Recursion Theoretic Properties of Frequency Computation and Bounded Queries *

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Abstract

The notion of *frequency computation* captures the class Ω of all sets A such that for some n, the n-fold characteristic function of A can be computed with fewer than n errors. We consider the recursion theoretic properties of Ω with special emphasis on recursively enumerable sets.

1 Introduction

In this paper we study properties of sets that possess an "effective structure" such that their characteristic function can be recursively approximated in a certain sense. We consider approximations given by frequency computation and bounded queries as explained below.

The notion of *frequency computation* was introduced by Rose [31] in the early sixties. For natural numbers $m, n \geq 1$, $m \leq n$, a set A is called (m,n)-recursive (in short $A \in \Omega(m,n)$) iff there is a recursive function $f : \omega^n \to \{0,1\}^n$, mapping n-tuples of numbers to n-tuples of bits, such that for any n pairwise distinct numbers $x_1, \ldots, x_n$:

$$f(x_1, \ldots, x_n) = (b_1, \ldots, b_n) \Rightarrow |\{i : \chi_A(x_i) = b_i\}| \geq m.$$

Informally, the characteristic function of A can be computed approximately on every n-tuple of pairwise distinct numbers such that at least m components of the approximation are correct, i.e., there are at most $n - m$ errors. There exist nonrecursive (1,2)-recursive sets. (For instance, take any nonrecursive semirecursive set.) Trakhtenbrot [33] proved that for $m > \frac{n}{2}$ every (m,n)-recursive set is already recursive (see [17, 24] for a recent survey and related results).

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A related notion emerged from the recent work on *bounded queries* (see [3, 4, 7, 16]). For $m \le 2^n$ we call a set A *strongly* (m,n)*-verbose* (in short $A \in V_S(m,n)$) iff there is a recursive function f such that for all $x_1, \dots, x_n$:

$$|D_{f(x_1,\dots,x_n)}| \le m \ \wedge \ \langle \chi_A(x_1), \dots, \chi_A(x_n) \rangle \in D_{f(x_1,\dots,x_n)}.$$

(D_i denotes the i-th finite set in a canonical indexing of all finite sets.) Informally, we can compute a set of at most m possibilities for $(\chi_A(x_1), \dots, \chi_A(x_n))$. The following alternative characterization follows from [4, Lemma 7]: A is strongly $(2^k, n)$-verbose iff there is an oracle X and an oracle Turing machine $M^{(-)}$ that computes a total function for every oracle, and M^X computes $F_n^A := \lambda x_1, \dots, x_n.\ (\chi_A(x_1), \dots, \chi_A(x_n))$ by making at most k queries to X.

Every set is strongly $(2^n, n)$-verbose (let $D_{f(x_1,\dots,x_n)} = \{\langle b_1, \dots, b_n \rangle : (b_1, \dots, b_n) \in \{0,1\}^n\}$). Beigel [3] (see [4, Theorem 9]) proved that every strongly (n,n)-verbose set is recursive. This is optimal since every semirecursive set is strongly $(n+1, n)$-verbose. It is easy to see that a set is $(1,n)$-recursive iff it is strongly $(2^n - 1, n)$-verbose. Therefore $\bigcup_{n \ge 1, 1 \le m \le n} \Omega(m,n) = \bigcup_{n \ge 1, m < 2^n} V_S(m,n)$. This class is denoted by Ω. In [7] the inclusion problem for V_S-classes is treated (i.e., the question for which m, n, m', n' the class $V_S(m,n)$ is contained in $V_S(m', n')$) and shown to be decidable. Partial results on the inclusion problem for frequency computation have been obtained in [14, 25].

In this paper we study the recursion theoretic properties of Ω. In particular, we show how well-known properties of r.e. sets as simplicity and completeness are related to frequency computation and verboseness. We obtain *quantitative* characterizations having a *complexity theoretic* flavor. Typically the relevant methods are a blend of combinatorics and recursion theory. It is characteristic for this field that often nonuniform proofs are required, which is already the case for Trakhtenbrot's Theorem, as Kinber noted [22]. In the present paper new nonuniform arguments are developed which may also be of interest from a methodological point of view.

Definitions and Notation. The recursion theoretic notation closely follows the books of Odifreddi [29] and Soare [32]. $\omega = \{0, 1, 2, \dots\}$. We use ":=" for "equal by definition". The following abbreviations are used: "$x_1 < \cdots < x_n \in E$" stands for "$x_1 < \cdots < x_n \wedge \{x_1, \dots, x_n\} \subseteq E$"; we often write $A(x)$ for $\chi_A(x)$; it is also convenient to identify finite sequences with their sequence numbers, e.g., we may write $(b_1, \dots, b_n) \in D_i$ instead of $\langle b_1, \dots, b_n \rangle \in D_i$.

The concatenation of strings v, w is denoted by $v \cdot w$ or simply by vw; $V_1 \cdot V_2 := \{u \cdot v : u \in V_1, v \in V_2\}$. For $v = (b_1, \dots, b_n) \in \{0,1\}^n$ and $i, i_1, \dots, i_j$ such that $1 \le i \le n$ and $1 \le i_1 \le \cdots \le i_j \le n$, $v[i]$ denotes the i-th component b_i, and $v[i_1, \dots, i_j]$ denotes the projection $(b_{i_1}, \dots, b_{i_j})$. $V[i_1, \dots, i_j] := \{v[i_1, \dots, i_j] : v \in V\}$, $\#v := |\{i : v[i] = 1\}|$.

φ_i is the i-th function in a standard enumeration of all partial recursive functions of one argument. $\varphi_{i,s}(x)$ denotes the result, if any, of performing s steps in the computation of $\varphi_i(x)$. $W_i = \mathrm{dom}(\varphi_i)$ is the i-th r.e. set. Let R_n denote the set of all total recursive n-place functions. $A \oplus B := \{2x : x \in A\} \cup \{2x+1 : x \in B\}$ is called the *join* of A and B. A set A is k-r.e. ($k > 0$) iff there is $h \in R_2$ such that $\chi_A(x) = \lim_{s \to \infty} h(x,s)$, $|\{s : h(x,s) \ne h(x,s+1)\}| \le k$, and $h(x,0) = 0$, for all x.

We are considering the following refinements of truth-table reduction (see [29] for further discussion). *Positive reducibility:* A is positively reducible to B ($A \leq_p B$) iff there is a recursive function f such that $x \in A \Leftrightarrow (\exists y)[y \in D_{f(x)} \ \wedge \ D_y \subseteq B]$, for all x. *Disjunctive reducibility:* A is disjunctively reducible to B ($A \leq_d B$) iff there is a recursive function f such that $x \in A \ \Leftrightarrow \ D_{f(x)} \cap B \neq \emptyset$, for all x. *Conjunctive reducibility:* A is conjunctively reducible to B ($A \leq_c B$) iff there is a recursive function f such that $x \in A \ \Leftrightarrow \ D_{f(x)} \subseteq B$, for all x.

A set $A \subseteq \omega$ is called *semirecursive* iff there is $h \in R_2$ such that for all x, y: $h(x,y) \in \{x,y\} \ \wedge \ (\{x,y\} \cap A \neq \emptyset \Rightarrow h(x,y) \in A)$. It easily follows that every semirecursive set is $(1,2)$-recursive. Equivalently, a set is semirecursive iff it is an initial segment of a recursive linear ordering of ω (cf. [29, p. 295]). From this characterization it easily follows that every semirecursive set is strongly $(n+1,n)$-verbose for all $n > 0$.

The following definitions and results are from [4]: A set A is called *terse* iff for all $n > 0$, F_n^A is not computable with fewer than n queries to A. A set A is called *superterse* iff for all $n > 0$, there is no oracle X such that F_n^A is computable with fewer than n queries to X. The halting problem K is not terse (F_n^K can be computed with $\lceil \log_2(n+1) \rceil$ queries to K) and every nonrecursive tt-cylinder is superterse [4, Theorem 14]. For semirecursive sets A, F_n^A can be computed with $\lceil \log_2(n+1) \rceil$ queries to A (using binary search w.r.t. a recursive linear ordering). Beigel's Nonspeedup Theorem states that for any nonrecursive set A the computation of F_n^A requires at least $\lceil \log_2(n+1) \rceil$ queries regardless of the oracle.

2 General Properties of Ω

In this section we consider properties that hold for all sets in Ω. First we recall the known facts.

It follows immediately from the definition that $\Omega(m,n)$ is closed under complement and 1-reduction, and that $\Omega(1,n)$ is closed under m-reduction. Dëgtev [14] noted the following easy properties: $\Omega(m,n) \subseteq \Omega(m,n+1)$; $\Omega(m+1,n+1) \subseteq \Omega(m,n)$; $\Omega(m,n) \cap \Omega(h,k) \subseteq \Omega(m+h,n+k)$. Every tt-degree contains a semirecursive set [19, Theorem 3.6], i.e., a set in Ω. On the other hand, as we mentioned above, every nonrecursive tt-degree contains a superterse set which of course is not in Ω.

Dëgtev proved that there is a proper hierarchy of (m,n)-recursive sets if we let the number of errors $n - m$ increase:

Fact 2.1 [14, Theorem 1] *For any numbers $h, k, m, n \geq 1$ such that $2h \leq k$, $2m \leq n$, and $k - h < n - m$ there exists an r.e. set $A \in \Omega(m,n) - \Omega(h,k)$.*

For $1 \leq k \leq n$, let $S(n,k) = \sum_{i=0}^{k-1} \binom{n}{i}$. For fixed k, $S(n,k)$ is a polynomial in n of degree $k-1$. The following fact first appeared in [3], see also [7, Section 2] and the discussion following Proposition 2.4.

Fact 2.2 *Let $1 \leq k \leq n$. Then $\Omega(1,k) \subseteq V_S(S(n,k),n)$. Note the special cases $\Omega(1,2) \subseteq V_S(n+1,n)$, $\Omega(1,3) \subseteq V_S(1+\frac{n(n+1)}{2},n)$.*

Theorem 2.3 *Ω is closed downwards under btt-reductions. Ω is a Boolean algebra (w.r.t. inclusion). Ω is closed under join and Cartesian product.*

Proof: a) Closure under *btt*-reductions: Assume we are given $A, B \subseteq \omega$, and $k, m \geq 1$ such that A is $(1,k)$-recursive and $B \leq_{btt(m)} A$. We show that there exists n such that B is $(1,n)$-recursive.

As $B \leq_{btt(m)} A$ there exist m total recursive functions $f_1, \ldots, f_m$ and a recursive function $g : \omega^m \to \{0,1\}$ such that $\chi_B(x) = g(\chi_A(f_1(x)), \ldots, \chi_A(f_m(x)))$, for all x. Choose n large enough such that $S(nm,k) < 2^n$. By Fact 2.2, A is strongly $(S(nm,k), nm)$-verbose. Thus, we get for every n-tuple $x_1, \ldots, x_n$ at most $S(nm,k)$ possibilities for the nm-vector

$$(\chi_A(f_1(x_1)), \ldots, \chi_A(f_m(x_1)), \ldots, \chi_A(f_1(x_n)), \ldots, \chi_A(f_m(x_n))).$$

Consequently, we can compute a set of at most $S(nm,k)$ possibilities for F_n^B. So B is strongly $(2^n - 1, n)$-verbose, i.e., B is $(1,n)$-recursive.

b) If $A \in \Omega(1,n)$ and $B \in \Omega(1,k)$ then $A \oplus B \in \Omega(1, n+k-1)$. Therefore Ω is closed under join.

c) Since $A \times B, A \cap B \leq_{btt(2)} A \oplus B$ it follows from a), b) that Ω is closed under Cartesian product and intersection. We already noted that Ω is closed under complement. Thus, Ω is a Boolean algebra (under inclusion). ∎

The Boolean algebra generated by the semirecursive sets is a subalgebra of Ω. By the following proposition and the fact that Ω contains cohesive sets (Theorem 4.4) it follows that it is a *proper* subalgebra of Ω. The proposition generalizes [19, Corollary 4.5].

Proposition 2.4 *No Boolean combination of semirecursive sets is strongly hyperimmune.*

Proof: Since semirecursive sets are closed under complement any Boolean combination of semirecursive sets can be written as the finite union of finite intersections of semirecursive sets. Therefore it suffices to show that the intersection of finitely many semirecursive sets is not strongly hyperimmune.

Suppose for a contradiction that there is a strongly hyperimmune intersection of finitely many semirecursive sets. Let k be minimal such that there exists a strongly hyperimmune set A with $A = A_1 \cap \cdots \cap A_k \cap C$ where each A_i is semirecursive and C is a recursive set. Choose recursive linear orderings $\prec_i$ such that A_i is an initial segment of $\prec_i$. It follows that for each i and each $x \in A$, $\{x' \in A : x' \prec_i x\}$ is finite: Otherwise we could replace A by its infinite subset $\{x' \in A : x' \prec_i x\}$ with the representation $A_1 \cap \cdots \cap A_{i-1} \cap A_{i+1} \cap \cdots \cap A_k \cap D$, an intersection of $k-1$ semirecursive sets and the recursive set $D = \{x' \in C : x' \prec_i x\}$. This would contradict the choice of k.

It follows that for every $x \in A$ there is $y \in A$ with $y > x$ and $x \prec_i y$ for $i = 1, \ldots, k$. Let $h(x)$ denote the maximal m such that there exist $x_1, \ldots, x_{m-1} \in C$ with $x_1 < \ldots < x_{m-1} < x_m := x$ such that $x_j \prec_i x_{j+1}$, for all i, j with $1 \leq j < m$, $1 \leq i \leq k$. h is a recursive function. If $x_1, \ldots, x_{m-1}$ witnesses that $h(x) = m$ then $h(x_j) = j$, for $1 \leq j < m$; furthermore, if $x \in A$ then $x_1, \ldots, x_{m-1} \in A$. If $x, y \in A$,

$x < y$, and $x \prec_i y$ for $i = 1, \ldots, k$ then $h(x) < h(y)$. It follows that $h(A) = \omega$. Hence the recursive sequence $\{h^{-1}(\{i\})\}_{i \in \omega}$ witnesses that A is not strongly hyperimmune, the final contradiction. ▮

There is a close connection between Ω and "Vapnik-Chervonenkis classes" as we now explain. This concept was introduced by Vapnik and Chervonenkis [35] in their work on uniformly approximating classes of events by their empirical distribution. Recently it has been applied in the theory of learning [8] and in computational geometry [18] (see also [2]).

Definition 2.5 [35] Let $\mathcal{S}$ be a class of subsets of ω. The *Vapnik-Chervonenkis (VC) dimension* of $\mathcal{S}$ is the maximal cardinality of a set $A \subseteq \omega$ such that for every $D \subseteq A$ there is $E \in \mathcal{S}$ with $D = A \cap E$. I.e., every subset of A can be obtained by intersecting A with a suitable set from $\mathcal{S}$. The VC dimension is infinite if a finite maximum does not exist.

Recall that $\mathcal{S}$ is called a Π^0_1 class iff there is a recursive tree $T \subseteq \{0,1\}^{<\omega}$ such that $\mathcal{S} = [T]$ where $[T]$ is the set of all infinite branches of T (see [32, VI.5.11]). The next result characterizes Ω in terms of Π^0_1 classes of finite VC dimension.

