Relativizations of Randomness and Genericity Notions

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Abstract

A set A is a basis for Schnorr randomness if and only if it is Turing reducible to a set R which is Schnorr random relative to A. One can define a basis for weak 1-genericity similarly. It is shown that A is a basis for Schnorr randomness if and only if A is a basis for weak 1-genericity if and only if the halting problem K is not Turing reducible to A. Furthermore, call a set A high for Schnorr randomness versus Martin-Löf randomness if and only if every set which is Schnorr random relative to A is also Martin-Löf random unrelativized. It is shown that A is high for Schnorr randomness versus Martin-Löf randomness if and only if K is Turing reducible to A. Other results concerning highness for other pairs of randomness notions are also included.

1 Introduction

Kučera and Terwijn [12] showed that there is a nonrecursive set A such that the notions of Martin-Löf randomness relative to A and Martin-Löf randomness unrelativized coincide. As every set is Turing reducible to a Martin-Löf random set [5, 11], A is also Turing reducible to a set which is Martin-Löf random relative to A. Later, this notion was systematically studied [19, 20] and characterized [7].

These studies were carried out for various notions $\mathcal{M}$. A set A is called a basis for a relativizable property $\mathcal{M}$ if there is a set $B \geq_T A$ that has the property $\mathcal{M}$ relative to A. For example, it is well known that every set is a basis for Kurtz randomness (see Remark 1.4 below). Furthermore, no nonrecursive set A is a basis for 1-genericity, since A is not Turing reducible to any set which is 1-generic relative to A. In the

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present work, the bases for the notions of Schnorr randomness and weak 1-genericity are investigated. It is shown that in both cases, the bases are the natural class of sets that are not Turing above the halting problem. This solves an open problem of Miller and Nies for the case of Schnorr randomness [18, Question 5.2].

There are several notions of algorithmic randomness [2, 14, 16, 25, 26]. A set A is Martin-Löf random if and only if there is no uniformly r.e. sequence of Σ_1^0 classes such that for every e, the e^{th} class has measure at most 2^{-e} and contains A [15]. A set is Schnorr random if and only if "at most 2^{-e}" is replaced by "exactly 2^{-e}" in the previous definition [25]. Alternatively, one can characterize these notions using martingales, where a martingale mg is a function defined on finite binary strings such that $\mathrm{mg}(\sigma 0)+\mathrm{mg}(\sigma 1) = 2\mathrm{mg}(\sigma) \geq 0$ for all σ. A martingale mg succeeds on A if and only if for every c, there is an n such that $\mathrm{mg}(A(0)A(1)\dots A(n)) > c$. A martingale mg is recursive (r.e.) if and only if the set $\{(\sigma, q) : \sigma \in \{0,1\}^*, q \in \mathbb{Q}, \mathrm{mg}(\sigma) > q\}$ is recursive (r.e.). One can characterize the Martin-Löf random sets as those on which no r.e. martingale is successful. Similarly, a set is recursively random if and only if no recursive martingale succeeds on this set. The martingale characterization of Schnorr randomness is more involved and there are various versions. Among these, the following is the most suitable for this paper. The notion of Kurtz randomness is presented here as well.

Property 1.1. *A set R is Schnorr random relative to A if for some A-recursive function r there is no A-recursive martingale* mg *and no A-recursive bound function f such that there are infinitely many n such that*

$$\mathrm{mg}(R(0)R(1)\dots R(f(n))) > r(n).$$

A set R is Kurtz random relative to A if for some A-recursive function r there is no A-recursive martingale mg *and no A-recursive bound function f such that*

$$\mathrm{mg}(R(0)R(1)\dots R(f(n))) > r(n)$$

for all n.

Furthermore, a set is called "weakly 2-random" [20] or "strongly random" [24] if and only if it is Martin-Löf random and forms a minimal pair with the halting problem.

Genericity notions [8, 22, 23] are complementary to randomness notions. One considers extension functions of certain types such that the generic set either meets some extension or strongly avoids all of them. For instance, a set G is 1-generic if and only if for every partial recursive extension function $f : \{0,1\}^* \to \{0,1\}^*$, either there are n and m such that $G(n)G(n+1)\dots G(m) = f(G(0)G(1)\dots G(n-1))$ ("G meets f") or $f(G(0)G(1)\dots G(n-1))$ is undefined for almost all n ("G strongly avoids f").

Weak 1-genericity is a variant in which one considers only total extension functions. There one can take f to depend only on the length of the input and not on the particular choice of input as below.

Property 1.2. A set G is weakly 1-generic relative to A if and only if for every A-recursive function $f : \mathbb{N} \to \{0,1\}^*$ there are numbers n and m such that $n \leq m$ and $f(n) = A(n)A(n+1)\ldots A(m)$.

The notion of bases of randomness is linked with lowness. For example, a set A is low for a property $\mathcal{M}$ if and only if the sets B that have the property $\mathcal{M}$ unrelativized are precisely those that have the property $\mathcal{M}$ relative to A. The most famous example of this type is that a set is low for Martin-Löf randomness if and only if it is a basis for Martin-Löf randomness [2, 19, 20].

