G(e)$. Then there is no $t \leq Q(e, s)$ such that $\Psi^B(e, s, t)$ is defined and thus B is not an infinite branch of T.

So it follows that A is an infinite branch of a recursive binary tree without recursive infinite branches. Since A has hyperimmune-free Turing degree, it follows from Theorem 10 that A is not low for Ω. ∎

The following example shows that this result does not capture all hyperimmune-free Turing degrees. The basic idea of the construction is from Miller and Martin [19]; in detail the construction is the combination of the just mentioned idea in the way it is presented by Odifreddi [22, Propositions V.5.5 and V.5.6] in combination with forcing out of a given list of trees. The method of Miller and Martin was improved by Jockusch [13] who built a nonrecursive set of biimmune-free degree. Note that every biimmune-free is also hyperimmune-free but not vice versa [13].

Example 13. *Given a list $T_0, T_1, \ldots$ of trees without recursive infinite branches, there is a nonrecursive set A of biimmune-free degree such that A is not on any of these trees.*

Proof. A perfect recursive tree is a nonempty recursive tree $T \subseteq \{0,1\}^*$ such that at least two infinite branches go through every node. Now one constructs a sequence of recursive perfect trees $P_0, P_1, \ldots$ such that P_0 is the full tree $\{0,1\}^*$ and P_{e+1} is chosen from P_e as follows:

– choose a node $\sigma_e \in P_e$ such that $\sigma_e \notin T_e$ and the e-th partial-recursive function does not compute an extension of σ_e;
– choose $P_{e+1} \subseteq P_e$ such that σ_e is below all branching nodes in P_{e+1} and either φ_e^A is partial or φ_e^A is not $\{0,1\}$-valued or φ_e^A is not biimmune for all infinite branches of P_{e+1}.

Since every perfect recursive tree has infinite recursive branches, $P_e \not\subseteq T_e$ and so there are several incomparable nodes of P_e outside T_e. At least one of them is not extended by φ_e and so one can choose σ_e according to the given requirements.

For the second step, the reader is referred to Jockusch's construction [13] which is not reproduced here; it is an improved method of the corresponding one for hyperimmune-free degrees [22, Proposition V.5.5].

The set A is then the unique infinite branch which is on all the trees P_e. The construction gives that A has biimmune-free degree [13] and furthermore A is not on any tree T_e. ∎

So known methods easily give the existence of a set of hyperimmune-free degree whose jump has hyperimmune degree relative to K. Indeed the function $F \leq_T A \oplus K$ given by

$$F(e) = \min\{x : A(x) \neq \varphi_e(x) \vee \varphi_e(x) \uparrow\}$$

is not bounded by any total K-recursive function. Downey and Milete [11] announced the following result which shows that Theorem 12 is not directly implied by a restriction on the jumps of the hyperimmune-free degrees.

Theorem 14 (Downey and Mileti). *There a nonrecursive set of hyperimmune-free degree such that the degree of its jump is hyperimmune-free relative to K.*

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