Theorem 2.6 *Let $d \in \omega$, $A \subseteq \omega$. $A \in \Omega(1, d+1)$ iff A is member of a Π^0_1 class $\mathcal{S} \subseteq \{0,1\}^\omega$ of VC dimension at most d. In particular, $A \in \Omega$ iff A is member of a Π^0_1 class $\mathcal{S} \subseteq \{0,1\}^\omega$ of finite VC dimension.*

Proof: ($\Rightarrow$) Assume that $A \in \Omega(1, d+1)$ via $f : \omega^{d+1} \rightarrow \{0,1\}^{d+1}$. Let $T \subseteq \{0,1\}^{<\omega}$ be the recursive tree T that consists of all strings $t \in \{0,1\}^{<\omega}$ that are *consistent* with f, i.e., for all $x_1 < \cdots < x_{d+1} < |t|$, $f(x_1, \ldots, x_{d+1})$ and $(t(x_1), \ldots, t(x_{d+1}))$ agree in at least one component. The set $[T]$ of all infinite branches of T is a Π^0_1 class of VC dimension at most d that contains A.
($\Leftarrow$) Given a Π^0_1 class $\mathcal{S} \subseteq \{0,1\}^\omega$ of VC dimension at most d. There is a recursive tree $T \subseteq \{0,1\}^\omega$ such that $[T] = \mathcal{S}$. Since $\mathcal{S}$ has VC dimension at most d, for all $x_1 < \cdots < x_{d+1}$ there exists $v \in \{0,1\}^{d+1}$ such that $v \neq (A(x_1), \ldots, A(x_{d+1}))$ for any infinite branch A of T. In other words, by Kőnig's Lemma, the subtree above any node $t \in T$ with $|t| = x_{d+1} + 1 \ \wedge \ (t(x_1), \ldots, t(x_{d+1})) = v$ is finite. Therefore we can effectively search for such a v and then define $f(x_1, \ldots, x_{d+1}) = (1 - v[1], \ldots, 1 - v[d+1])$ which agrees with every member of $\mathcal{S}$ in at least one component. ▮

In view of Theorem 2.6, Fact 2.2 and Theorem 2.3 are restatements of well-known facts on VC classes [2, Lemma 1.8, Propositions 2.4-2.6]. There are other facts that also have some interesting consequences for frequency computation. For instance from [18, Theorems 3.3, 3.8] one can deduce:

Fact 2.7 (a) *If $A \in \Omega(1, d)$ via f then for every $\epsilon > 0$ there is a constant $c > 0$ such that one can compute for any n and $x_1 < \cdots < x_n$ an approximation $(b_1, \ldots, b_n)$ of $F^A_n(x_1, \ldots, x_n)$ by c parallel queries to A such that at most ϵn components are incorrect. The algorithm is uniform in the index of f, and c can be computed from ϵ and d (independently of f).*
(b) *Let $k(d,n)$ be the greatest k such that $\Omega(1, d) \subseteq \Omega(k, n)$. Then $\frac{k(d,n)}{n}$ approaches $\frac{1}{2}$ (from below) as $n \rightarrow \infty$.*

Part (a) shows in a sense that every set in Ω is "approximately non-terse". This stands in contrast to Theorem 5.8 where we show that there exist terse sets in Ω.

In [25] we gave a direct proof of part (b) which we now sketch: Since $\Omega(1,d) \subseteq V_S(S(n,d),n)$ we need to find the least $r = r(n,d)$ such that any $S(n,d)$ points from $\{0,1\}^n$ can be covered by a Hamming sphere H of radius r. Each point excludes, as possible centers of H, just those points of Hamming distance greater than r. There are $\binom{n}{r+1} + \binom{n}{r+2} + \cdots + \binom{n}{n} = S(n, n-r)$ such excluded points. Hence it suffices to require that $S(n,d) \cdot S(n,n-r) < 2^n$ (in terms of coding theory we have applied the "sphere bound"). Using Stirling's formula one can show that $\frac{r(n,d)}{n}$ approaches $\frac{1}{2}$ from above as $n \to \infty$. Since $V_S(S(n,d),n) \subseteq \Omega(n - r(n,d), n)$ part (b) follows. $\square$

We now show that the bounds from Fact 2.2 are optimal.

Theorem 2.8 *For $1 \leq 2h \leq k$, let $t_{h,k}(n)$ denote the least m such that every (h,k)-recursive set is strongly (m,n)-verbose. Then $t_{h,k}(n) = S(n, k-h+1)$. Note that for fixed h, k, $t_{h,k}(n)$ is a polynomial in n of degree $k-h$.*

Proof: Let $l = k - h + 1$. As $\Omega(h,k) \subseteq \Omega(1,l)$ and $\Omega(1,l) \subseteq V_S(S(n,l),n)$ it remains to show that $\Omega(h,k) \not\subseteq V_S(S(n,l)-1,n)$. This proof is similar to the proof of Theorem 4.3, d) $\Rightarrow$ b) in [7]. Partition ω into disjoint adjacent intervals $\{I_j\}_{j\in\omega}$ with $|I_j| = n$, for all j. We construct a set A by recursive approximation and satisfy the following requirements (we write $\varphi_i(x_1,\ldots,x_n)$ for $\varphi_i(\langle x_1,\ldots,x_n\rangle)$).

$$(P_i)\ A \text{ is not strongly } (S(n,l)-1,n)\text{-verbose via } \varphi_i$$

Our construction will automatically produce a (h,k)-recursive set.

Construction

Stage 0: Let $A(x,0) = 0$, for all x. Let $p(i,0) = i$, for all i.

Stage $s+1$: If there is a least i such that $p(i,s) < s$, $I_{p(i,s)} = \{x_1 < \cdots < x_n\}$, $|D_{\varphi_{i,s}(x_1,\ldots,x_n)}| < S(n,l)$, and $(A(x_1,s),\ldots,A(x_n,s)) \in D_{\varphi_{i,s}(x_1,\ldots,x_n)}$ then choose $v \in \{0,1\}^n$ containing at most $l-1$ 1s such that $v \notin D_{\varphi_{i,s}(x_1,\ldots,x_n)}$ (note that v exists because there are $S(n,l)$ strings of length n that contain at most $l-1$ 1s). Furthermore, let $(A(x_1,s+1),\ \ldots, A(x_n,s+1)) = v$, and let $A(x,s+1) := 0$, for all $x > x_n$, let $p(j,s+1) := s+j$, for all $j > i$. (Note that the intervals $I_{p(j,s+1)}$ for $j > i$ are by now unused.) All other parameters remain unchanged.

End of construction

By induction on i it is easy to show that each i is chosen at only finitely many stages and consequently (P_i) is satisfied. If $A(x,s+1)$ is reset from 1 to 0 then it is also thrown out of any diagonalization interval (by the way in which $p(j,s+1)$ is updated), thus $A(x,s') = 0$ for all $s' > s$. This proves that A is 2-r.e. We now show that A is (h,k)-recursive. For given $x_1 < \cdots < x_k$ let $s = x_k$ and define

$$f(x_1,\ldots,x_k) := (A(x_1,s),\ldots,A(x_h,s),0,0,\ldots,0).$$

We show that at least h components of $(A(x_1),\ldots,A(x_k))$ agree with the corresponding components of $f(x_1,\ldots,x_k)$. Let I denote the interval to which x_h belongs. If some interval to the left of I changes after stage s then $A(x_i) = 0$ for $i = h,\ldots,k$, and we get at least $k - h \geq h$ correct components. If no interval to the left of I changes after stage s but I changes after stage s, then $A(x, s) = 0$ for all $x \in I$ and at most $l - 1$ 0s in I are changed to 1s, also $A(x_i, s+1) = 0 = A(x_i)$ for all x_i to the right of I. This gives at most $l - 1$ errors in I, so at least $k - (l - 1) = h$ components are correct (note that the i-th component is correct for $x_i \notin I$). In the remaining case, when only intervals to the right of I may change after stage s, the first h components are correct. It follows that A is (h, k)-recursive. ∎

Remark: Can we modify the construction such that A becomes r.e.? This appears to be difficult. Indeed we shall show that Theorem 2.8 is no longer true when restricted to r.e. sets (see Theorem 3.10). □

Let $p(n)$ be the least number such that there exist T-incomparable sets A, B and $A \oplus B \in V_S(p(n), n)$. In [7, Theorem 3.3] it is shown that $p(n) = (\lceil \frac{n}{2} \rceil + 1)(\lfloor \frac{n}{2} \rfloor + 1)$. We now consider the analogous question for frequency computation. It is easy to see that the join of any two semirecursive sets is (m, n)-recursive if $\frac{m}{n} < \frac{1}{2}$. The next result gives a corresponding lower bound and generalizes [19, Corollary 6.3].

Theorem 2.9 *For all $n > 0$, if A, B are nonrecursive sets such that $A \oplus B$ is $(n, 2n)$-recursive then A and B are T-equivalent.*

Proof: Let A, B be given nonrecursive sets and assume that $A \oplus B$ is $(n, 2n)$-recursive via $f \in R_{2n}$. We distinguish two cases:
(a) For all x_1 there exist $y_2, \dots, y_{2n}$ such that $2x_1 < 2y_2 + 1 < \cdots < 2y_{2n} + 1$, $f(2x_1, 2y_2 + 1, \dots, 2y_{2n} + 1) = (b_1, \dots, b_{2n})$, and at most $n - 1$ of the $2n - 1$ equalities $\chi_B(y_i) = b_i$ (for $2 \leq i \leq 2n$) are correct. Then b_1 must be correct (i.e., $b_1 = \chi_A(x_1)$) since there is a total of at least n correct components.

Hence in this case A is recursive in B (for given x_1 search for suitable $y_2, \dots, y_{2n}$ as above using the B-oracle).
(b) There is x_1 such that for all $y_2, \dots, y_{2n}$ with $2x_1 < 2y_2 + 1 < \cdots < 2y_{2n} + 1$, if $f(2x_1, 2y_2 + 1, \dots, 2y_{2n} + 1) = (b_1, \dots, b_{2n})$ then at least n of the $2n - 1$ equalities $\chi_B(y_i) = b_i$ are correct.

In this case B is $(n, 2n - 1)$-recursive and hence recursive by Trakhtenbrot's Theorem. But this contradicts the hypothesis on B.

We have shown that A is recursive in B. By symmetry, B is recursive in A which completes the proof. ∎

Remark: The conclusion of Theorem 2.9 cannot be strengthened to "*wtt*-equivalent" since one can construct r.e. *wtt*-incomparable sets A, B such that $A \oplus B$ is semirecursive. This has been observed by Cholak and Downey [10, p. 638]. They ask whether there are r.e. sets A and B such that A and B have incomparable Turing degree and $\overline{A}$, $\overline{B}$, and $\overline{A \oplus B}$ are all retraceable. A negative answer follows from Theorem 2.9 and the fact that every co-r.e. retraceable set is semirecursive [29, III.5.6]. Owings [30] has also answered this question using a different approach.

The next definition is due to Trakhtenbrot [34]: $A \subseteq \omega$ is *autoreducible* iff $\chi_A(x)$ is A-recursively computable without ever asking "$x \in A$?".

Theorem 2.10 *Every set in Ω is autoreducible.*

Proof: We show by induction on n that every $(1,n)$-recursive set is autoreducible. The case $n = 1$ is trivial. For the inductive step assume that A is $(1,n+1)$-recursive via $f \in R_{n+1}$. We distinguish two cases:
(a) For all x_1 there are $x_2 < \cdots < x_{n+1}$ with $x_1 < x_2$ such that $f(x_1,\ldots,x_{n+1}) = (b_1,\ldots,b_{n+1})$ and none of the n equalities $\chi_A(x_i) = b_i$, for $2 \le i \le n+1$, is correct. Then $\chi_A(x_1) = b_1$ since at least one component is correct.

In this case A is autoreducible (on input x_1 search for suitable $x_2,\ldots,x_{n+1} > x_1$ as above; the oracle is queried only for numbers greater than x_1).
(b) There is x_1 such that for all $x_2 < \cdots < x_{n+1}$ with $x_1 < x_2$ and $f(x_1,\ldots,x_{n+1}) = (b_1,\ldots,b_{n+1})$, at least one of the n components $(b_2,\ldots,b_{n+1})$ is correct.

In this case it follows that A is $(1,n)$-recursive and so is autoreducible by the induction hypothesis. ▮

Remark: 1) The converse does not hold even for r.e. sets: K is autoreducible (since it is a cylinder) but $K \notin \Omega$, by Fact 3.1 (a).
2) By a similar proof it can be shown that any non-terse set is autoreducible: Use induction on n. In the inductive step we are given a machine M^A that computes F^A_{n+1} with at most n queries. Distinguish two cases. The first case is: For all x_1 there exist $x_2,\ldots,x_{n+1} > x_1$ such that $M^A(x_1,\ldots,x_{n+1})$ does not query x_1. Then A is autoreducible. In the second case it follows that F^A_n can be computed with at most $n-1$ queries to A and we are done by the induction hypothesis.
This argument also works for $Parity^A_n(x_1,\ldots,x_n) := x_1 + \cdots + x_n \pmod 2$ instead of F^A_n. So, if $Parity^A_n$ can be computed by fewer than n queries to A then A is autoreducible. □

3 General Properties of R.E. Sets in Ω

We begin with a couple of facts on what is known about r.e. sets in Ω.

Fact 3.1 (a) *$K \notin \Omega$.* (b) *For each $n \ge 1$ there is an r.e. set $A \in \Omega(1,n+1) - \Omega(1,n)$ and there is an r.e. set $B \in \Omega(n+1, 2(n+1)) - \Omega(n, 2n)$.* (c) *The r.e. sets in Ω can be arranged in a uniformly recursively enumerable sequence.* (d) *There exist r.e. sets $A, B \notin \Omega$ such that $\{C \text{ r.e.} : C \in \Omega\} = \{D \text{ r.e.} : D \le_m A \ \wedge \ D \le_m B\}$. In other words, the (nonprincipal) ideal of the r.e. m-degrees in Ω has an exact pair of r.e. m-degrees.* (e) *Every r.e. tt-degree contains an r.e. set in Ω.* (f) *There are nonrecursive r.e. tt-degrees such that all of their r.e. sets are contained in Ω.* (g) *Any nonrecursive r.e. wtt-degree contains an r.e. set $A \notin \Omega$.*

Proof: (a) This follows from (b), but there is an easy direct proof by contradiction using the recursion theorem, cf. [16, Theorem 23].
(b) This follows from Fact 2.1. Alternatively, the first part follows from results in [7] which imply that there is an r.e. set $A \in V_S(2^{n+1}-1, n+1) - V_S(2^n - 1, n)$.
(c) We can uniformly enumerate a sequence of sets $M_{i,j,n}$ such that $M_{i,j,n} = W_i$ if W_i is $(1,n)$-recursive via φ_j, and $M_{i,j,n}$ is finite otherwise. Since every finite set is in Ω

this gives us the required sequence.
(d) Since Ω is closed under join and m-reduction the r.e. degrees in Ω form an ideal of the r.e. m-degrees. This ideal is not principal: otherwise all sets in Ω are $(1,n)$-recursive for some n contradicting (b) since they are m-reducible to the $(1,n)$-recursive set generating the ideal. The existence of an exact pair follows from (a) and (c) using a general exact pair lemma for r.e. m-degrees (see [1, Theorem 6] for the wtt-version).
(e) Jockusch [19, Corollary 3.7] proved that every r.e. tt-degree contains an r.e. semirecursive set.
(f) Dëgtev [12, Theorem 3] proved that there exist nonrecursive r.e. tt-degrees that contain a single r.e. m-degree. By Jockusch's result (and since semirecursive sets are closed under $\leq_m$) this m-degree consists entirely of semirecursive sets.
(g) This can be shown by combining a straightforward diagonalization with permitting and coding, cf. the proof of [4, Theorem 11] for a similar construction. ∎

An r.e. set A is *r-separable* iff for any disjoint r.e. set B there is a recursive set C with $A \subseteq C \ \wedge \ C \cap B = \emptyset$. It is not difficult to show that every r.e. semirecursive set is r-separable. This is generalized by the next result:

Theorem 3.2 *Every r.e. set in Ω is r-separable.*

Proof: The following definition is due to Kinber [23]: If A, B are disjoint sets we say that they are $(1,n)$-separable iff there is $f \in R_n$, $f : \omega^n \to \{0,1\}^n$ such that for all $x_1, \ldots, x_n \in A \cup B$: there is $x_i \in A$ with $f(x_1, \ldots, x_n)[i] = 1$, or there is $x_i \in B$ with $f(x_1, \ldots, x_n)[i] = 0$. Note that A, B are $(1,1)$-separable iff they are recursively separable. This can be generalized for r.e. sets as follows:

Claim: If A, B are disjoint r.e. $(1,n)$-separable sets then they are recursively separable.
Proof: We already noted that the claim holds for $n = 1$. By induction we assume that it is true for n and prove it for $n+1$. Let A, B be disjoint r.e. sets that are $(1, n+1)$-separable via $f \in R_{n+1}$. We distinguish two cases:
1) There is z such that $\lambda x_1, \ldots, x_n.\ f(x_1, \ldots, x_n, z)[1, \ldots, n]$ $(1,n)$-separates A, B. Then we are done by the induction hypothesis.
2) Otherwise, for every z there exist $x_1, \ldots, x_n \in A \cup B$ such that $f(x_1, \ldots, x_n, z)[i] = 0$ iff $x_i \in A$, for $i = 1, \ldots, n$. Then for every z we enumerate A and B and search for $x_1, \ldots, x_n$ as above. Since f $(1, n+1)$-separates A, B we get:
$(f(x_1, \ldots, x_n, z)[n+1] = 1 \ \Rightarrow \ z \notin B)$ and $(f(x_1, \ldots, x_n, z)[n+1] = 0 \ \Rightarrow \ z \notin A)$.
Let $C = \{z : f(x_1, \ldots, x_n, z)[n+1] = 1\}$. C is a recursive superset of A that is disjoint from B, i.e., A, B are recursively separable. □

Now suppose that A, B are r.e. disjoint sets, and $A \in \Omega$, say A is $(1,n)$-recursive via f. Then A, B are $(1,n)$-separable via f and, by the claim, they are recursively separable. Thus A is r-separable. ∎

Remark: Using a straightforward diagonalization (see [23]) one can show that Theorem 3.2 does not generalize to non-r.e. sets. □

Corollary 3.3 *The r.e. sets in Ω are not closed under splitting.*

Proof: Let A be any r.e. nonrecursive set in Ω. Let (B_1, B_2) be a Friedberg splitting of A (see [32, X.2.1]). Then B_1 and B_2 are recursively inseparable. By Theorem 3.2 they do not belong to Ω. ▮

Let $\preceq$ denote the canonical ordering on $\{0,1\}^n$ defined as: $(a_1, \ldots, a_n) \preceq (b_1, \ldots, b_n)$ iff $a_i \leq b_i$ for $i = 1, \ldots, n$. The next result is an important technical tool that isolates a characteristic property of r.e. sets in Ω which will be exploited in Theorems 3.10 and 5.8. The result is also used in [7].

Theorem 3.4 *Assume that A is r.e. and strongly $(2^n - 1, n)$-verbose via $f \in R_n$. Then there is $g \in R_n$ such that for all $x_1, \ldots, x_n$:*
$D_{g(x_1,\ldots,x_n)} \subseteq D_{f(x_1,\ldots,x_n)}$, $(A(x_1), \ldots, A(x_n)) \in D_{g(x_1,\ldots,x_n)}$, and $D_{g(x_1,\ldots,x_n)}$ contains a least and a greatest element (w.r.t. $\preceq$).

Proof: Suppose that A is r.e. and strongly $(2^n - 1, n)$-verbose via $f \in R_n$. Let $\{A_s\}_{s\in\omega}$ be a recursive enumeration of A with $A_s \subseteq \{0, \ldots, s\}$. For any given $x_1, \ldots, x_n$ we can enumerate A until $(A_s(x_1), \ldots, A_s(x_n))$ appears in $D_{f(x_1,\ldots,x_n)}$. All $v \in \{0,1\}^n$ with $(A_s(x_1), \ldots, A_s(x_n)) \not\preceq v$ can then be omitted from $D_{f(x_1,\ldots,x_n)}$. Thus we can obtain $f' \in R_n$ such that $D_{f'(x_1,\ldots,x_n)} \subseteq D_{f(x_1,\ldots,x_n)}$, $(A(x_1), \ldots, A(x_n)) \in D_{f'(x_1,\ldots,x_n)}$, and $D_{f'(x_1,\ldots,x_n)}$ contains a least element (w.r.t. $\preceq$).
Claim: If A is strongly $(2^n - 1, n)$-verbose via $f \in R_n$ then there is $f' \in R_n$ such that for all $x_1, \ldots, x_n$: $(A(x_1), \ldots, A(x_n)) \in D_{f'(x_1,\ldots,x_n)}$, $D_{f'(x_1,\ldots,x_n)} \subseteq D_{f(x_1,\ldots,x_n)}$ and either $D_{f(x_1,\ldots,x_n)}$ contains a greatest element or $D_{f'(x_1,\ldots,x_n)}$ is a proper subset of $D_{f(x_1,\ldots,x_n)}$. In the latter case $D_{f'(x_1,\ldots,x_n)}$ is missing only one maximal element of $D_{f(x_1,\ldots,x_n)}$.
Proof: We say that a finite set $E \subseteq \{0, \ldots, s\}$ is *s-consistent* iff $A_s \subseteq E$ and $(E(x_1), \ldots, E(x_n)) \in D_{f(x_1,\ldots,x_n)}$, for all $x_1, \ldots, x_n \leq s$. We say that a finite set $S = \{y_1 < \cdots < y_k\}$ is *independent* if for all $v \in \{0,1\}^k$ and all $s > y_k$ there is an s-consistent set E such that $(E(y_1), \ldots, E(y_k)) = v$. Intuitively, no information about $(A(y_1), \ldots, A(y_k))$ is derivable from f and the enumeration of A.

$A \upharpoonright (s+1)$ is s-consistent for all s. Note that every subset of an independent set is independent. There is no independent set of cardinality n (because for all $y_1, \ldots, y_n$, there is $v \in \{0,1\}^n$ such that $v \notin D_{f(y_1,\ldots,y_n)}$). Thus we can choose an independent set S of maximal cardinality, say $S = \{y_1 < \cdots < y_k\}$, $k < n$. Note that $S \subseteq \overline{A}$.

Now consider any given $x_1, \ldots, x_n$. If $D_{f(x_1,\ldots,x_n)}$ does not contain a greatest element (w.r.t. $\preceq$) then it contains two maximal incomparable elements v_1 and v_2. Hence there exists a component i such that $v_1[i] < v_2[i]$.

If $x_i \in S$ then $A(x_i) = 0$, so v_2 is not correct. Hence we set $D_{f'(x_1,\ldots,x_n)} := D_{f(x_1,\ldots,x_n)} - \{v_2\}$.

If $x_i \notin S$ then $S \cup \{x_i\}$ is not independent (by the choice of S). Hence we can effectively find $s_0 > \max\{y_k, x_i\}$ and $v \in \{0,1\}^{k+1}$ such that for all s_0-consistent sets E we have:

$$(E(y_1), \ldots, E(y_k), E(x_i)) \neq v.$$

It follows that $(A(y_1), \ldots, A(y_k), A(x_i)) \neq v$. Choose $b \in \{1, 2\}$ such that $v[k+1] = v_b[i]$. Suppose for a contradiction $(A(x_1), \ldots, A(x_n)) = v_b$. Choose $s_1 > s_0$ such that $(A_{s_1}(x_1), \ldots, A_{s_1}(x_n)) = v_b$. As v_b is maximal it follows that $(E(x_1), \ldots, E(x_n)) = v_b$, for every s_1-consistent set E. By the independence of S there is an s_1-consistent set E with $(E(y_1), \ldots, E(y_k), E(x_i)) = v$. But then $E \upharpoonright (s_0 + 1)$ is s_0-consistent, a contradiction. As v_b is not correct we set $D_{f'(x_1,\ldots,x_n)} := D_{f(x_1,\ldots,x_n)} - \{v_b\}$. $\Box$

By repeated applications (at most $2^n - 2$ times) of the claim we finally get the required $g \in R_n$. ∎

Corollary 3.5 *An r.e. set is semirecursive iff it is* $(1, 2)$*-recursive.*

Proof: If A is r.e. and $(1, 2)$-recursive then it is strongly $(3, 2)$-verbose. By Theorem 3.4, A is strongly $(3, 2)$-verbose via $f \in R_2$ such that $D_{f(x_1,x_2)}$ contains a least and a greatest element (w.r.t. $\preceq$). Thus $\{(0, 1), (1, 0)\} \not\subseteq D_{f(x_1,x_2)}$, for all x_1, x_2. Therefore the function h defined as

$$h(x_1, x_2) := \begin{cases} x_1 & \text{if } (0,1) \notin D_{f(x_1,x_2)}; \\ x_2 & \text{otherwise} \end{cases}$$

witnesses that A is semirecursive. ∎

Remark: 1) By a straightforward application of the recursion theorem it can be shown that there is no uniform method to compute from an index of A and an index of f such that A is $(1, 2)$-recursive via f an index of h such that A is semirecursive via h. Hence the proof of Theorem 3.4 is necessarily nonuniform.
2) Using Corollary 3.5 it is shown in [7, Corollary 9.2] that an r.e. set A is semirecursive iff there is $n > 1$ such that $A \in V_S(\lfloor \frac{3}{2}n \rfloor, n)$. Furthermore, for any $m > \lfloor \frac{3}{2}n \rfloor$ there is an r.e. strongly (m, n)-verbose set which is not semirecursive.
3) By Fact 3.1 (b), there exist r.e. $(2, 4)$-recursive sets which are not semirecursive. However, we shall see that the r.e. $(n, 2n)$-recursive sets share many properties with semirecursive sets which do not generalize to r.e. (m, n)-recursive sets with $2m < n$. $\Box$

Beigel, Gasarch and Owings [6] investigated the behavior of nondeterministic query bounded Turing machines. They called A *locally n-subjective* iff $F^A_{n+1}(x_1, \ldots, x_{n+1})$ can be nondeterministically computed from A by asking at most n of the questions "$x_1 \in A$?", "$x_2 \in A$?", …, "$x_{n+1} \in A$?" (i.e., the outputs on all terminating computation paths are correct and there is a path on which at most n of the above questions are asked; the phrase "locally" refers to the fact that no queries other than $x_1, \ldots, x_{n+1}$ are allowed). They proved that any r.e. semirecursive set is locally 1-subjective, and that locally 1-subjective sets are closed downwards under $btt(1)$-reductions. Using Corollary 3.5 we can immediately answer two open questions from [6]:

Corollary 3.6 (a) *Any r.e. locally 1-subjective set is semirecursive.*
(b) *Any locally 1-subjective set is* $btt(1)$*-reducible to some r.e. semirecursive set.* [1]

[1] Hence the class of all locally 1-subjective sets is exactly KLOS (as defined in [6]).

Proof: (a) We shall show that any locally 1-subjective set is $(1,2)$-recursive. Then part (a) follows using Corollary 3.5. Assume that A is locally 1-subjective and let M be a nondeterministic Turing machine which demonstrates this. We may assume w.l.o.g. that M asks exactly one question on each terminating computation path. For any given x_1, x_2 one can effectively find $i \in \{1,2\}$, $(b_1, b_2) \in \{0,1\}^2$ and a computation path of $M(x_1, x_2)$ which terminates with output (b_1, b_2) such that on this path the query "$x_i \in A$?" is asked and the query answer b_i is used. Let $j = 3 - i$. Then one can conclude that $A(x_i) = 1 - b_i$ or $A(x_i) = b_i \ \wedge \ A(x_j) = b_j$. Hence if we define $f(x_i, x_j) = (1 - b_i, b_j)$ then at least one component is correct. Thus A is $(1,2)$-recursive.
(b) In [6, Corollary 20] it is shown that every locally 1-subjective set A is weakly r.e., in the sense that there is a recursive approximation $A(x,s)$ of χ_A such that:

$$A(x,s) \in \{0,1\} \ \wedge \ \lim_{s\to\infty} A(x,s) = \chi_A(x) \ \wedge \ |\{s : A(x,s) \neq A(x,s+1)\}| \leq 1.$$

Then $B := \{x : (\exists s)(A(x,s) \neq A(x,s+1))\}$ is r.e. and $A \equiv_{btt(1)} B$. As was remarked above, locally 1-subjective sets are closed under $btt(1)$-reductions. Thus B is locally 1-subjective and, using (a), B is semirecursive. ∎

Remark: Similarly as in part (a) one can show that every locally n-subjective set is $(1, n+1)$-recursive. □

Using Theorem 3.4 it can be shown that Theorem 2.8 does not hold for *r.e.* sets. For instance, we verified with computer assistance that every r.e. $(2,4)$-recursive set is already strongly $(12,5)$-verbose. Furthermore, as we will now show, restricted to r.e. sets $t_{k,2k}(n)$ is linearly bounded in n, for each fixed k, whereas in the general case $t_{k,2k}(n)$ is a polynomial in n of degree k. To this end we need to take a closer look at the combinatorics behind frequency computation.

If A is (m,n)-recursive via f then we can compute for any finite set D, $|D| = s \geq n$ the set $V \subseteq \{0,1\}^s$ of all vectors that are locally consistent with f, among them $\chi_A \restriction D$. The following definition of Dëgtev isolates the combinatorial property of these sets of vectors.

Definition 3.7 [14] Let $s \geq n \geq m \geq 1$. A set $V \subseteq \{0,1\}^s$ is called (m,n)-*admissible* iff for every n numbers $1 \leq x_1 < \cdots < x_n \leq s$ there exists a vector $(b_1, \ldots, b_n) \in \{0,1\}^n$ such that for every $v \in V$:

$$|\{i : v[x_i] = b_i\}| \geq m.$$

In other words, there exists a function $f : \{1, \ldots, s\}^n \to \{0,1\}^n$ such that for all pairwise distinct $x_1, \ldots, x_n \in \{1, \ldots, s\}$, $|\{i : v[x_i] = f(x_1, \ldots, x_n)[i]\}| \geq m$. In this case we say that V is (m,n)-admissible via f.

For $s < n$ we call $V \subseteq \{0,1\}^s$ (m,n)-admissible iff $V \cdot \{0^{n-s}\}$ is (m,n)-admissible.