In the case of bases of Schnorr randomness, there are some parallels to this result if one considers notions of bases of randomness with respect to truth-table reducibility [4]. In the case of Turing reducibility, though, the class of the sets which are low for Schnorr randomness forms a proper subclass of the class of the bases for Schnorr randomness. However, one can obtain a connection to a notion of highness for Schnorr randomness versus Martin-Löf randomness. This is the dual of lowness for a pair of randomness notions. The concept of lowness for a pair of randomness notions was introduced by Kjos-Hanssen, Nies and Stephan [9]. A set A is said to be low for a notion $\mathcal{M}$ versus a notion $\mathcal{N}$ if and only if every set which has the property $\mathcal{M}$ also has the property $\mathcal{N}$ relative to A. This notion has also been explicitly studied by Downey, Nies, Weber and Yu [3], Nies [19], and Greenberg and Miller [6]. The concept of highness for a pair of randomness notions is formalized as follows.

Definition 1.3. A set A is high for a notion $\mathcal{M}$ versus a notion $\mathcal{N}$ if and only if every set which has the property $\mathcal{M}$ relative to A also has the property $\mathcal{N}$ unrelativized.

A is therefore high for Schnorr randomness versus Martin-Löf randomness if and only if every set which is Schnorr random relative to A is also Martin-Löf random. Miller [17] showed that a set A is high for Martin-Löf randomness versus strong randomness if and only if there is no K-recursive function f such that $f(x) \neq \varphi_x^A(x)$ whenever the latter is defined; that is, A is high for Martin-Löf randomness versus strong randomness if and only if K does not compute a function that is diagonally nonrecursive relative to A.

The main result of this paper is that this notion is antithetical to being a basis for Schnorr randomness and characterizes the Turing degrees above the halting problem. The following properties are shown to be equivalent to $A \geq_T K$.

- A is not a basis for Schnorr randomness; that is, there is no $R \geq_T A$ such that R is Schnorr random relative to A (Theorems 2.1 and 2.2).

- A is high for Schnorr randomness versus Martin-Löf randomness; that is, every set which is Schnorr random relative to A is also Martin-Löf random unrelativized (Theorems 2.1 and 2.2).

- A is not a basis for weak 1-genericity; that is, there is no $G \geq_T A$ which is weakly 1-generic relative to A (Theorem 3.1).

- A is high for weak 1-genericity versus 1-genericity; that is, every set which is weakly 1-generic relative to A is also 1-generic unrelativized (Corollary 3.2).

- A is high for 1-genericity versus weak 2-genericity; that is, every set which is 1-generic relative to A is also weakly 2-generic unrelativized (Theorem 3.3).

Some results on recursive randomness are also presented. No set which is a basis for recursive randomness has PA-complete Turing degree. Furthermore, if $A \leq_T K$ and A does not compute a diagonally nonrecursive function, then A is a basis for recursive randomness [7]. The following two partial characterizations of the sets which are high for recursive randomness versus Martin-Löf randomness can be proven similarly.

- If A is PA-complete, then A is high for recursive randomness versus Martin-Löf randomness.

- If A is high for recursive randomness versus Martin-Löf randomness, then there is a Martin-Löf random set that is Turing reducible to A.

The question remains open for the sets that compute a Martin-Löf random set but not a complete extension of Peano Arithmetic. The results for Kurtz randomness are summarized in the following remark, as they are quite straightforward and mostly known.

Remark 1.4. For every set A there is a A'-recursive sequence $a_0, a_1, a_2, \ldots$ of numbers such that R is Kurtz random whenever it is chosen outside the intervals $I_n = \{x : 2^{a_n} \leq x < 2^{a_n+1}\}$ such that betting according to the universal A-r.e. martingale will not increase one's capital, regardless of the values of R on the intervals I_n. Hence, for all $x \in I_n$, one can define $R(x) = A(n)$. As there are only finitely many $m \notin \{a_0, a_1, a_2, \ldots\}$ such that R is constant on the interval $2^m \leq x < 2^{m+1}$, one can compute the positions of the I_n from R and then compute $A(n)$. As R is Kurtz random relative to A and Turing above A, A is a basis for Kurtz randomness [10].

Furthermore, the set R constructed here is neither Schnorr random nor weakly 1-generic. In the case of Schnorr randomness, this follows from the fact that R is constant on all I_n. In the case of weak 1-genericity, this follows from the fact that R is either random or constant on the intervals $\{x : 2^m \leq x < 2^{m+1}\}$ but does not meet any other extension requirement. It follows that there is no set A such that A is high for Kurtz randomness versus Schnorr randomness, recursive randomness, Martin-Löf randomness, weak 1-genericity, 1-genericity and weak 2-genericity.

Tables 1 and 2 contain a summary of this information. All of these results appear in this paper except the characterization of high for Martin-Löf randomness versus strongly randomness, which was proven by Miller [17].