If A is r.e. and (m,n)-recursive then we can compute for all $x_1 < \ldots < x_n$ a set $V(x_1, \ldots, x_n) \subseteq \{0,1\}^n$ which is (m,n)-admissible and contains $(A(x_1), \ldots, A(x_n))$. Applying Theorem 3.4 we may assume that $V(x_1, \ldots, x_n)$ contains a least and a greatest element. Now assume we are given a finite set $D = \{x_1 < \cdots < x_s\}$, $s > n$. Let $V := \{v \in \{0,1\}^s : (\forall 1 \le i_1 < \cdots < i_n \le s)(v[i_1,\ldots,i_n] \in V(x_{i_1},\ldots,x_{i_n}))\}$. V is (m,n)-admissible and contains $(A(x_1),\ldots,A(x_s))$. Let E be the set of all i, $1 \le i \le s$, such that $v[i] = v'[i]$, for all $v, v' \in V$. Hence for all $i \in E$ we already know the value of $\chi_A(x_i)$. Let v_0 [v_1] denote the vector of length s that is all zero [one] except at components $i \in E$ where it is $\chi_A(x_i)$. Consider the least and greatest elements v_l, v_g of any $V(x_{i_1},\ldots,x_{i_n})$ and observe that for all $i_j \notin E$ we have $v_l[j] = 0$ and $v_g[j] = 1$ (otherwise $v[j]$ would be unique for all $v \in V(x_{i_1},\ldots,x_{i_n})$ and hence $i_j \in E$). It follows that $V \cup \{v_0, v_1\}$ is also (m,n)-admissible which gives us the following lemma:

Lemma 3.8 *If A is (m,n)-recursive and r.e. then we can compute for each finite set of numbers D_x a set of vectors $V_x \subseteq \{0,1\}^{|D_x|}$ such that V_x is (m,n)-admissible, contains a least and a greatest element, and contains $\chi_A \restriction D_x$.*

The following combinatorial lemma shows that $(n,2n)$-admissible sets which possess a least and a greatest element are quite restricted. If $V \subseteq \{0,1\}^k$ and π is a permutation of $\{1,\ldots,k\}$ ("a permutation of the columns") let $\pi(V) = V[\pi(1),\ldots,\pi(k)]$. We call V *c-linear* iff $V \subseteq 1^*\{0,1\}^c 0^*$.

Lemma 3.9 *For each $n \ge 1$ there is a constant c_n such that for every $(n,2n)$-admissible set $V \subseteq \{0,1\}^k$ which contains a least and a greatest element there is a permutation π of the columns such that $\pi(V)$ is c_n-linear.*

Proof: By the finite Ramsey theorem there is a constant $\hat{c}_n$ such that for every function $h : \{1,\ldots,\hat{c}_n\}^{2n-1} \to \{0,1\}^{2n}$ there is $E \subseteq \{1,\ldots,\hat{c}_n\}$, $(b_1,\ldots,b_{2n}) \in \{0,1\}^{2n}$ such that $|E| = 8n$ and $h(x_1,\ldots,x_{2n-1}) = (b_1,\ldots,b_{2n})$, for all $x_1 < \cdots < x_{2n-1} \in E$. We shall prove the lemma for $c_n = 2\hat{c}_n + 2$.
Assume that $V \subseteq \{0,1\}^k$ is $(n,2n)$-admissible and contains a least element $v_l = 0^k$ and a greatest element $v_g = 1^k$. (The case when $v_l[i] = v_g[i]$ for some i's easily reduces to this one.) Let $f : \{1,\ldots,k\}^{2n} \to \{0,1\}^{2n}$ witness that V is $(n,2n)$-admissible. Let $i \sqsubseteq j$ iff $v[i] \ge v[j]$, for all $v \in V$ (i.e., we take the canonical ordering on the columns). We claim that for each i there are at most $\hat{c}_n$ columns j incomparable with i.

Otherwise, by the choice of $\hat{c}_n$ we find a set E, $|E| = 8n$, of columns that are incomparable with i and such that $f(i,i_2,\ldots,i_{2n})$ takes on a fixed value $(b_1,\ldots,b_{2n})$ for all $i_2 < \cdots < i_{2n} \in E$. Since $\{0^k, 1^k\}$ is $(n,2n)$-admissible via f it follows that n of the b_i's are zero and n are one. W.l.o.g. assume that $b_1 = 0$. Let e be the $4n$-th element of E. We claim that for every $v \in V$ either $v[i] = 0$ or $v[e] = 1$, hence $e \sqsubseteq i$ which contradicts the hypothesis. Indeed, if $v[i] = 1$ then the first component of $f(i,i_2,\ldots,i_{2n})$ is incorrect and 0 for all $i_2 < \cdots < i_{2n} \in E$. Hence there are at most $n-1$ correct 0s, so there is a correct 1, i.e., there is i_j with $v[i_j] = b_j = 1$. It follows that there are at most $2n-2$ components $j \in E$ with $v[j] = 0$. Fix j_0 with $b_{j_0} = 1$ and choose $i_2 < \cdots < i_{2n}$ such that $i_{j_0} = e$ and $v[i_j] = 1$ for $j \ne j_0$. Then there are only $n-1$ correct components $j \ne j_0$ in $f(i,i_2,\ldots,i_{2n})$, so component j_0 has to be correct in order to get at least n correct components, i.e., $v[e] = 1$.

Consider any linear extension of $\sqsubseteq$ and choose a corresponding permutation π such that $i \sqsubset j \Rightarrow \pi(i) < \pi(j)$. If $\pi(i) < \pi(j) - 2\hat{c}_n - 1$ then $i \sqsubseteq j$ since, by the claim, there is a column m, $\pi(i) < \pi(m) < \pi(j)$ which is comparable to both i and j, hence $i \sqsubseteq m \sqsubseteq j$, i.e., $i \sqsubseteq j$. It follows that $v \in 1^*\{0,1\}^{c_n}0^*$, for $c_n = 2\hat{c}_n + 2$ and every $v \in \pi(V)$. This proves the lemma. ∎

The previous Lemma is a further indication of the close relationship between semirecursive and r.e. $(n, 2n)$-recursive sets because in the semirecursive case the possible characteristic functions on a k-element set yield (modulo a permutation) the 0-linear set $\{1^i 0^{k-i} : 0 \leq i \leq k\}$.

Theorem 3.10 *There is a constant d_n such that every r.e. set $A \in \Omega(n, 2n)$ is (uniformly) strongly $(d_n(k+1), k)$-verbose for all $k \geq 1$.*

Proof: Assume that $A \in \Omega(n, 2n)$ is an r.e. set. By Lemma 3.8 we can compute for any $x_1 < \cdots < x_k$ an $(n, 2n)$-admissible set $V \subseteq \{0,1\}^k$ which contains a least and a greatest element and contains $(A(x_1), \ldots, A(x_k))$. By Lemma 3.9, V contains at most $(k+1)2^{c_n}$ vectors. Hence the Theorem follows with $d_n = 2^{c_n}$. ∎

Let us now consider k-r.e. sets. Jockusch and Owings [21, Theorem 6] have shown that every semirecursive k-r.e. set is r.e. or co-r.e. This does not generalize to Ω. For instance take any r.e. nonrecursive semirecursive set A. Then $A \oplus \overline{A}$ is 2-r.e. and $A \oplus \overline{A} \in \Omega(1, 2)$ but $A \oplus \overline{A}$ is neither an r.e. nor a co-r.e. set. This can be strengthened as follows:

Theorem 3.11 *For any $k \geq 1$ there is a $(k+1)$-r.e. $(1, 2)$-recursive set which is not k-r.e.*

Proof: We construct a $(k+1)$-r.e. set A which is not k-r.e. by diagonalizing all recursive approximations which change their value at most k times. The strategy for diagonalizing φ_e is to specify a witness x_e and to change $\lambda s.A(x_e, s)$ once more than $\lambda s.\varphi_e(x_e, s)$. The different strategies are combined as follows: In each stage s only the least strategy which wants to act is allowed to act, say it changes $A(x_e, s)$. Then we cancel all witnesses $x_i > x_e$ and move them to $s + x_i$. For all y such that $x_e < y \leq s$ we define $A(y, s) = (k+1) \pmod 2$. $A(x, s)$ retains its value for $x < x_e$. By a standard finite injury argument every requirement is satisfied. So A is $(k+1)$-r.e. and not k-r.e. We claim that A is $(1, 2)$-recursive via the recursive function $f : \omega^2 \to \{0,1\}^2$ defined as follows for $x < s$: $f(x, s) := (A(x, s), (k+1) \pmod 2)$.

If $A(x, s) = A(x)$ then the first component is correct. Otherwise there is $t \geq s$ with $A(x, t+1) \neq A(x, t)$. In this case $A(s, t') = A(s, t+1) = (k+1) \pmod 2$, for all $t' > t$, i.e., the second component is correct. ∎

The previous result seems to suggest that membership in Ω and being k-r.e. are independent properties. However, our next result shows that membership in Ω forces the Turing degree of any k-r.e. set to be r.e. This stands in contrast to the fact that there exist $(k+1)$-r.e. sets that do not have k-r.e. Turing degree [32, VII.2.4].

Theorem 3.12 *Every k-r.e. set in Ω has r.e. degree.*

Proof: (Induction on k) The case $k = 1$ is trivial. Now assume that $k > 1$ and that every $(k-1)$-r.e. set $A \in \Omega$ has r.e. degree. Given a k-r.e. $(1,m)$-recursive set M we need to show that M has r.e. degree. There is $h \in R_2$ witnessing that M is k-r.e.: Let $B_s := \{x : (\exists t < s)(h(x,t) = 1)\}$. $B = \bigcup_{s\in\omega} B_s$ is an r.e. set and $M \subseteq B$. Let $D := \{\langle x, s\rangle : x \in B_s \ \wedge \ x \notin M\}$. D is $(k-1)$-r.e. and $D \leq_m \overline{M}$. Thus D is $(1,m)$-recursive, and, by the induction hypothesis, there is an r.e. set A with $D \equiv_T A$. Note that $A \leq_T M$. Choose a Turing reduction $\Psi^{()}$ such that $\chi_D = \Psi^A$.

Claim: Assume there is a $\{0,1\}^n$-valued function $f \in R_n$ such that

$$f(x_1,\dots,x_n) \neq (1-\chi_M(x_1),\dots,1-\chi_M(x_n)) \text{ for all } x_1,\dots,x_n \in B.$$

Then M has r.e. degree.

Since the hypothesis of the claim is satisfied for $n = m$ it follows that M has r.e. degree. This completes the inductive step.

Proof of the Claim: (Induction on n) Given $f \in R_n$ such that the hypothesis of the claim is satisfied. If $n = 1$ then $M = \{x : x \in B \ \wedge \ f(x) = 1\}$ is an r.e. set. Now assume that $n > 1$ and that the claim is true for $n-1$. We distinguish two complementary cases:

1) $(\exists \overline{x}_1)(\forall x_2,\dots,x_n \in B)(f(\overline{x}_1,x_2,\dots,x_n)[2,\dots,n] \neq (1-\chi_M(x_2),\dots,1-\chi_M(x_n)))$. Then let $\overline{f}(x_2,\dots,x_n) = f(\overline{x}_1,x_2,\dots,x_n)[2,\dots,n]$, for all $x_2,\dots,x_n \in \omega$. As $\overline{f} \in R_{n-1}$, we obtain by the induction hypothesis that M has r.e. degree.

2) $(\forall x_1)(\exists x_2,\dots,x_n \in B)(f(x_1,x_2,\dots,x_n)[2,\dots,n] = (1-\chi_M(x_2),\dots,1-\chi_M(x_n)))$. The following observations are crucial:

(I) If $f(x_1,\dots,x_n) = (0, 1-\chi_M(x_2),\dots,1-\chi_M(x_n))$ for some $x_2,\dots,x_n \in B$ then, by the hypothesis on f, $(x_1 \in B \Rightarrow x_1 \notin M)$. Since $(x_1 \notin B \Rightarrow x_1 \notin M)$ holds trivially we get $x_1 \notin M$.

(II) If $f(x_1,\dots,x_n) = (1, 1-\chi_M(x_2),\dots,1-\chi_M(x_n))$ for some $x_2,\dots,x_n \in B$ then, by the hypothesis on f, $(x_1 \in B \Rightarrow x_1 \in M)$. Since $(x_1 \in M \Rightarrow x_1 \in B)$ holds trivially we get $(x_1 \in B \Leftrightarrow x_1 \in M)$.

We define an r.e. set E as follows:

$$\begin{aligned} E := \ & \{\langle x_1,\dots,x_n,s,t\rangle : (\exists b_2,\dots,b_n \in \{0,1\})(f(x_1,\dots,x_n) = (1,b_2,\dots,b_n) \ \wedge \\ & s \leq t \ \wedge \ x_2,\dots,x_n \in B_s \ \wedge \ (\forall i;\, 2 \leq i \leq n)(\Psi_t^{A_t}(\langle x_i, s\rangle) = b_i) \ \wedge \\ & [x_1 \in B \ \vee \ (\exists i;\, 2 \leq i \leq n)(A \restriction u(A,\langle x_i,s\rangle,t) \neq A_t \restriction u(A_t,\langle x_i,s\rangle,t))]\}. \end{aligned}$$

Here $u(-,-,-)$ denotes the use function of $\Psi^{()}$ and $\{A_t\}_{t\in\omega}$ is a recursive enumeration of A.

$A \oplus E \leq_T M$: This follows from (II) and the facts that $A \leq_T M$ and $\Psi^A(\langle x, s\rangle) = 1 - \chi_M(x)$ for $x \in B_s$.

$M \leq_T A \oplus E$: We describe an algorithm for computing χ_M using $A \oplus E$ as an oracle. Given x_1, we search for $x_2,\dots,x_n,s,t$ such that one of the following cases holds:

a) $x_2,\dots,x_n \in B_s \ \wedge \ f(x_1,\dots,x_n) = (0, \Psi^A(\langle x_2,s\rangle),\dots,\Psi^A(\langle x_n,s\rangle))$. Then, by (I), $x_1 \notin M$.

b) $x_2,\dots,x_n \in B_s \ \wedge \ s \leq t \ \wedge \ f(x_1,\dots,x_n) = (1, \Psi_t^{A_t}(\langle x_2,s\rangle),\dots,\Psi_t^{A_t}(\langle x_n,s\rangle)) \ \wedge$ $(\forall i;\, 2 \leq i \leq n)(A \restriction u(A,\langle x_i,s\rangle,t) = A_t \restriction u(A_t,\langle x_i,s\rangle,t))$. Then, by (II), $x_1 \in M$ iff

$x_1 \in B$ iff $\langle x_1, \ldots, x_n, s, t\rangle \in E$.
By the hypothesis of 2) one of these cases holds, so the search terminates.

Thus we have shown that $M \equiv_T A \oplus E$, i.e., M has r.e. degree. This completes the proof of the claim. $\square$ ▮

Remark: The previous theorem cannot be transferred to stronger reducibilities since one can construct a $(1,2)$-recursive 2-r.e. set A which does not have r.e. wtt-degree. We give a brief outline of the construction. Let $(\Psi_e, \Theta_e, u_e, B_e)_{e\in\omega}$ be a simultaneous recursive enumeration of all quadruples of T-reductions Ψ, Θ, partial recursive functions u, and r.e. sets B. We need to satisfy all requirements

(R_e) If u_e is total and bounds the use of Ψ_e, Θ_e then $A \neq \Psi_e^{B_e}$ or $B_e \neq \Theta_e^A$.

The strategy for a single requirement (R_e) is as follows (we omit the subscript):

(1) Choose a witness x and wait until $n = \max\{u(z) : z \leq u(x)\}$ converges.
(2) Let $A(y) = 0$ for all $x \leq y \leq n$, and wait until $\Psi^B(x) = 0$ with use $\leq u(x)$ and $\Theta^A(z) = B(z)$ with use $\leq u(z)$ for all $z \leq u(x)$.
(3) Let $A(x) = 1$ and wait until a new $z' \leq u(x)$ is enumerated into B.
(4) Let $A(x) = 0$ (this restores the old computation of $\Theta^A(z') = 0 \neq 1 = B(z')$).