Table 1: Highness for pairs of randomness notions

	Kurtz random	Schnorr random	recursively random	Martin-Löf random	strongly random
Kurtz random	$\mathcal{A}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
Schnorr random	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}$
recursively random	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{P}$	????
Martin-Löf random	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{D}$
strongly random	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$

Table 2: Highness for pairs of genericity notions

	Kurtz random	weakly 1-generic	1-generic	weakly 2-generic	2-generic
Kurtz random	$\mathcal{A}$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
weakly 1-generic	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{K}$	$\mathcal{K}$	$\mathcal{K}'$
1-generic	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{K}$	$\mathcal{K}$
weakly 2-generic	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{H}$
2-generic	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$	$\mathcal{A}$

In these two tables, the entry in row $\mathcal{M}$ and column $\mathcal{N}$ represents the class $\mathcal{C}$ of sets A which are high for $\mathcal{M}$ versus $\mathcal{N}$; that is, $\mathcal{C} = \{A : \text{every set } R \text{ satisfying } \mathcal{M} \text{ relative to } A \text{ also satisfies } \mathcal{N}\}$. The class $\mathcal{C}$ of sets is one of the following: the class $\mathcal{A}$ of all sets, the class $\mathcal{K}$ of all $A \geq_T K$, the class $\mathcal{K}'$ of all $A \geq_T K'$, the class $\mathcal{H}$ of all sets which are high ($A' \geq_T K'$), the empty class $\emptyset$, the partially known class $\mathcal{P}$ or the class $\mathcal{D}$ of all A such that there is no K-recursive function f which is diagonally nonrecursive relative to A, that is, which satisfies $\varphi_e^A(e) \neq f(e)$ whenever $\varphi_e^A(e)$ is defined. Although $\mathcal{P}$ is not completely determined, it is known that $\mathcal{P}$ contains every PA-complete set and that every set $A \in \mathcal{P}$ is Turing above some Martin-Löf random set; furthermore, not every Martin-Löf random set is in $\mathcal{P}$. Note that $\mathcal{A}$ is the class of all sets which are high for $\mathcal{M}$ versus $\mathcal{N}$ iff $\mathcal{M}$ implies $\mathcal{N}$. For example, the entry for recursively random versus Schnorr random is $\mathcal{A}$ as every recursively random set is also Schnorr random. If the entry for $\mathcal{M}$ versus $\mathcal{N}$ is $\emptyset$, then there is a very strong form of nonimplication which cannot be bridged by relativizing $\mathcal{M}$.

2 Schnorr Randomness

The proof of the following theorem is the basis for several of the results in this paper.

Theorem 2.1. *For every set $A \not\geq_T K$ and every set B, there is a set R such that $B \leq_T R$, R is not recursively random and R is Schnorr random relative to A.*

Proof. First, define a recursive injective enumeration $\langle a_m, b_m \rangle$ of all pairs such that $a_m > 0$ and either $b_m = 0$ or some element below a_m is enumerated into K at stage b_m. The enumeration is chosen such that $b_m \leq m$ for all m. Therefore, for each n, there are at most $n+2$ indices m with $a_m = n$. The largest of these m satisfies $m \geq c_K(n)$, where $c_K(n) = \min\{s \geq n : \forall m \leq n\,[K_s(m) = K(m)]\}$ is the convergence module of K. Divide the integers into intervals I_m of length $3a_m + 1$ such that $\min(I_0) = 0$ and $\min(I_{m+1}) = \max(I_m) + 1$ for every m. Let mg be a weighted sum of all total A-recursive martingales. Note that mg itself is not A-recursive.

Let $F(x) = \max\{m : a_m = x\}$. Note that F majorizes c_K and F is K-recursive. Let $f_0, f_1, f_2, \ldots$ be a list of all A-recursive functions. Now let $E = \{x_0, x_1, x_2, \ldots\}$, where

$$x_n = \min\{y : \forall m < n\,[x_m < y \wedge f_m(y) < F(y)]\}.$$

Note that every x_n can be defined, as otherwise $F(y) \leq f_0(y) + f_1(y) + \ldots + f_n(y)$ for almost all y. This would contradict the fact that $K \not\leq_T A$. Using E, one can now define the set R inductively on all intervals I_m as follows.

- If there is $k > m$ with $a_m = a_k$ or if $a_m \notin E$, then choose R on I_m such that R is not 0 on all of the least $2a_m$ elements of I_m and mg grows on I_m by at most the factor $4^{a_m}/(4^{a_m} - 1)$.

- Otherwise (that is, if there is no $k > m$ with $a_m = a_k$ and if $a_m \in E$), choose $R(\min(I_m) + u) = 0$ for $u \in \{0, 1, \ldots, 2a_m - 1\}$ and choose $R(\min(I_m) + u) = B(u - 2a_m)$ for $u \in \{2a_m, 2a_m + 1, \ldots, 3a_m\}$.

Now it is shown that R has the desired properties.