Note that $A(x)$ changes at most twice. The different strategies are combined in a finite injury argument such that the witnesses are ordered by decreasing priority. If the strategy for (R_e) causes a change in stage $s+1$ then we set $\chi_A(z) = 0$ for all z with $x_e < z \leq s$ and move the lower priority witnesses to new numbers greater than s. It can be shown that A is $(1,2)$-recursive as in the proof of Theorem 3.11. $\square$

In [4, Theorem 16] it is shown that A' is superterse if A is nonrecursive. We now consider $H_A := \{e : W_e \cap A \neq \emptyset\}$, the "weak jump" of A [32, IV.4.6]. Morris [28, Theorem 1] proved that $H_{\overline{A}}$ is not k-r.e. if A is r.e. and nonrecursive. This can be strengthened as follows:

Theorem 3.13 *If A is r.e. and nonrecursive then $H_{\overline{A}}$ is superterse.*

Proof: Suppose for a contradiction that there is an r.e. nonrecursive set A such that $H_{\overline{A}}$ is not superterse. By a simple fact from [4, Lemma 7] there is $n > 0$ and $f \in R_n$ such that for all $x_1, \ldots, x_n$: $W_{f(x_1,\ldots,x_n)} \subset \{0,1\}^n$ and $(H_{\overline{A}}(x_1), \ldots, H_{\overline{A}}(x_n)) \in W_{f(x_1,\ldots,x_n)}$.

We enumerate r.e. sets M_j^i for $i \in \omega$, $j = 1, \ldots, n$. By the recursion theorem, we assume that we are given a recursive 1-1 function $\lambda i, j. e_j^i$ such that $M_j^i = W_{e_j^i}$. Start with any $(b_1, \ldots, b_n) \in \{0,1\}^n$ and enumerate i in M_j^i iff $b_j = 1$. Since A is nonrecursive the set $\{i : (b_1, \ldots, b_n) \in W_{f(e_1^i,\ldots,e_n^i)} \ \wedge \ i \in A\}$ is r.e. and infinite. One can uniformly compute an infinite recursive subset L. Hence, using $M_j^{i'}$ instead of M_j^i, where i' is the i-th element of L, we can iterate this step for another n-tuple from $\{0,1\}^n$, etc. After 2^n iterations we obtain indices $e_1, \ldots, e_n$ such that $W_{f(e_1,\ldots,e_n)} = \{0,1\}^n$, a contradiction. ▮

4 Characterizations of Simple Sets

In this section we consider variants of simple sets and determine whether such sets can be (m,n)-recursive (or strongly (m,n)-verbose, resp.). Since there are hypersimple semirecursive sets [19, Corollary 3.3], we get $(1,2)$-recursive (strongly $(n+1,n)$-verbose) hypersimple sets. Every simple semirecursive set is hypersimple [19, Theorem 4.2. (v)]. The question arises whether there is at all any simple non-hypersimple set in Ω. Theorem 4.3 gives a positive answer. The proof uses the following "interval sets".

Definition 4.1 Partition ω into a sequence of adjacent intervals $\{I_n\}_{n\in\omega}$ such that $|I_n| = n!$. We identify in some canonical way the elements of I_n with the permutations of $\{0,\ldots,n-1\}$.
For every r.e. set A and every recursive 1-1-enumeration $a_0, a_1, \ldots$ of A we define an r.e. set B as follows:

$$B := \bigcup_{n\in\omega} \{\pi \in I_n : n \in A \vee (\exists i,j < n)(\exists k)(\pi(i) < \pi(j) \wedge j = a_k \wedge i \notin \{a_0,\ldots,a_k\})\}.$$

We call B the *interval set* of A (w.r.t. to enumeration $a_0, a_1, \ldots$).

Discussion: Consider the enumeration of B that corresponds to the enumeration of A. In the i-th step the interval I_{a_i} is enumerated completely into B. In addition, certain elements from intervals I_n with $n > a_i$ are enumerated into B. The intuition behind the latter enumeration is as follows: Each $\pi \in I_n$ guesses some order $(\pi(0),\ldots,\pi(n–1))$ in which the numbers $m < n$ are enumerated into A. Note that the intervals I_m with $m < n$ are enumerated into B in the same order. As soon as this guess turns out to be incorrect, π is enumerated into B. Each interval contains at least one element whose guess is always correct. Thus if $n \notin A$ then $I_n \not\subseteq B$. It follows that $A \leq_c B$, as $(\forall n)[n \in A \Leftrightarrow I_n \subseteq B]$. Conversely, $B \leq_{wtt} A$.

For each $x = \pi \in I_n$ let $\prec_x$ denote the order of enumeration of the elements from $\bigcup_{m\leq n} I_m$ that is obtained if we let $a_0 = \pi(0),\ \ldots,\ a_{n-1} = \pi(n-1),\ a_n = n$ (assuming that the elements in step i are enumerated in order of magnitude).

Note that $\prec_x$ is a linear ordering of $\bigcup_{m\leq n} I_m$ and that $B \cap (\bigcup_{m\leq n} I_m)$ is an $\preceq_x$-initial segment of $\bigcup_{m\leq n} I_m$ for each $x \in \overline{B}$.

It follows that B must be simple if A is not semirecursive. (If $\overline{B}$ has an infinite recursive subset E then we can compute for all x,y an element $z \in E$ with $z > \max(x,y)$. Say $z \in I_n$. Then x,y are ordered by $\preceq_z$ and we let $h(x,y)$ be the minimum of x,y w.r.t. this ordering. Since $B \cap (\bigcup_{m\leq n} I_m)$ is an $\preceq_z$-initial segment of $\bigcup_{m\leq n} I_m$ we have $\{x,y\} \cap B \neq \emptyset \Rightarrow h(x,y) \in B$. Thus B is semirecursive via h.) This observation can be used to show that in general the wtt-reduction from B to A cannot be turned into a tt-reduction: Jockusch [20, Theorem 3] constructed a nonrecursive r.e. set A whose tt-degree contains no simple set. By a slight modification one can ensure that A is not semirecursive. Hence any interval set B of A is simple. Thus B is not contained in the tt-degree of A and, as $A \leq_c B$, it follows that $B \not\leq_{tt} A$. □

Proposition 4.2 *Assume that B is an interval set of an r.e. set A. Then B is $(n, 2n+1)$-recursive and strongly $(1 + \frac{n(n+1)}{2}, n)$-verbose for all $n > 0$.*

Proof: We define $f_{2n+1} \in R_{2n+1}$ inductively and show that $B \in \Omega(n, 2n+1)$ via f_{2n+1} for all $n > 0$. Let $2n+1$ pairwise distinct numbers $x_1, \ldots, x_{2n+1}$ be given.
If $n = 0$ then $f_1(x_1) := 1$.
If $n > 0$, assume that x_{2n+1} is the greatest element. Let $b_{2n+1} := 1$. $x_1, \ldots, x_{2n}$ is linearly ordered by $\prec_{x_{2n+1}}$. We may assume that x_{2n} is the maximal element in this ordering. Let $b_{2n} := 0$. Let $(b_1, \ldots, b_{2n-1}) := f_{2n-1}(x_1, \ldots, x_{2n-1})$ and $f_{2n+1}(x_1, \ldots, x_{2n+1}) := (b_1, \ldots, b_{2n+1})$.
We claim that at least n of the b_i equal $\chi_B(x_i)$. This holds trivially for $n = 0$. For $n > 0$ we have two cases.
a) $b_{2n} = \chi_B(x_{2n}) \ \vee \ b_{2n+1} = \chi_B(x_{2n+1})$. By the induction hypothesis, at least $n-1$ of the b_i with $1 \leq i \leq 2n-1$ are correct. Thus altogether we get at least n correct values.
b) $b_{2n} \neq \chi_B(x_{2n}) \ \wedge \ b_{2n+1} \neq \chi_B(x_{2n+1})$. This means $\chi_B(x_{2n}) = 1$ and $\chi_B(x_{2n+1}) = 0$. Recall that $\prec_{x_{2n+1}}$ is a linear ordering of $\bigcup_{m' \leq m} I_{m'}$, where I_m is the interval containing x_{2n+1}. Since $x_{2n+1} \notin B$, $B \cap \bigcup_{m' \leq m} I_{m'}$ is an $\preceq_{x_{2n+1}}$-initial segment of $(\bigcup_{m' \leq m} I_{m'})$. Since $x_{2n} \in B$ it follows that $x_1, \ldots, x_{2n} \in B$. By the definition of f_{2n-1}, n of $b_1, \ldots, b_{2n-1}$ are equal to 1. So we get at least n correct values.
Since B is $(1,3)$-recursive it follows from Fact 2.2 that B is strongly $(1 + \frac{n(n+1)}{2}, n)$-verbose for all $n > 0$. ∎

Theorem 4.3 *There is a simple non-hypersimple set which is* $(n, 2n+1)$*-recursive and strongly* $(1 + \frac{n(n+1)}{2}, n)$*-verbose for all* $n > 0$.

Proof: Let A be any simple non-hypersimple set and consider an interval set B of A. Since $(x \in A \Leftrightarrow I_x \subseteq B)$ we conclude that B is simple. If $\{D_{f(i)}\}_{i \in \omega}$ is a strong array witnessing that A is not hypersimple then $\{\bigcup_{n \in D_{f(i)}} I_n\}_{i \in \omega}$ is a strong array witnessing that B is not hypersimple. Now the theorem follows by Proposition 4.2. ∎

Remark: By a standard diagonalization one can construct a simple non-hypersimple terse set. The question whether there is a simple nonhypersimple non-terse set appears to be more difficult. However, it is answered positively by the observation that no interval set B is terse:

F_6^B can be computed with fewer than 6 queries to B. Assume we are given pairwise distinct numbers $x_1, \ldots, x_6$. Let $x = \max\{x_1, \ldots, x_6\}$ and assume w.l.o.g. that $x_1 \prec_x \cdots \prec_x x_6$. Note that, by the definition of $\prec_x$, x is the greatest element in the ordering $\prec_x$. Thus $x_6 = x$. Let y be the largest number of $x_2, \ldots, x_5$. Query the oracle on y. If the answer is "no" then $B \cap \{x_2, \ldots, x_5\}$ is an $\prec_y$-initial segment of $\{x_2, \ldots, x_5\}$. Hence we can determine $F_4^B(x_2, \ldots, x_5)$ with only two additional queries, which saves one query. If the answer is "yes" then either $x_1 \in B$ or $x_6 \in B$. (If $x \notin B$ then $B \cap \{x_1, \ldots, x_6\}$ is an $\prec_x$-initial segment of $\{x_1, \ldots, x_6\}$, so $x_1 \in B$ since $x_1 \prec_x y$ and $y \in B$.) We can enumerate B until x_1 or x_6 appears and again we are saving one query to B. □

Theorem 4.4 *There is a maximal set which is* $(n, 2n+1)$*-recursive, and consequently strongly* $(1 + \frac{n(n+1)}{2}, n)$*-verbose for all* $n > 0$.

Proof: We suppose that the reader is familiar with the standard Yates-construction of a maximal set [32, X.3]. We slightly modify the Yates construction in the following way: In step s we are allowed to move the least marker Γ_e which wants to maximize its e-state. If this happens then we will also move all markers Γ_i with $i > e$ downwards to numbers greater than s. (This is the only change of the standard construction.) Any number $x \leq s$ that is not covered by a marker is enumerated into A. By induction on e, we can prove that every marker Γ_e moves at most finitely often and A is maximal by the usual argument.

Now fix m, n with $1 \leq m < \frac{n}{2}$. We claim that A is (m,n)-recursive via f defined as follows: For $x_1 < x_2 < \cdots < x_n$ simulate the construction until the end of step x_n. Let $Y = \{x_i : x_i \text{ is covered by a marker at the end of step } x_n\} = \{y_1 < \cdots < y_k\}$. Note that $\{x_1, \ldots, x_n\} - Y \subseteq A$. Let $D = \{y_j : 1 \leq j \leq \frac{k}{2}\}$. We define $f(x_1, \ldots, x_n) = (b_1, \ldots, b_n)$ such that $b_i = 0$ if $x_i \in D$, and $b_i = 1$ otherwise.

Clearly, f is recursive. If $D \subseteq \overline{A}$ then at least $n - k + |D| = n - k + \lfloor \frac{k}{2} \rfloor = n - \lceil \frac{k}{2} \rceil \geq \lfloor \frac{n}{2} \rfloor \geq m$ components are correct. If $D \cap A \neq \emptyset$ let y denote the least number in $A \cap D$. y is enumerated into A at some step $t > x_n$. By construction, each $y' \in Y$ such that $y' > y$ is also enumerated into A with at most one exception (if a marker is moved to some y' in step t). In particular, $Y - D$ is enumerated into A with at most one exception. Thus, at least $n - k + |Y - D| - 1 = n - k + \lceil \frac{k}{2} \rceil - 1 \geq \lceil \frac{n}{2} \rceil - 1 \geq m$ components are correct. This completes the verification. ∎

Remark: 1) By a straightforward combination of the Yates-construction with diagonalization one can get a maximal set $M \notin \Omega$. Alternatively, the existence of M can be deduced from a result of Marchenkov [27] that for any m-incomplete nonrecursive r.e. set A there exists a maximal set M such that A and M are incomparable w.r.t. m-reduction. If we choose A as in Fact 3.1 (i.e., as half of an exact pair for the r.e. m-degrees in Ω) then $M \notin \Omega$. One may also use a result of Ladner [26, Theorem 6] who constructed a maximal set which is not autoreducible. By Theorem 2.10 this set is not in Ω.

Since any two maximal sets are automorphic [32, Chap. XV] it follows that the class of all r.e. sets in Ω is not lattice theoretic.

2) By combining the Yates-construction with diagonalization, a terse maximal set can be constructed. The maximal set A from the proof of Theorem 4.4 is not terse since F_5^A can be computed with fewer than 5 queries to A: Given $x_1 < \cdots < x_5$ we simulate the construction of A until the end of step x_5. If one x_i has since been enumerated then we save the query "$x_i \in A$?". Now suppose that none of the x_i's is enumerated until the end of step x_5. Then we query the oracle on x_2, x_3. If one of them is in A then one of x_4, x_5 is in A and we can find out which one by enumerating A. If $\{x_2, x_3\} \subseteq \overline{A}$ then we know that $x_1 \notin A$. In any case we save one query to A.

This argument can be easily generalized to show that F_n^A can be computed with $O(\log_2(n))$ queries to A. □

We now turn to lower bounds.

Theorem 4.5 *If A is simple and $(n, 2n)$-recursive then A is hypersimple.*

Proof: Suppose that $A \in \Omega(n, 2n)$ is r.e., infinite, and not hypersimple. We shall prove that A is not simple. Choose a strong array $\{U_i\}_{i\in\omega}$ of disjoint finite sets that intersect $\overline{A}$. We may assume that $|U_i \cap \overline{A}| \geq c_n + 1$, where c_n is the constant from Lemma 3.9. Using Lemma 3.8 we can compute for each i an $(n, 2n)$-admissible set $V_i \subseteq \{0,1\}^{|U_i|}$ containing a least and a greatest element and containing $\chi_A \restriction U_i$. By Lemma 3.9 there is a permutation π of the columns of V_i such that $\pi(V_i)$ is c_n-linear. Since $\chi_A \restriction U_i$ contains at least $c_n + 1$ zeros it follows that the $\pi^{-1}(|U_i|)$-th element of U_i is in $\overline{A}$. Hence we can compute for each i an element in $U_i \cap \overline{A}$, i.e., A is not simple. ∎

It is well-known that no semirecursive set is hyperhyperimmune [19, Corollary 4.5]. There exist hyperhyperimmune retraceable sets retraced by total recursive functions [36, Section 3]. It is easy to see that every retraceable set retraced by a total recursive function is $(1, 2)$-recursive. Thus, there are hyperhyperimmune $(1, 2)$-recursive sets. This stands in contrast to the following result.