$B \leq_T R$: To compute $B(n)$, search for the first interval I_m such that $a_m \geq n+1$ and $R(\min(I_m) + u) = 0$ for all $u \in \{0, 1, \ldots, a_m - 1\}$. As E contains a number larger than n, the search will terminate. It can be seen that $B(n) = R(\min(I_m) + 2a_m + n)$.

R is not recursively random: One can construct a recursive martingale mh that succeeds on R as follows. The initial capital of mh is set as 2 and for each interval

I_m, mh invests 4^{-a_m}, which is then bet on R being 0 for the first $2a_m$ members of I_m. If all bets are true, then mh doubles the invested capital $2a_m$ times and makes a profit of $2^{2a_m} \cdot 4^{-a_m} - 4^{-a_m} = 1 - 4^{-a_m}$. Otherwise, mh loses the invested 4^{-a_m}. On one hand, all potential losses can be bounded by $\sum_m 4^{-a_m} \leq \sum_{n>0}(n+2) \cdot 4^{-n} = \frac{3}{4} + \frac{4}{16} + \frac{5}{64} + \frac{6}{256} + \ldots < 2$ and therefore the martingale never takes the value of 0. On the other hand, there are infinitely many intervals I_m such that R is 0 on the least $2a_m$ members, so the profit is at least 3/4 on these intervals and the value of mh goes to infinity on R. Thus mh witnesses that R is not recursively random.

R is Schnorr random relative to A: To see this, consider the following function $\tilde{r}(n)$.

$$\tilde{r}(n) = n \cdot \left(\prod_{m<n} 2^{2^{3m+1}} \right) \cdot \left(\prod_{m>0} \left(\frac{4^m}{4^m - 1} \right)^{m+2} \right)$$

Note that an infinite product $\prod_k q_k$ such that $q_k > 1$ satisfies $\prod_k q_k < \infty$ if and only if $\sum_k (q_k - 1) < \infty$. To adjust for the fact that some intervals I_m are copies of each other as described in the first component of the definition of R, let $q_k = 4^m/(4^m - 1)$ as appropriate. Since $4^m/(4^m - 1) - 1 = 1/(4^m - 1)$, this inequality can be applied here. For each m, there are at most $m + 2$ values of k for which $q_k = 4^m/(4^m - 1)$. Hence

$$\sum_{m>0} (m+2) \cdot \frac{1}{4^m - 1} \leq \sum_{m>0} \frac{2^{m+2}}{4^m} \leq \sum_{m>0} 2^{2-m} = 4$$

and $(\prod_{m>0} (\frac{4^m}{4^m-1})^{m+2})$ is a positive real number. Therefore, the function $\tilde{r}$ has a recursive upper bound r such that $r(n) \in \mathbb{N}$ for all n.

Assume now that mg_k is a total A-recursive martingale and f_k is an A-recursive bound function for r as in Property 1.1 such that, in addition, $n < f_k(n) < f_k(n+1)$ for all n. For almost all n,

$$\mathrm{mg}_k(R(0)R(1) \ldots R(f_k(n))) \leq n \cdot \mathrm{mg}(R(0)R(1) \ldots R(f_k(n))).$$

Now consider $n > x_0 + x_1 + \ldots + x_k$. Then for each $u < n$, there is at most one interval I_m such that $m \leq f_k(n)$, $a_m = u$, $F(a_m) = u$ and $u \in E$; for $u \geq n$ there is no interval I_m satisfying these conditions. On the intervals that satisfy these conditions, the martingale mg can increase its capital by at most a factor of 2^{3a_m+1}; on all other intervals I_m below $f_k(n)$, mg can increase its capital by at most a factor of $4^{a_m}/(4^{a_m} - 1)$. Hence, one has that

$$\mathrm{mg}(R(0)R(1) \ldots R(f_k(n))) \leq r(n)/n.$$

It can be seen from the two previous inequalities that for almost all n,

$$\mathrm{mg}_k(R(0)R(1) \ldots R(f_k(n))) \leq r(n)$$

and hence R is not Schnorr random relative to A by Property 1.1. ∎

The next result is based on this construction. The equivalence of the first two conditions solves an open problem of Miller and Nies for the special case of Schnorr randomness [18, Question 5.2].

Theorem 2.2. *The following conditions are equivalent for every set A.*

- $A \not\geq_T K$.
- *A is a basis for Schnorr randomness.*
- *A is not high for Schnorr randomness versus recursive randomness.*
- *A is not high for Schnorr randomness versus Martin-Löf randomness.*

Proof. If $A \geq_T K$, then every set which is Schnorr random relative to A is already recursively random and Martin-Löf random unrelativized.

Furthermore, as A is above a low Martin-Löf random set R, there is no set which is Martin-Löf random relative to R above R: As every set which is Schnorr random relative to A is also Martin-Löf random relative to R, there is no set which is Schnorr random relative to A above A.

If $A \not\geq_T K$, then by Theorem 2.1, there is a set which is above A, Schnorr random relative to A and not recursively random. Clearly, it is not Martin-Löf random. ∎

Remark 2.3. It should be noted that this characterization can be extended to strong randomness: It holds that $A \geq_T K$ if and only if A is high for Schnorr randomness versus strong randomness. In contrast to this, Miller [17] showed that A is high for Martin-Löf randomness versus strong randomness if and only if there is a K-recursive function which is diagonally non-recursive relative to A.

3 Genericity

The weakly 1-generic sets are a generalization of the 1-generic sets. Their behaviour with respect to Turing degrees can be characterized easily: A Turing degree contains a weakly 1-generic set if and only if it contains a hyperimmune set [13]. It is now shown that the bases for weak 1-genericity also admit a nice characterization.

Theorem 3.1. *A set A is a basis for weak 1-genericity if and only if $A \not\geq_T K$.*

Proof. As mentioned in Property 1.2, it is sufficient to consider extension functions that depend only on the length of the string extended. Let $f_0, f_1, f_2, \ldots$ be a list of all total A-recursive functions from $\mathbb{N}$ to $\{0,1\}^*$ and let c_K be the convergence module of K.

First, suppose that $A \not\geq_T K$. Define G via a sequence $a_0, a_1, a_2, \ldots$ starting with $a_0 = 0$ inductively as follows:

- $a_{n+1} = a_n + 2 + c_K(n)$;
- $G(a_n) = K(n)$;
- $G(a_n + 1) = A(n)$;
- $G(a_n + 2)G(a_n + 3) \ldots G(a_{n+1} - 1)$ is $f_k(a_n + 2)0^{c_K(n) - |f_k(a_n+2)|}$ for the first k not used at previous stages such that $|f_k(a_n + 2)| \leq c_K(n)$.

As there are infinitely many k that map $a_n + 2$ to the empty string, a corresponding extension can always be found and the process goes through all stages of the construction.

The set G satisfies $A \leq_T G$ and $K \leq_T G$, as one can compute $A(n)$ and $K(n)$ inductively from G given a_n, then $c_K(n)$ from $K(0)K(1) \ldots K(n)$ and, finally, a_{n+1} from a_n and $c_K(n)$.

Assume now for a contradiction that some f_k is never used in this construction. Let k be its index. Then, from some n onwards, no $k' < k$ is selected due to the nature of the finite injury construction and hence k does not qualify as it could not be such an index. In other words, for all $n' \geq n$, $|f_k(a_{n'} + 2)| > c_K(n')$. As one can approximate $c_K(n)$ by $c_{K,s}(n) = \max\{t \leq s : \exists m \leq n\, [t = 0 \text{ or } m \text{ goes into } K \text{ at stage } t]\}$, one can compute for $n' \geq n$ the values

- $c_K(n')$ as $c_{K, f_k(a_{n'})}(n')$ and
- $a_{n'+1}$ as $a_{n'} + 2 + c_{K, f_k(a_{n'})}(n')$.

This gives $K \leq_T A$, which produces a contradiction, so every f_k will be built into the construction of G eventually and G is weakly 1-generic relative to A. Therefore, there is a $G \geq_T A$ such that G is weakly 1-generic relative to A and A is a basis for weak 1-genericity.

Second, suppose that $A \geq_T K$. Every set G which is weakly 1-generic relative to A is also 1-generic unrelativized. There is no 1-generic set above K, so A is not a basis for weak 1-genericity. ∎

The following corollary can be seen immediately.

Corollary 3.2. *A set A is high for weak 1-genericity versus 1-genericity iff $A \geq_T K$.*

The A such that every set which is 1-generic relative to A is also weakly 2-generic unrelativized have the same characterization.

Theorem 3.3. *A set A is high for 1-genericity versus weak 2-genericity iff $A \geq_T K$.*

Proof. If $A \geq_T K$, then every set which is 1-generic relative to A is 2-generic, so assume that $A \not\geq_T K$. For a given set G, define $\text{next}_G(n) = \min\{m - n : m \geq n \wedge m \in G\}$. This represents the distance to the next element of G after n. The basic idea of the proof is to show that there is a 1-generic set $G \leq_T A'$ such that $\text{next}_G(n) \leq c_K(n)$ for all n. This set will not be weakly 2-generic since it does not meet the K-recursive extension function $f(n) = 0^{c_K(n)+1}1$. The A'-recursive algorithm to produce G is the following. At stage 0, let $e = 0$, $n = 0$ and $G(0) = 1$. At each subsequent stage, proceed as follows.

1. While there is no extension of $G(0)G(1)\ldots G(n)$ in W_e^A, update $e = e + 1$.
2. If there is $\sigma \in \{0,1\}^*$ such that $G(0)G(1)\ldots G(n)\sigma \in W_e^A$ and $|\sigma| \leq c_K(n)$, then take the length-lexicographic first such σ, let $G(n+m+1) = \sigma(m)$ for all $m < |\sigma|$ and update $n = n + |\sigma|$ and $e = e + 1$.
3. Let $n = n + 1$ and $G(n) = 1$.