Theorem 4.6 *There is no strongly hyperimmune $(n, 2n)$-recursive set. In particular, there is no hyperhypersimple $(n, 2n)$-recursive set.*

Proof: Assume that A is infinite and $(n, 2n)$-recursive via $f : \omega^{2n} \to \{0,1\}^{2n}$. For $b \in \{0,1\}$ let $F_b(x_1, \ldots, x_{2n}) := \{x_i : f(x_1, \ldots, x_{2n})[i] = b\}$. We distinguish two cases:
(a) $(\forall x)(\exists x_1, \ldots, x_{2n})(x < x_1 < \cdots < x_{2n} \;\wedge\; \#f(x_1, \ldots, x_{2n}) > n)$. Then there is a recursive sequence $x_{1,1} < x_{1,2} < \cdots < x_{1,2n} < \cdots < x_{i,1} < \cdots < x_{i,2n} \cdots$ such that $\#f(x_{i,1}, \ldots, x_{i,2n}) > n$. Let $U_i = \{x_{i,1}, \ldots, x_{i,2n}\}$. Then $U_i \cap A \neq \emptyset$ and $\{U_i\}_{i\in\omega}$ witnesses that A is not (strongly) hyperimmune.
(b) $(\exists x)(\forall x_1, \ldots, x_{2n})[x < x_1 < \cdots < x_{2n} \Rightarrow \#f(x_1, \ldots, x_{2n}) \leq n]$. We may assume that $\#f(x_1, \ldots, x_{2n}) \leq n$ for all $x_1 < \cdots < x_{2n}$. Now we proceed similarly as in the proof that no infinite retraceable set is strongly hyperhyperimmune [29, III.4.8]. We inductively define a recursive "height"-function u as follows (with $\max \emptyset := 0$):

$$\begin{aligned} u(x) := \max\{1 + \;\min\{u(y) : y \in F_1(x_1, \ldots, x_{2n})\} :& \\ x_1 < \cdots < x_{2n} = x \;\wedge\; \#f(x_1, \ldots, x_{2n}) = n \;\wedge\; x \in F_0(x_1, \ldots, x_{2n})\}& \end{aligned}$$

Let $U_i := \{x : u(x) = i\}$. We first show that $U_i \cap A$ is recursive for all i. Let $x_1 < \cdots < x_{2n-1}$ be the $2n - 1$ least numbers from $U_i \cap A$ (if they do not exist then $|U_i \cap A| < 2n - 1$ and we are done). Assume we are given $x = x_{2n} > x_{2n-1}$ with $u(x) = i$. Since $x_1, \ldots, x_{2n-1} \in A$ there must be at least $n - 1$ indices $j < 2n$ with $f(x_1, \ldots, x_{2n})[j] = 1$.

If $\#f(x_1, \ldots, x_{2n}) < n$ then $f(x_1, \ldots, x_{2n})[2n] = 0$ and only $n-1$ of the first $2n-1$ components are correct. Thus $\chi_A(x) = 0$.

If $\#f(x_1, \ldots, x_{2n}) = n$ then $x \in F_1(x_1, \ldots, x_{2n})$ since otherwise $u(x) > i$. As only $n - 1$ of the first $2n - 1$ components are equal to 1 (and hence correct) it follows that the last component is correct, i.e., $\chi_A(x) = 1$.

So $U_i \cap A$ is recursive. If there is i such that U_i has infinite intersection with A then A is not immune and we are done. For the rest of the proof assume that $U_i \cap A$ is finite for all i. We show that every U_i intersects A, thus $\{U_i\}_{i\in\omega}$ witnesses that A is not strongly hyperimmune.

Since $\{U_i\}_{i\in\omega}$ partitions ω the set $B := \{x : (\exists j > i)(x \in U_j \cap A)\}$ is nonempty. Let x be the least element of B. By the definition of u there are $x_1 < \cdots < x_{2n} = x$ such that $i \leq \min\{u(y) : y \in F_1(x_1,\ldots,x_{2n})\}$, $\#f(x_1,\ldots,x_{2n}) = n$, and $x \in F_0(x_1,\ldots,x_{2n})$. Since the last component of $f(x_1,\ldots,x_{2n})$ is 0 and incorrect, there is $z \in F_1(x_1,\ldots,x_{2n}) \cap A$. Since $z < x$ we have $z \notin B$, i.e., $u(z) \leq i$. By the choice of $x_1,\ldots,x_{2n-1}$ we get $u(z) \geq i$. Thus $z \in U_i \cap A$, which completes the proof. ▮

The next result shows that the upper bounds from Theorems 4.3 and 4.4 are also optimal for strongly verbose sets.

Theorem 4.7 *If A is simple and strongly $(\frac{n(n+1)}{2}, n)$-verbose for some $n > 0$, then A is hypersimple but not hyperhypersimple.*

Proof: For the proof we need the following technical definition:
Given $A \subseteq \omega$. We say that $E \subseteq \overline{A}$ is *A-flat* if there is no infinite subset E' of E and no binary r.e. relation r such that:
(1) $(\forall x, y)((x < y \ \wedge \ r(x,y) \ \wedge \ x \in A) \Rightarrow \ y \in A)$,
(2) $(\forall x, y \in E')(x < y \Rightarrow r(x,y))$.
The next lemma provides the link between A-flat sets and verboseness. Its proof is interesting since it uses the *infinite* Ramsey theorem (a highly nonconstructive tool) in connection with r.e. sets.

Lemma 4.8 *If A is simple and $\overline{A}$ has an infinite A-flat subset then A is not strongly $(\frac{n(n+1)}{2}, n)$-verbose, for all $n > 0$.*

Proof: Assume that A is simple and that E is an infinite A-flat subset of $\overline{A}$. Let $\Gamma_A(n)$ denote the least m such that A is strongly (m,n)-verbose. Fix $n > 0$. In this proof it is convenient to consider *co-r.e.* indices instead of canonical indices. We call $f_k \in R_k$ an *(l,k)-operator* for A if for all $x_1,\ldots,x_k$: $(A(x_1),\ldots,A(x_k)) \in \{0,1\}^k - W_{f_k(x_1,\ldots,x_k)}$ and $|\{0,1\}^k - W_{f_k(x_1,\ldots,x_k)}| \leq l$. Observe that there is an (l,k)-operator for A iff A is strongly (l,k)-verbose.
Claim 0: There is an infinite subset E_1 of E, and for each $1 \leq k \leq n$ there is a $(\Gamma_A(k), k)$-operator $f_k \in R_k$ for A such that:
There is no infinite subset E_2 of E_1 and no $(2^k, k)$-operator $g_k \in R_k$ for A such that:

$$\max\{|G_k(x_1,\ldots,x_k)| : x_1 < \cdots < x_k \in E_2\}$$

$$< \max\{|F_k(y_1,\ldots,y_k)| : y_1 < \cdots < y_k \in E_1\}$$

where $G_k(x_1,\ldots,x_k) = \{0,1\}^k - W_{g_k(x_1,\ldots,x_k)}$, $F_k(x_1,\ldots,x_k) = \{0,1\}^k - W_{f_k(x_1,\ldots,x_k)}$.
Proof: We start with any collection of $(\Gamma_A(k), k)$-operators for $1 \leq k \leq n$ and $E_1 = E$. If there exists g_k and E_2 as above then we replace F_k by $F_k \cap G_k$ (this determines a $(\Gamma_A(k), k)$-operator) and E_1 by E_2. This operation is iterated until the claim is satisfied. Note that there are only finitely many iterations since $\sum_{k=1}^{n} \max\{|F_k(y_1,\ldots,y_k)| : y_1 < \cdots < y_k \in E_1\}$ strictly decreases in each iteration. □

Now fix $F_1,\ldots,F_n$ and E_1 with the properties mentioned above. By n applications of the infinite Ramsey theorem there are $V_1,\ldots,V_n$, with $V_i \subseteq \{0,1\}^i$, $1 \leq i \leq n$, and an infinite subset E_3 of E_1 such that for all $k = 1, \ldots, n$ and $x_1 < \cdots < x_k \in E_3$: $F_k(x_1, \ldots, x_k) = V_k$. Now we establish several properties of V_k that will imply $|V_n| \geq 1 + \frac{n(n+1)}{2}$ which proves the lemma.

Claim 1: $V_1 = \{0, 1\}$.

Proof: We have $0 \in V_1$ because $E_3 \subseteq \overline{A}$. $\{x : 1 \notin F_1(x)\}$ is an r.e. subset of $\overline{A}$. It is finite since A is simple. As E_3 is infinite it follows that $1 \in V_1$. □

Claim 2: For $1 \leq k < n$: $V_k \cdot \{1\} \subseteq V_{k+1}$.

Proof: Consider the uniformly co-r.e. sets

$$\begin{aligned} G_k(x_1, \ldots, x_k) := \{ & v \in F_k(x_1, \ldots, x_k) : \\ & (\forall x > x_k)[x \in A \Rightarrow v \cdot 1 \in F_{k+1}(x_1, \ldots, x_k, x)]\}. \end{aligned}$$

By Claim 0 there are $x_1 < \cdots < x_k \in E_3$ such that $G_k(x_1, \ldots, x_k) = F_k(x_1, \ldots, x_k) = V_k$. Thus, by the definition of G_k, if $x \in A$, $x > x_k$, then $v \cdot 1 \in F_{k+1}(x_1, \ldots, x_k, x)$ for all $v \in V_k$. Hence the set $\{x : x > x_k \ \wedge \ (\exists v \in V_k)(v \cdot 1 \notin F_{k+1}(x_1, \ldots, x_k, x)\}$ is an r.e. subset of $\overline{A}$. It is finite since A is simple. As E_3 is infinite we get $V_k \cdot \{1\} \subseteq F_{k+1}(x_1, \ldots, x_k, x) = V_{k+1}$, for some $x \in E_3$ with $x > x_k$. □

Claim 3: For $1 \leq k < n$: $\{0\} \cdot V_k \subseteq V_{k+1}$.

Proof: Let $b = \min(E_3)$, and define uniformly co-r.e. sets

$$G_k(x_1, \ldots, x_k) := \begin{cases} F_k(x_1, \ldots, x_k) & \text{if } x_1 \leq b, \\ \{v \in F_k(x_1, \ldots, x_k) : 0 \cdot v \in F_{k+1}(b, x_1, \ldots, x_k)\} & \text{if } x_1 > b. \end{cases}$$

By Claim 0 (with $E_2 = E_3 - \{b\}$) there exist $x_1 < \cdots < x_k \in E_3$, $b < x_1$ such that $G_k(x_1, \ldots, x_k) = F_k(x_1, \ldots, x_k)$. Then $V_k = G_k(x_1, \ldots, x_k) = \{v : 0 \cdot v \in F_{k+1}(b, x_1, \ldots, x_k)\}$, i.e., $\{0\} \cdot V_k \subseteq V_{k+1}$. □

Claim 4: For $2 \leq k \leq n$ there is $v \in \{0, 1\}^{k-2}$ with $1 \cdot v \cdot 0 \in V_k$.

Proof: Define the r.e. relation r as follows:

$$r(x_1, x_k) :\Leftrightarrow (\exists x_2, \ldots, x_{k-1})(x_1 < x_2 < \cdots < x_{k-1} < x_k \ \wedge$$

$$(\forall v \in \{0, 1\}^{k-2})(1 \cdot v \cdot 0 \notin F_k(x_1, \ldots, x_k))).$$

Note that r satisfies (1). Let $a_0, a_1, \ldots$ be the elements of E_3 in order of magnitude. Suppose for a contradiction that V_k does not contain a string $1 \cdot v \cdot 0$ with $v \in \{0, 1\}^{k-2}$. Then $r(a_i, a_j)$ holds for all i, j with $j \geq i + k$, and the infinite set $\{a_k, a_{2k}, a_{3k}, \ldots\}$ satisfies (2). This contradicts the hypothesis that E is A-flat. □

Claim 4 together with Claims 2,3 implies that for all $i, j \geq 0$ with $i + j \leq n - 2$ there is a string $v \in \{0, 1\}^{n-i-j-2}$ such that $0^i \cdot 1 \cdot v \cdot 0 \cdot 1^j \in V_n$. All of these $\frac{n(n-1)}{2}$ strings are pairwise distinct. Claim 1 together with Claims 2, 3 gives $n+1$ additional distinct strings of the form $0^i 1^{n-i} \in V_n$ (for $0 \leq i \leq n$). Thus $\Gamma_A(n) \geq |V_n| \geq \frac{n(n-1)}{2} + n + 1 = 1 + \frac{n(n+1)}{2}$ which completes the proof of the lemma. □

For a proof of Theorem 4.7 it now suffices to show that the hypothesis of Lemma 4.8 is satisfied if A is hyperhypersimple, or simple non-hypersimple, respectively. This is provided by the following two results.

Lemma 4.9 *If A is r.e. and $\overline{A}$ is not A-flat then A is not hyperhypersimple.*

Proof: Let A be an r.e. set with recursive enumeration $\{A_s\}_{s\in\omega}$. Assume that $\overline{A}$ is not A-flat. Then there is an r.e. binary relation r and an infinite set $E' \subseteq \overline{A}$ such that (1) and (2) are satisfied.
We define a partial recursive function $\psi = \bigcup_{s\in\omega} \psi_s$ such that the uniformly r.e. sets $U_i := \{x : \psi(x) = i\}$ $(i \in \omega)$ witness that A is not hyperhypersimple, i.e., each U_i intersects $\overline{A}$. Let $\{r_s\}_{s\in\omega}$ be some recursive enumeration of r.
Stage 0: $\psi_0 := \lambda x. \uparrow$.
Stage $s+1$: Let $a_s := \min\{a : (\forall x \in \mathrm{dom}(\psi_s) - A_s)(\psi_s(x) \neq a)\}$.
If there is $x \leq s$ such that for some $Q \subseteq \{0, \ldots, x\}$:
(a) $x = \max(Q)$
(b) $|Q| \geq (1 + a_s)^2$
(c) $x \notin \mathrm{dom}(\psi_s)$
(d) $(\forall y, z \in Q)(y < z \Rightarrow r_s(y, z))$
then let $\psi_{s+1} = \psi_s \cup \{(x, a_s)\}$ for the least such x, else let $\psi_{s+1} = \psi_s$. □
Claim 1: ψ is injective on $\overline{A}$.
Proof: By induction on s, ψ_s is injective on $\overline{A_s}$. □
Claim 2: $\psi(\overline{A})$ is an initial segment of ω.
Proof: Suppose for a contradiction that $\psi(\overline{A})$ is not an initial segment. Then there exist $a = \min(\omega - \psi(\overline{A}))$ and $x = \min\{y \notin A : \psi(y) > a\}$. Note that $\psi(z) < a$ for all $z \in \mathrm{dom}(\psi) \cap \overline{A} \cap \{0, \ldots, x-1\}$. Let s be the first stage such that $\psi_s(x)$ is defined, and choose a corresponding $Q \subseteq \{0, \ldots, x\}$ such that conditions (a)-(d) are satisfied at stage s. Since $x \notin A$, we get $Q \subseteq \overline{A}$, by (d) and (1). $|Q| \geq (1 + \psi(x))^2 \geq (2 + a)^2$, by (b). Claim 1 and the choice of x imply that $|\mathrm{dom}(\psi) \cap \overline{A} \cap \{0, \ldots, x\}| \leq a + 1$. Hence $Q' := Q - \mathrm{dom}(\psi)$ has at least $(a+2)^2 - (a+1) \geq (a+1)^2$ elements. Choose s_0 such that $\mathrm{dom}(\psi_s)$ does not change below x at any stage $s \geq s_0$. Since $a \notin \psi(\overline{A})$ and ψ_s is injective on $\overline{A_s}$ for all s, there exists $s_1 \geq s_0$ such that $a \notin \psi(\overline{A_{s_1}})$. At stage $s_1 + 1$, $a_{s_1} \leq a$, and $x' = \max(Q') < x$ satisfies conditions (a)-(d). Hence an element less than x enters $\mathrm{dom}(\psi_{s_1+1})$ at stage $s_1 + 1$, contradicting the choice of s_0. □
Claim 3: $\psi(\overline{A})$ is infinite.
Proof: Suppose for a contradiction that $\psi(\overline{A})$ is finite. By Claim 2 there is $a \in \omega$ such that $\psi(\overline{A}) = \{0, \ldots, a-1\}$. Since E' is infinite there is $Q' \subseteq E' - \mathrm{dom}(\psi)$ such that $|Q'| = (1+a)^2$. Now we obtain the same contradiction as in the proof of Claim 2, for $x' = \max(Q')$. □
Claims 1–3 imply that $\psi(\overline{A}) = \omega$. Hence every U_i intersects $\overline{A}$ and the proof is complete. □

Lemma 4.10 *If A is simple non-hypersimple then $\overline{A}$ has an infinite A-flat subset.*

Proof: Assume that A is a simple non-hypersimple set. Let $\{U_i\}_{i\in\omega}$ be a strong array witnessing that A is not hypersimple (i.e., the U_i's are mutually disjoint and intersect $\overline{A}$). Let $\{r_e\}_{e\in\omega}$ be an enumeration (not necessarily uniformly r.e.) of all binary r.e. relations that satisfy (1).