Note that there are infinitely many e with $W_e^A = \{0,1\}^*$, so the algorithm never loops in the first step for an infinite time.

Note that the current value $\tilde{e}$ of the variable e is only abandoned if the corresponding value $\tilde{n}$ of n is such that either $G(0)G(1)\ldots G(\tilde{n})$ has no extension in $W_{\tilde{e}}^A$ or after the first $\tilde{n}$ bits, G takes the values of a selected string $\tilde{\sigma}$ such that $G(0)G(1)\ldots G(\tilde{n})\tilde{\sigma}$ is in $W_{\tilde{e}}^A$. Furthermore, there is no constant $\tilde{e}$ such that the variable e equals $\tilde{e}$ from some point on, as that would mean that, for the corresponding value $\tilde{n}$, the extension of $G(0)G(1)\ldots G(\tilde{n})1^m$ in $W_{\tilde{e}}^A$ found first (relative to A) always has a length greater than $c_K(\tilde{n}+m)$. This would imply that $K \leq_T A$, contradicting the assumption on A. Therefore, every possible value $\tilde{e}$ of e is eventually taken and eventually abandoned and G is 1-generic relative to A. It can be seen from the construction that every $\tilde{\sigma}$ added after $\tilde{n}$ has length at most $c_K(\tilde{n})$ and is followed by a 1, so $\text{next}_G(\tilde{n}) \leq c_K(\tilde{n})$ for all $\tilde{n}$. This completes the proof. ▮

On one hand, if G is 1-generic relative to K, then G is already 2-generic. On the other hand, if $A \not\geq_T K$, then the preceding result shows that there is a set G which is 1-generic relative to A but not weakly 2-generic. Obviously, G is not 2-generic in this case. Hence one obtains the following corollary.

Corollary 3.4. *A set A is high for 1-genericity versus 2-genericity iff $A \geq_T K$.*

Note that G is weakly 2-generic relative to A iff it is weakly 1-generic relative to A'. Furthermore, G is 2-generic iff G is 1-generic relative to K. Hence one can relativize Corollary 3.2 such that whenever $A' \not\geq_T K'$, there is a set G which is weakly 1-generic relative to A' but not 1-generic relative to K; it follows that G is weakly 2-generic relative to A but not 2-generic. On the other hand, if $A' \geq_T K'$, then every set which is weakly 2-generic relative to A is also weakly 1-generic relative to A', weakly 1-generic relative to K', weakly 3-generic and 2-generic. Hence one has the following corollary.

Corollary 3.5. *A set A is high for weak 2-genericity versus 2-genericity iff A is high; that is, iff $A' \geq_T K'$.*

For the next result, assume that A is high for weak 1-genericity versus 2-genericity. As every 2-generic set is 1-generic, $A \geq_T K$ by Corollary 3.2. Hence $A \equiv_T B'$ for some B by the Jump Inversion Theorem and the sets which are weakly 1-generic relative to A are precisely those which are weakly 2-generic relative to B. It follows from Corollary 3.5 that $B' \geq_T K'$. Hence $A \geq_T K'$.

Conversely, consider any $A \geq_T K'$. Every set which is weakly 1-generic relative to A is also weakly 3-generic unrelativized. Hence A is high for weak 1-genericity versus 2-genericity. This is summarized in the following corollary.

Corollary 3.6. *A set A is high for weak 1-genericity versus 2-genericity iff $A \geq_T K'$.*

4 Recursive Randomness

It has already been shown in Theorem 2.2 that a set A is high for Schnorr randomness versus recursive randomness if and only if $A \geq_T K$. Furthermore, if A is PA-complete, then the universal r.e. martingale is majorized by a martingale which is recursive relative to A. Thus, if R is recursively random relative to A, then this martingale is not successful on R, so the first martingale does not succeed on R either and the set R is Martin-Löf random. Thus one obtains the following well known result.

Property 4.1. *If A is PA-complete, then A is high for recursive randomness versus Martin-Löf randomness.*

One might ask what is known about the other direction. Indeed, the above result is not known to be a characterization and the Turing degrees of many Martin-Löf random sets are not PA-complete. Hence, the next result is not a full characterization.

Theorem 4.2. *If A is high for recursive randomness versus Martin-Löf randomness, then there is a Martin-Löf random set $R \leq_T A$.*

Proof. Let A be a set that does not bound any Martin-Löf random set. It will be shown that A is not high for Martin-Löf randomness versus recursive randomness. This will be done by constructing a function $F \leq_T A'$ such that no Martin-Löf random set is Turing reducible to $A \oplus F$ and $A \oplus F$ has high Turing degree relative to A. Then there will be a set $Q \leq_T A \oplus F$ which is recursively random relative to A [21]. As Q is not Martin-Löf random, it follows automatically that A is not high for recursive randomness versus Martin-Löf randomness.