We now define an A-flat set $E = \{a_1 < a_2 < \ldots\} \subseteq \overline{A}$ together with auxiliary r.e. sets B_k, E_k, by induction on k. The E_k's will form an increasing sequence of coinfinite supersets of A such that each $\overline{E_k}$ intersects every U_i.

$E_0 := A$; $a_0 := -1$.
$B_k := \{x : (\exists i)(\forall y \in U_i - E_k)(x < y \ \wedge \ (\exists e \leq k)(r_e(x,y)))\}$.
$a_{k+1} := \ \min\{y : y > a_k \ \wedge \ y \notin E_k \cup B_k\}$.
$E_{k+1} := E_k \cup \{y : y > a_{k+1} \ \wedge \ (\exists e \leq k)(r_e(a_{k+1}, y))\}$.
Clearly B_k and E_k are r.e. sets and $A = E_0 \subseteq E_1 \subseteq E_2 \subseteq \ldots$. By induction assume that $\overline{E_k}$ intersects every U_i (in particular E_k is coinfinite). Since each r_e satisfies (1), $A \subseteq E_k$, and $U_i \not\subseteq E_k$ for all i, we get $B_k \subseteq \overline{A}$. As A is simple, B_k is finite. It follows that a_{k+1} is defined. Suppose for a contradiction that U_i does not intersect $\overline{E_{k+1}}$. By the definition of E_{k+1} it follows that

$$(\forall y \in U_i - E_k)[y > a_{k+1} \ \wedge \ (\exists e \leq k)(r_e(a_{k+1}, y))].$$

By the definition of B_k this implies $a_{k+1} \in B_k$, a contradiction. Hence $\overline{E_{k+1}}$ intersects every U_i.
We claim that $r_e(a_i, a_j)$ does not hold for $e < i < j$. Suppose for a contradiction that $r_e(a_i, a_j)$ does hold for some $e < i < j$. Then, by the definition of E_i, $a_j \in E_i \subseteq E_{j-1}$. But this contradicts the definition of a_j. It follows that there is no e and no infinite subset of E such that (2) is satisfied for $r = r_e$. Hence E is A-flat. □ ▮

Remark: There is a more direct (but less general) version of the proof that no hyperhypersimple set is strongly $(\frac{n(n+1)}{2}, n)$-verbose. Assume that A is hyperhypersimple and strongly (m,n)-verbose via f. By the infinite Ramsey theorem there is an infinite set $E \subseteq \overline{A}$ and $V \subseteq \{0,1\}^n$ such that $D_{f(x_1,\ldots,x_n)} = V$ for all $x_1 < \cdots < x_n \in E$. Of course we must have $0^n \in V$. If $1^n \notin V$ then A is not hypersimple. Now one shows that $0^i 1^j 0 1^{n-i-j-1} \in V$ for all i,j with $0 \leq i \leq n-2$, $0 \leq j \leq n-i-1$. Thus V contains at least $2 + \sum_{i=0}^{n-2}(n-i) = 1 + \frac{n(n+1)}{2}$ elements which completes the proof.

If $0^i 1^j 0 1^{n-i-j-1} \notin V$ for some i,j with $0 \leq i \leq n-2$, $0 \leq j \leq n-i-1$ then one could use a similar construction as in the proof of Lemma 4.9 to show that A is not hyperhypersimple. This is based on the following observation: Let $y_1 < \cdots < y_i$ be the i least elements of E. If $D = \{x_1 < \cdots < x_{2t+1}\}$, $y_i < x_1$, $x_{t+1} \in \overline{A}$, and $0^i 1^j 0 1^{n-i-j-1} \notin D_{f(z_1,\ldots,z_n)}$ for all $z_1 < \cdots < z_n \in \{y_1, \ldots, y_i\} \cup D$, then either $|\{x_1, \ldots, x_t\} \cap A| < j$ or $|\{x_{t+2}, \ldots, x_{2t+1}\} \cap A| < n-i-j-1$. In any case, $|\overline{A} \cap D| > t - n$. Hence we can modify the construction from the proof of Lemma 4.9 in such a way that in stage $s+1$ we look for a sufficiently large (depending on a_s) set D as above (instead of Q as in the original proof) with $x_{t+1} \notin A_s \cap \mathrm{dom}(\psi_s)$ and then let $\psi_{s+1}(x_{t+1}) = a_s$. In addition we do not allow that at any later stage s', $\psi_{s'}(x) := a$ for any $x \in D$ and $a > a_s$. The remaining details are left to the reader. □

5 Characterizations of Complete Sets

For each of the usual reducibilities $\leq$ (see [13, 29] for the definitions) we investigate whether there is a complete set (w.r.t. $\leq$) in Ω. It is known that there exist semirecursive complete sets for $\leq \in \{\leq_{tt}, \leq_Q, \leq_L\}$. (Bulitko [9, Corollary 2] showed that every semirecursive tt-complete set is L-complete.)

It remains to consider $\leq_m$, $\leq_{btt}$, $\leq_d$, $\leq_c$, and $\leq_p$. Recall that $K \notin \Omega$ (Fact 3.1). Since all m-complete sets are recursively isomorphic to K we get:

Fact 5.1 *There is no m-complete set in Ω.*

The next result is an immediate consequence of Fact 5.1 and Theorem 2.3.

Theorem 5.2 *There is no btt-complete set in Ω.*

Remark: It is easy to see that no btt-complete set A is terse: Assume that $K \leq_{btt(c)} A$. F_n^A can be computed with $\lceil \log_2(n+1) \rceil$ queries to K, each query to K can be answered by c queries to A. Hence, F_n^A can be computed with $c\lceil \log_2(n+1) \rceil$ queries to A. □

Theorem 5.3 *There is no d-complete set in Ω.*

Proof: We prove for all $n \geq 1$: If A is d-complete and $(1, n+1)$-recursive then A is $(1, n)$-recursive. Then, by induction on n, every d-complete set in Ω is recursive. Hence there is no such set.

Now suppose that A is d-complete and $(1, n+1)$-recursive via $f \in R_{n+1}$. Let $s \in R_1$ be a d-reduction from K to A, i.e., $(\forall x)(x \in K \Leftrightarrow D_{s(x)} \cap A \neq \emptyset)$.

Define a recursive function $g \in R_1$ as follows: Enumerate $(x_1, \ldots, x_n)$ into $W_{g(a)}$ iff $\varphi_a(\langle x_1, \ldots, x_n \rangle)$ is defined, say equal to k, and $(\forall z \in D_{s(k)})(f(x_1, \ldots, x_n, z)[n+1] = 0)$.

Take $h \in R_1$ such that $W_i \leq_m K$ via $\varphi_{h(i)}$. By the recursion theorem there is an $\overline{a}$ such that $\varphi_{\overline{a}} = \varphi_{h(g(\overline{a}))}$. Note that $\varphi_{\overline{a}} \in R_1$ is an m-reduction from $W_{g(\overline{a})}$ to K. For $k = \varphi_{\overline{a}}(\langle x_1, \ldots, x_n \rangle)$ we have:

$$(\forall z \in D_{s(k)})(f(x_1, \ldots, x_n, z)[n+1] = 0) \Leftrightarrow (\exists z \in D_{s(k)})(z \in A).$$

Now we can define $g \in R_n$ such that A is $(1, n)$-recursive via g: Given $x_1, \ldots, x_n$ we compute $k = \varphi_{\overline{a}}(\langle x_1, \ldots, x_n \rangle)$. If there is $z \in D_{s(k)}$ such that $f(x_1, \ldots, x_n, z)[n+1] = 1$ then $z \notin A$, so one of the first n components of $f(x_1, \ldots, x_n, z)$ must be correct. In this case we define $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, z)[1, \ldots, n]$.

If $f(x_1, \ldots, x_n, z)[n+1] = 0$ for all $z \in D_{s(k)}$ then we enumerate A until we find $z_0 \in D_{s(k)} \cap A$. Also in this case the last component is incorrect, hence we can define $g(x_1, \ldots, x_n) = f(x_1, \ldots, x_n, z_0)[1, \ldots, n]$ and obtain at least one correct component. So A is $(1, n)$-recursive which completes the proof. ∎

Remark: 1) It can be shown that the r.e. sets in Ω are not closed downwards under d-reductions.
2) The proof of [4, Theorem 11] shows that for every r.e. set A there is an r.e. terse set B with $A \leq_d B$. Thus there exist d-complete terse sets (let $A = K$). □

Theorem 5.4 *There is no c-complete $(n, 2n)$-recursive set. No c-complete set is strongly $(\frac{n(n+1)}{2}, n)$-verbose.*

Proof: Both parts follow from Theorems 4.5 and 4.7, respectively, using the following claim:
For every c-complete set A there is a simple non-hypersimple set H such that $H \leq_1 A$.
The theorem now follows using the fact that (m, n)-recursive (strongly (m, n)-verbose) sets are closed under $\leq_1$.

Proof of the claim: Let S be any simple non-hypersimple set and assume that $S \leq_c A$ via $f \in R_1$, i.e., $x \in S \Leftrightarrow D_{f(x)} \subseteq A$, for all x. Let $B = \bigcup_{x\in\omega} D_{f(x)}$ and let $h \in R_1$ be a 1-1 enumeration of B. Then $H := h^{-1}(A)$ is 1-reducible to A via h and simple non-hypersimple as we now show.

If L is an infinite r.e. subset of $\overline{H}$ then $\{y : (\exists x \in L)(h(x) \in D_{f(y)})\}$ is an infinite r.e. subset of $\overline{S}$ which does not exist. Thus H is simple.

Let $\{I_k\}_{k\in\omega}$ be a strong array witnessing that S is not hypersimple. Let $\{A_s\}_{s\in\omega}$ be a recursive enumeration of A. We define a strong array $\{J_k\}_{k\in\omega}$ inductively as follows: $J_0 := h^{-1}(\bigcup_{x\in I_0} D_{f(x)})$. In step $k+1$ we search for n, s such that

$$(\bigcup_{i\leq k} J_i) \cap h^{-1}(\bigcup_{x\in I_n} (D_{f(x)} - A_s)) = \emptyset,$$

and let $J_{k+1} := h^{-1}(\bigcup_{x\in I_n}(D_{f(x)} - A_s))$. Note that the search terminates because there are only finitely many $z \in \overline{S}$ such that $\overline{A} \cap D_{f(z)} \cap \bigcup_{x\in I_i, i\leq k} D_{f(x)} \neq \emptyset$ (otherwise S is not simple). $\{J_k\}_{k\in\omega}$ witnesses that H is not hypersimple. ∎

Remark: This approach does not work for p-complete sets since one can construct a d-complete set A to which no simple set is m-reducible. □

Theorem 5.5 *There is a c-complete set which is $(n, 2n+1)$-recursive and strongly $(1 + \frac{n(n+1)}{2}, n)$-verbose for all $n > 0$.*

Proof: Let B denote an interval set of K. Then we have for all x: $x \in K \Leftrightarrow I_x \subseteq B$. So B is c-complete. The theorem now follows using Proposition 4.2. ∎

Remark: One can construct a c-complete terse set. The basic idea is to set up a sequence of adjacent intervals $B_0, C_0, B_1, C_1, \ldots$ and enumerate a set A such that $(x \in K \Leftrightarrow C_x \subseteq A)$. The interval $B_{\langle i,n\rangle}$ is used to diagonalize against the i-th oracle Turing machine M to ensure that F_n^A is not computable with fewer than n queries to A via M. One chooses $x_1 < \cdots < x_n \in B_{\langle i,n\rangle}$ and simulates M^A on $x_1, \ldots, x_n$. If numbers to the right of $B_{\langle i,n\rangle}$ are queried they are enumerated into A. If the machine halts by making fewer than n queries we diagonalize by enumerating into A at most one x_i which has not been queried. This diagonalization may be injured finitely often by the action of higher priority strategies. Then it starts again with new numbers from $B_{\langle i,n\rangle}$. If $B_{\langle i,n\rangle}$ will be chosen large enough (one can inductively compute an explicit upper bound) then we will have enough diagonalization points left. Also C_n is to be chosen larger than the number of elements that are enumerated by strategies of higher priority. The remaining details are left to the reader. □

It is well-known that semirecursive sets are closed under p-reductions [19, Theorem 4.2] and therefore no p-complete set is semirecursive. This generalizes to $(n, 2n)$-recursive sets:

Theorem 5.6 *If A, B are r.e., $A \leq_p B$, and B is $(n, 2n)$-recursive, then $A \leq_{btt} B$. In particular, there is no p-complete $(n, 2n)$-recursive set.*

Proof: The second part of the theorem follows from the first part since we already know that there is no btt-complete set in Ω. Assume that the hypothesis of the first part is satisfied. We can compute for each x a monotone Boolean function p_x and $x_1, \ldots, x_k$ such that $x \in A$ iff $p_x(u) = 1$ where $u = (B(x_1), \ldots, B(x_k))$. By Lemmas 3.8 and 3.9 we may assume w.l.o.g. that we are given in addition a c_n-linear set $V \subseteq \{0,1\}^k$ containing u. First determine the least i such that $v[i] = 0$ and $p(v) = 1$ for some $v \in V$. If i does not exist, then $p(u) = 1$ only if $u = 1^k$ and we can determine this by checking $B(x_j)$ for $k - c_n < j \leq k$. Otherwise we fix i, v as above and compute $B(x_j)$ for $i - c_n - 1 \leq j \leq i + 2c_n + 1$. If $B(x_{i-1}) = \cdots = B(x_{i+c_n}) = 1$ then it follows that $v \preceq u$ (since V is c_n-linear), hence $p(u) = 1$. If $B(x_{i-1}) = 0$ then $p(u) = 0$ by the choice of v. If $B(x_{i-1}) = 1$ and one of the B-values is zero then u is fully determined by the other $2c_n + 1$ neighboring values. This defines a btt-reduction of norm $3c_n + 3$ from A to B. ▮

Remark: There is an r.e. p-degree that does not contain any set in Ω. This can be shown in two ways. First one can devise a direct finite injury construction with a set-up similar to Jockusch's proof that there is an r.e. nonrecursive tt-degree that does not contain a simple set [20]. Alternatively the result follows from:

Theorem 5.7 *If A is maximal and $A \equiv_p B$, then $A \leq_m B$.*

Choose any maximal set $A \notin \Omega$. Since Ω is closed under m-reduction and A is m-reducible to every member of the p-degree of A, it follows that the p-degree of A does not contain any set in Ω.
Theorem 5.7 generalizes the following result of Dëgtev [14, Lemma 1]:

If A is maximal and B is nonrecursive with $B \leq_c A$, then $A \leq_m B$.

Note however that this does not hold if we replace $\leq_c$ by $\leq_p$: Jockusch [19, Theorem 3.6] showed for any set A there is a semirecursive set B with $B \leq_p A \wedge A \leq_{tt} B$, but no maximal set is p-reducible to a semirecursive set [19, Corollary 4.6].

We include the proof of Theorem 5.7 in the appendix. □

There is a $(1,2)$-recursive terse set (construct a retraceable set with the total recursive retracing function $f(\langle x, y\rangle) = x$ and diagonalize). However, *r.e.* terse sets are more restricted as we now show.

Theorem 5.8 (a) *No r.e. $(k, 2k)$-recursive set is terse.*
(b) *There is an r.e. terse set A with $A \in V_S(5,3) \cap \bigcap_{n \geq 1} \Omega(n, 2n+1)$.*

Proof: (a) Assume that A is r.e. and $(k, 2k)$-recursive. We claim that F_n^A can be computed with $O(\log_2(n))$ queries to A. Using Lemmas 3.8 and 3.9 it suffices to show that given a c_k-linear set $V \subseteq \{0,1\}^n$ we can determine any vector $v \in V$ by $O(\log_2(n))$ queries of the form "Is $v[i] = 0$?". First we test whether $v[1] = 1$ and $v[n] = 0$. If this is not the case then either $v \in \{0,1\}^{c_k} 0^{n-c_k}$ or $v \in 1^{n-c_k}\{0,1\}^{c_k}$ and v can be determined by $c_k - 1$ additional queries. Now suppose that $v[1] = 1$ and $v[n] = 0$. Then, using binary search with $\lceil \log_2(n-1) \rceil$ queries, we find $i, i+1$ such that $v[i] = 1$, $v[i+1] = 0$. Then $v \in 1^j\{0,1\}^{c_k}0^{n-j-c_k}$ for some $j \in \{i - c_k, \ldots, i\}$. It

follows that v is fully determined by the $2c_k$ neighboring values. Hence we need only $2c_k$ additional queries. In total, $\lceil \log_2(n-1) \rceil + 2c_k + 2 = O(\log_2(n))$ queries suffice.
(b) The proof closely resembles the proof of Theorem 2.8. Therefore we shall only present a proof sketch and leave all further details to the reader.

Let $M_i^{()}$ be a recursive enumeration of all oracle Turing machines. We construct an r.e. set A that satisfies the following requirements

$$(P_{i,n}) \quad F_n^A \text{ is not computable with fewer than } n \text{ queries to } A \text{ via } M_i^{()}.$$

The strategy for a single requirement $(P_{i,n})$ is the following. We set up a diagonalization interval $I = \{x_1, \ldots, x_n\}$ with $x_j = x_1 + j - 1$ and simulate $M_i^A(x_1, \ldots, x_n)$. At each step s when a new number x with $x_n < x < s$ is queried we enumerate all intervals to the right of I into A if they contain a number less than s. If at least n queries are made we are done. Otherwise we reach a step s where $M_i^A(x_1, \ldots, x_n)$ outputs $(b_1, \ldots, b_n)$ and at least one x_i has not been queried. Then we enumerate x_i into A iff $(b_1, \ldots, b_n) = 0^n$. In addition we enumerate all intervals to the right of I into A if they contain a number less than s. Clearly this strategy satisfies $(P_{i,n})$ and acts only finitely often. Note that $F_n^A(x_1, \ldots, x_n) \in \{0^n, 1^n, 0^i 1 0^{n-i-1} : 0 \leq i \leq n-1\}$. The value 1^n occurs if the strategy is reinitialized for the sake of some higher priority strategy.

The different strategies are combined in a standard finite injury construction. The intervals are ordered such that the interval for $P_{i,n}$ is to the left of the interval for $P_{e,m}$ iff $\langle i, n \rangle < \langle e, m \rangle$. When $P_{i,n}$ acts in stage s then all strategies of lower priority are initialized with new intervals of numbers greater than s.

By a standard argument all requirements are satisfied, i.e., A is terse.
$A \in V_S(5, 3)$: Given $x_1 < x_2 < x_3$ we simulate the construction until the end of step x_3. If one x_i has been enumerated we are left with at most 4 possibilities for $v := F_3^A(x_1, x_2, x_3)$. Otherwise we have at most 5 possibilities: If all x_i's belong to the same interval then $v \in \{0^3, 1^3, 100, 010, 001\}$. If x_1 and x_2 belong to different intervals then $v \in \{0^3, 1^3, 010, 001, 011\}$. If x_2 and x_3 belong to different intervals then $v \in \{0^3, 1^3, 001, 011, 101\}$.
$A \in \Omega(n, 2n+1)$: Given $x_1 < \cdots < x_{2n+1}$ let $s = x_{2n+1}$ and define

$$f(x_1, \ldots, x_{2n+1}) = (A_s(x_1), \ldots, A_s(x_{n+1}), 1, 1, \ldots, 1).$$

At least n components are correct: Suppose that at least two of the first $n+1$ components are incorrect, i.e., some numbers $x_i < x_j$ are both enumerated into A after step s. They either belong to different intervals and hence the interval of x_j is completely enumerated into A, or they belong to the same interval which is completely enumerated into A (since otherwise at most one number would have been enumerated). In any case all x_k with $k \geq j$ are also enumerated into A and so the last n components are correct. ▮

Remark: In [7, Theorem 6.1, Corollary 9.2] it is shown that $V_S(5,3) \subseteq V_S(1 + \lfloor \frac{3}{2} n \rfloor, n)$ for all $n \geq 1$ and that every r.e. strongly $(\lfloor \frac{3}{2} n \rfloor, n)$-verbose set is semirecursive. Hence Theorem 5.8 implies that $m = 1 + \lfloor \frac{3}{2} n \rfloor$ is the least number m such that there is an r.e. terse set which is strongly (m, n)-verbose. □

6 Summary

For any property $\mathcal{P}$ of sets of natural numbers let $f(\mathcal{P}, n)$ denote the greatest m such that there exists an (m, n)-recursive set having property $\mathcal{P}$ ($m = 0$ if there is no such set), and let $v(\mathcal{P}, n)$ denote the least number m such that there exists a strongly (m, n)-verbose set having property $\mathcal{P}$. The table summarizes the known results. In the fourth column we give a reference for each result.

Property $\mathcal{P}$	$f(\mathcal{P}, n)$	$v(\mathcal{P}, n)$	Reference
nonrecursive	$\lfloor \frac{n}{2} \rfloor$	$n+1$	[4, 33]
nonrecursive semirecursive	$\lfloor \frac{n}{2} \rfloor$	$n+1$	[4, 17]
(k+1)-r.e. and not k-r.e.	$\lfloor \frac{n}{2} \rfloor$	$n+1$	Thm 3.11
2-r.e. and not of r.e. degree	0	2^n	Thm 3.12
join of nonrecursive sets of different T-degrees	$\lfloor \frac{n-1}{2} \rfloor$	$(\lceil \frac{n}{2} \rceil + 1)(\lfloor \frac{n}{2} \rfloor + 1)$	Thm 2.9, [7]
creative	0	2^n	[14, 16]
btt-complete	0	2^n	Thm 5.2
d-complete	0	2^n	Thm 5.3
c-complete	$\lfloor \frac{n-1}{2} \rfloor$	$1 + \frac{n(n+1)}{2}$	Thm 5.4
p-complete	$\lfloor \frac{n-1}{2} \rfloor$	?	Thm 5.6
tt-complete	$\lfloor \frac{n}{2} \rfloor$	n+1	[4]
not autoreducible	0	2^n	Thm 2.10
r.e. and not *r*-separable	0	2^n	Thm 3.2
r.e. and not semirecursive	$\begin{cases} 0 & \text{if } n \leq 2 \\ \lfloor \frac{n}{2} \rfloor & \text{if } n > 2 \end{cases}$	$1 + \lfloor \frac{3}{2} n \rfloor$	[7]
r.e. and terse	$\lfloor \frac{n-1}{2} \rfloor$	$1 + \lfloor \frac{3}{2} n \rfloor$	Thm 5.8
strongly hyperimmune	$\lfloor \frac{n-1}{2} \rfloor$	?	Thm 4.6
simple and not hypersimple	$\lfloor \frac{n-1}{2} \rfloor$	$1 + \frac{n(n+1)}{2}$	Thm 4.3, 4.7
hypersimple	$\lfloor \frac{n}{2} \rfloor$	n+1	[4]
hyperhypersimple	$\lfloor \frac{n-1}{2} \rfloor$	$1 + \frac{n(n+1)}{2}$	Thm 4.4, 4.7
maximal	$\lfloor \frac{n-1}{2} \rfloor$	$1 + \frac{n(n+1)}{2}$	Thm 4.4, 4.7

A Appendix: Proof of Theorem 5.7

Theorem A.1 *If M is maximal and $M \equiv_p A$ then $M \leq_m A$.*

Proof: The following lemma on p-self-reductions of maximal sets is the crucial part of the proof.

Lemma A.2 *Let Φ be any p-reduction and M a maximal set such that $M = \Phi^M$.*
(a) *There is a finite set $E \subseteq \overline{M}$ such that for almost all $x \in \overline{M}$ and all finite sets D:*

$$\Phi^{D-\{x\}}(x) = 1 \Rightarrow D \cap E \neq \emptyset.$$

(b) *$\Phi^{M \cup \{x\}}(x) = 1$ for almost all $x \in \overline{M}$.*

Proof: Let $g \in R_1$ be a recursive function that bounds the use of Φ. Since $\overline{M}$ is strongly uniformly hyperimmune [11, Proposition 3] it follows that for almost all $x \in \overline{M}$ the interval $\{x+1, \ldots, g(x)\}$ is contained in M. (1)
(a) Suppose for a contradiction that for every finite set $E \subseteq \overline{M}$ there are infinitely many $x \in \overline{M}$ and a finite set D such that $\Phi^{D-\{x\}}(x) = 1 \;\wedge\; D \cap E = \emptyset$. (2)
Note that if $\Phi^{D-\{x\}}(x) = 1$ for $x \in \overline{M}$ then $(D - \{x\}) \cap \overline{M} \neq \emptyset$ (otherwise $\Phi^M(x) = 1$ since Φ is a positive reduction). Using (1) it follows that there is x_0 such that for all $x \in \overline{M}$, if $x > x_0$ and $\Phi^{D-\{x\}}(x) = 1$ then $D \cap \overline{M} \cap \{0, \ldots, x-1\} \neq \emptyset$. (3)
Now we can enumerate an r.e. set B such that $B \cap \overline{M}$ and $\overline{B} \cap \overline{M}$ are both infinite contradicting the fact that M is maximal.
Step s: If there is a least $x \notin M_s$ and $D \subseteq \{0, \ldots, g(x)\}$ such that

$$\Phi^{D-\{x\}}(x) = 1 \;\wedge\; \min(D \cap \overline{M_s}) > \max(B_s \cap \overline{M_s} \cap \{0, \ldots, x\})$$

then enumerate x into B. □
$B \cap \overline{M}$ is infinite: Otherwise let $b := \max(B \cap \overline{M})$. Using (2) there is $x \in \overline{M}$, $x > b$, and a finite set D such that $\Phi^{D-\{x\}}(x) = 1$ and $D \cap \overline{M} \cap \{0, \ldots, b\} = \emptyset$. Hence $\min(D \cap \overline{M}) > b = \max(B \cap \overline{M} \cap \{0, \ldots, x\})$, so x is eventually enumerated into B, a contradiction.
$\overline{B} \cap \overline{M}$ is infinite: Otherwise let $b_0 := x_0 + \max(\overline{B} \cap \overline{M})$, and choose any $b_1 \in \overline{M} \cap B, b_1 > b_0$. Let s be the stage at which b_1 is enumerated into B. Let $x \in \overline{M}$ be minimal such that $x > b_1$ and x is enumerated into B at some stage $t > s$. Note that there is D with $\Phi^{D-\{x\}}(x) = 1$, $y = \min(D \cap \overline{M}) > b_1$. By (3) it follows that $y < x$, hence $y \notin B_t$. By the choice of b_0, y is enumerated into B at some stage $t' > t$ contradicting the choice of x. This completes the proof of part (a).
(b) Suppose for a contradiction that $\Phi^{M \cup \{x\}}(x) = 0$ for infinitely many $x \in \overline{M}$. Since $L = \{x : \Phi^{M \cup \{x\}}(x) = 1\}$ is an r.e. coinfinite superset of M it follows by the maximality of M that $\Phi^{M \cup \{x\}}(x) = 0$ for almost all $x \in \overline{M}$. (4)
We distinguish two cases:
(I) There is a finite set $E \subseteq \overline{M}$ such that for infinitely many $x \in \overline{M}$ and all D, if $\Phi^{D \cup \{x\}}(x) = 1$ then $D \cap E \neq \emptyset$. Thus $\{x : (\forall D \text{ finite})(\Phi^{D \cup \{x\}}(x) = 1 \;\Rightarrow\; D \cap E \neq \emptyset)\}$ is an infinite recursive subset of $\overline{M}$, i.e., M is not simple, a contradiction.
(II) For every finite set $E \subseteq \overline{M}$ and almost all $x \in \overline{M}$ there is D such that $\Phi^{D \cup \{x\}}(x) =$

1 and $D \cap E = \emptyset$. Note that, by (1) and hypothesis (4), if $\Phi^{D\cup\{x\}}(x) = 1$ then $D \cap \{0,\ldots,x-1\} \cap \overline{M} \neq \emptyset$, for almost all $x \in \overline{M}$. Now we can proceed in just the same way as in part (a) and enumerate a set B which splits the complement of M in a nontrivial way contradicting the maximality of M. We leave the details to the reader.
This completes the proof of the lemma. □

Now suppose we are given a maximal set M and a set A which is p-equivalent to M. Choose p-reductions Ψ, Θ such that $M = \Psi^A$ and $A = \Theta^M$. There is a p-reduction Φ such that $\Phi^X = \Psi^{\Theta^X}$ for all X, in particular $M = \Phi^M$. Choose a finite set $E \subseteq \overline{M}$ as in part (a) of the lemma. Note that there is x_0 such that $\Phi^{\overline{E-\{x\}}}(x) = 0$ for all $x \in \overline{M}, x > x_0$. On the other hand, for all $x > x_0$ there is a finite set $D \subseteq M$ such that $\Phi^{D\cup\{x\}}(x) = 1$, by part (b) of the lemma (it trivially holds for $x \in M$).

Now we can describe an m-reduction from M to A: For each $x > x_0$ first check whether $\Phi^{\overline{E-\{x\}}}(x) = 0$. If not then $x \in M$. Otherwise compute a finite set $D \subseteq M$ such that $\Phi^{D\cup\{x\}}(x) = 1$. Since Ψ, Θ are positive reductions and $\Phi^X = \Psi^{\Theta^X}$ there is y such that $\Theta^{D\cup\{x\}}(y) = 1$ and $\Theta^{\overline{E-\{x\}}}(y) = 0$. Therefore ($x \in M \Leftrightarrow y \in A$) follows: if $x \in M$ then $A(y) = \Theta^M(y) \geq \Theta^{D\cup\{x\}}(y) = 1$; if $x \notin M$ then $A(y) = \Theta^M(y) \leq \Theta^{\overline{E-\{x\}}}(y) = 0$. ▮

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