In order to code highness in F, one considers an A'-recursive injective enumeration $e_0, e_1, e_2, \ldots$ of all indices of partial A-recursive functions such that for all k, there is an $x \leq k$ with $\varphi_{e_k}^A(x)$ undefined. Furthermore, mg denotes the universal r.e. martingale and, for computations relative to a partial function ψ as an oracle, $\varphi_e^\psi(x)$ is undefined whenever the computation asks for some value of ψ outside the domain of ψ.

The function $F : \mathbb{N} \to \mathbb{N}$ is defined by stepwise extensions, starting with $\sigma_0 = 000$ and by building $\sigma_{n+1} = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,n}$ where each string $\tau_{n,m}$ is chosen from $\{e_n, e_n+1, e_n+2, \ldots\}^*$ such that $\eta = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,m}$ satisfies one of the following two conditions for all $m \leq n$.

- $\mathrm{mg}(\rho) > n$ for some $\rho \preceq \varphi_{m,|\eta|}^{A \oplus \eta}$.
- There is $x < |\eta|$ with $\neg(\varphi_m^{A \oplus \eta G}(x)\downarrow \in \{0,1\})$ for all $G \in \{e_n, e_n+1, e_n+2, \ldots\}^\infty$.

To verify the construction, one first shows that the algorithm never terminates, so there is always an extension. Let n and m be given, and let $\vartheta = \sigma_n e_n \tau_{n,0} \tau_{n,1} \ldots \tau_{n,m-1}$. Given A, one can iteratively search the strings $\gamma_0, \gamma_1, \gamma_2, \ldots \in \{e_n, e_n+1, e_n+2, \ldots\}^*$ such that

$$E(k) = \varphi_m^{A \oplus \vartheta \gamma_0 \gamma_1 \gamma_2 \ldots \gamma_k}(k)\downarrow \in \{0,1\}$$

for each k where γ_k is found.

If this goes through for all k, then $E \leq_T A$ and E is not Martin-Löf random. Therefore, there is a k such that $\mathrm{mg}(E(0)E(1)E(2)\ldots E(k)) > n$. Furthermore, one can choose $\tau_{n,m} = \gamma_0\gamma_1\gamma_2 \ldots \gamma_k$ and satisfy the first condition in the definition of $\tau_{n,m}$.

If this construction goes through up to some k but not beyond, it is impossible to define $E(k+1)$ with a value in $\{0,1\}$. In this case, one can choose $\tau_{n,m} = \gamma_0\gamma_1\gamma_2 \ldots \gamma_k$ for this k and satisfy the second condition in the definition of $\tau_{n,m}$.

Second, one shows that the resulting F is such that $F \oplus A$ has a Turing degree which is high relative to A. Given m, one can, relative to F, find the largest k such that $F(m) \leq k$ in the limit. It follows from the construction that whenever there is an n such that $e_n = m$, then $n \leq |\sigma_n| \leq k$. Hence one can check whether there is an $n \leq k$ such that $e_n = m$ relative to A in the limit. Then the overall algorithm is recursive in $F' \oplus A'$ and thus $A'' \leq_T (A \oplus F)'$. In other words, $A \oplus F$ has high Turing degree relative to A.

Third, one shows that there is no Martin-Löf random set recursive in $A \oplus F$.

To see this, consider any m such that $\varphi_m^{A\oplus F}$ is total and $\{0,1\}$-valued. Furthermore, consider the infinitely many $n \geq m$ satisfying $\forall k > n\,[e_n < e_k]$. These n must exist, as $e_0, e_1, e_2, \ldots$ is a injective enumeration of an infinite set. Now the extension $\tau_{n,m}$ cannot be selected as in the second item above, as then $\varphi_m^{A\oplus F}$ would either be partial or not $\{0,1\}$-valued. Therefore, the extension $\tau_{n,m}$ is chosen according to the first condition and there is some $\rho \preceq \varphi_m^{A\oplus\eta}$ such that $\mathrm{mg}(\rho) > n$ for $\eta = \sigma_n e_n \tau_{n,0}\tau_{n,1}\ldots\tau_{n,m}$. As ρ is a prefix of $\varphi_m^{A\oplus F}$, it follows that mg succeeds on $\varphi_m^{A\oplus F}$ and $\varphi_m^{A\oplus F}$ is not Martin-Löf random. This completes the proof. ∎

The next result shows that the above result is not optimal.

Theorem 4.3. *There is a Martin-Löf random set A which is not high for recursive randomness versus Martin-Löf randomness.*

Proof. Cholak, Greenberg and Miller [1] constructed an r.e. set $B <_T K$ and a function $f \leq_T B$ such that for a subclass of $\{0,1\}^\infty$ of measure 1, every function recursive relative to a member of this class is dominated by f. Such a function f is called "almost everywhere dominating".

A recursive martingale can be given by a distribution function φ_e which computes a rational number q between 0 and 2 on every input σ that says how to bet on the next bit. In other words, the capital at $\sigma 1$ is q times the old capital and the capital at $\sigma 0$ is $(2-q)$ times the old capital. It is known that the notion of recursive randomness (relative to some oracle) is the same whether one uses real-valued or rational-valued martingales [25], so one can describe the martingales using the functions φ_e. In the case of an oracle E, one considers the distribution function φ_e^E.

Now one produces an B-recursive martingale mg (the superscript B is omitted here and from now on in order to keep notation simple) which follows the following strategy: For each oracle E and each index e, mg_e^E computes the capital $\mathrm{mg}_e^E(\sigma)$ using the base case $\mathrm{mg}_e^E(\sigma) = 1$ when $|\sigma| \leq e$. If $|\sigma| \geq e$ and $a \in \{0,1\}$, then one defines mg_e^E inductively using the following formula:

$$\mathrm{mg}_e^E(\sigma a) = \begin{cases} q \cdot \mathrm{mg}_e^E(\sigma) & \text{if } a = 1 \text{ and } q = \varphi_e^E(\sigma) \text{ is in } \mathbb{Q},\ 0 \leq q \leq 2 \\ & \text{and } q \text{ is computed with time and use } f(|\sigma|); \\ (2-q) \cdot \mathrm{mg}_e^E(\sigma) & \text{if } a = 0 \text{ and } q = \varphi_e^E(\sigma) \text{ is in } \mathbb{Q},\ 0 \leq q \leq 2 \\ & \text{and } q \text{ is computed with time and use } f(|\sigma|); \\ \mathrm{mg}(\sigma) & \text{otherwise.} \end{cases}$$

The martingale mg is defined as

$$\mathrm{mg}(\sigma) = \sum_{e=0,1,2,\ldots} 2^{-e-1} \int_E \mathrm{mg}_e^E(\sigma)\, dE$$

and mg is B-recursive since $\mathrm{mg}_e^E(\sigma)$ can be computed from the first $f(|\sigma|)$ bits of E for each σ and E and, furthermore, $\mathrm{mg}_e^E(\sigma)$ can only differ from 1 when $e \leq |\sigma|$.

One can now choose a B-recursive set R on which mg is not successful; mg does not make any profit on this set and $\mathrm{mg}(B(0)B(1)\dots B(n)) \leq 1$ for all n. The set R is not Martin-Löf random as B is r.e. and Turing incomplete [22, 23].

Now it is shown that R is recursively random relative to every member A of a class of measure 1. Assume for a contradiction that this is not the case. Then there must be a fixed martingale mh such that mh^A is A-recursive and mh^A succeeds on R for a set of oracles A which does not have measure 0. Using arguments given by Mihailović [16] as well as Franklin and Stephan [4], one can assume that mh has the savings property and that $\mathrm{mh}^A(\sigma\tau) \geq \mathrm{mh}^A(\sigma) - 2$ for all $\sigma, \tau \in \{0,1\}^*$. The class

$$\mathcal{A} = \{A : \mathrm{mh}^A \text{ is total and } \forall c \exists n\, [\mathrm{mh}^A(R(0)R(1)\dots R(n)) > c]\}$$

is measurable and hence has positive measure. Note that one can compute φ_e^A from mh^A for all these A. Therefore, if f dominates all A-recursive functions, then there is a constant r_A such that $\mathrm{mh}^A(\sigma) \leq r_A \cdot \mathrm{mg}_e^A(\sigma)$ for all σ, as mg_e^A is computed using the function $\varphi_e^A(\sigma)$ for almost all σ. Since one can require that $r_A \in \mathbb{N}$, there are only countably many choices for each A and so there must be one fixed constant r and some $\epsilon > 0$ such that the class

$$\mathcal{B} = \{A \in \mathcal{A} : \forall \sigma \in \{0,1\}^*\, [\mathrm{mh}^A(\sigma) \leq r \cdot \mathrm{mg}_e^A(\sigma)]\}$$

has measure ϵ. Due to the savings property of mh, there is a function g such that the measure of each class

$$\mathcal{C}_n = \{A \in \mathcal{B} : \mathrm{mh}^A(R(0)R(1)\dots R(g(n))) > n+1\}$$

is at least $\epsilon \cdot \frac{n}{n+1}$: $g(n)$ is simply the first m such that for sufficiently many members of $\mathcal{B}$, mg has already reached a value above $n+3$ after processing $R(0)R(1)\dots R(m)$ and, due to the savings property, is therefore still above $n+1$. It follows for all n that

$$\mathrm{mg}(R(0)R(1)\dots R(g(n))) \geq \epsilon \cdot \tfrac{n}{n+1} \cdot 2^{-e-1} \cdot \tfrac{1}{r} \cdot (n+1) = \epsilon \cdot 2^{-e-1} \cdot \tfrac{1}{r} \cdot n,$$

which contradicts the fact that mg does not succeed on R. Hence R is recursively random relative to all members of a class of measure one. One of these A is also Martin-Löf random. In other words, R witnesses that there is a Martin-Löf random A which is not high for recursive randomness versus Martin-Löf randomness. ∎

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