Graphs realised by r.e. equivalence relations*

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Abstract. We investigate dependence of recursively enumerable graphs on the equality relation given by a specific r.e. equivalence relation on ω. In particular we compare r.e. equivalence relations in terms of graphs they permit to represent. This defines partially ordered sets that depend on classes of graphs under consideration. We investigate some algebraic properties of these partially ordered sets. For instance, we show that some of these partial ordered sets possess atoms, minimal and maximal elements. We also fully describe the isomorphism types of some of these partial orders.

1 Introduction

Recursively enumerable (r.e.) structures are given by a domain, recursive functions representing basic operators in the structure, and some recursively enumerable predicates, among which there is a predicate E representing the equality relation in the structure. When E is fixed, various algebraic properties of r.e. structures with the equality relation E depend heavily on the equivalence relation E. Furthermore, various computability-theoretic properties of E depend on algebraic properties of structures in which the equality relation is E. For example, Novikov constructed a finitely generated group with undecidable word-problem; in other words, there is a group which can be represented using an r.e. but nonrecursive equivalence relation E (as equality of the group) but not using a recursive equivalence relation E. On the other hand, for Noetherian rings [29], Baur [3] showed that every r.e. Noetherian ring is a recursive ring, implying that the underlying equality E is always a recursive relation. So only recursive equality relations E can be used to represent Noetherian rings.

Our aim is to investigate recursively enumerable graphs emphasising the role of the r.e. equivalence relation E representing the equality. In the paper [18] we initiated this program

* A. Gavruskin is supported by the FRDF grant of the University of Auckland; S. Jain is supported in part by NUS grants C252-000-087-001 and R252-000-420-112; B. Khoussainov is partially supported by Marsden Fund of the Royal Society of New Zealand and he did part of this work while on Sabbatical Leave to the School of Computing, National University of Singapore; F. Stephan is supported in part by NUS grant R252-000-420-112.

and studied general properties of r.e. structures, particularly various classes of algebras and linear orders. In this paper, we study recursively enumerable graphs, their properties, and their dependence on the equality relation. Later we will define various classes of graphs, but for the meantime for the reader by graph we mean a set of vertices together with a set of edges between the vertices where self-loops are allowed.

Our focus on graphs is motivated by the fact that for every algebraic structure one can construct a graph such that the structure and the graph are interpretable in each other through the first-order logic [20]. Thus, consideration of graphs gives us a general framework under which we can investigate the relationship between various properties of r.e. structures and their dependence on underlying equality relation E. We expect that the general framework provided in this paper will be further developed for studying the relationship between r.e. equivalence relations and various types of structures such as groups, Boolean algebras, partial orders and so on. In the next section, we proceed more formally.

1.1 Basic Definitions

Here is a simple graph-theoretic terminology. A *directed graph* is a pair $(V;\mathrm{Edge})$ where Edge is a binary relation on the set V of vertices. By a *graph* we mean a pair $(V;\mathrm{Edge})$ where Edge is a binary symmetric and irreflexive relation. Thus, for graphs $(V;\mathrm{Edge})$ the set of edges is simply a collection of unordered pairs. Finally, a pair $(V;\mathrm{Edge})$ is called a *pseudograph* if Edge is a binary symmetric relation on V. Thus, pseudographs can be viewed as graphs in which self-loops, that is pairs (x,x), are allowed as edges. We denote the classes of graphs, directed graphs and pseudographs by Graph, Dgraph, and Pgraph, respectively. For any graph $(V;\mathrm{Edge})$, and $x \in V$, we let $\mathrm{Edge}(x) = \{x' : (x,x') \in \mathrm{Edge}\}$.

We start with the following example in order to provide a central definition that connects r.e. equivalence relations with graphs. Let G be a finitely presented group with a presentation P and generators $g_1,\ldots,g_n$. Consider the Cayley graph $\Gamma(G)$ of the group with respect to the generators. Recall that the vertices of the Cayley graph $\Gamma(G)$ are elements of the group, and an edge is put between vertices x and y if and only if there is a generator g_i such that $x = yg_i$ or $x = yg_i^{-1}$. The vertices of the graph can be represented as words over the alphabet $\Sigma = \{g_1,\ldots,g_n,g_1^{-1},\ldots,g_n^{-1}\}$. On the set $\Sigma^\star$ of all words over the alphabet Σ consider the following relation E:

$$E = \{(u,v) : \text{the words } u \text{ and } v \text{ represent the same element in } G\}.$$

This relation is an equivalence relation on $\Sigma^\star$. Since G is finitely presented, the relation E is recursively enumerable. Thus, the Cayley graph $\Gamma(G)$ can be defined as follows:

1. The vertices of the graph are E-equivalence classes.
2. The edge relation Edge consists of all pairs $([u]_E,[v]_E)$ such that for some generator g_i we have either $(ug_i,v) \in E$ or $(ug_i^{-1},v) \in E$, where $[x]_E$ denotes the equivalence class containing x.

An important observation here is that the edge relation Edge is independent of the representatives of equivalence classes. This example suggests to single out those graphs whose vertices are E-equivalence classes and whose edge relations Edge are independent of the representatives of the equivalence classes. Note that we can code the set of all words $\Sigma^\star$ by natural numbers. Therefore, the equality relation E and the edge relation Edge can be viewed as r.e. binary relations on ω the set of natural numbers.

In this paper, we *always* assume that our equivalence relations E are r.e. equivalence relations on the set of natural numbers ω. We note that Ershov [10], and following him Odifreddi [30], call r.e. equivalence relations *positive* equivalence relations.

Let E be an r.e. equivalence relation on ω. We say that an n-ary relation R on ω *respects* E if for all $x_1, y_1, x_2, y_2, \ldots, x_n, y_n \in \omega$ such that $(x_1, y_1), \ldots, (x_n, y_n) \in E$ we have $(x_1, \ldots, x_n) \in R$ if and only if $(y_1, \ldots, y_n) \in R$. Note that if $n = 1$, then R is simply a unary relation of ω, and R respects E if and only if R is a union of E-equivalence classes.

Let $[x]_E$ denote the equivalence class of x with respect to the equivalence relation E, that is, $\{y : (x, y) \in E\}$.

For graphs, we will be using a binary r.e. relation Edge. An r.e. binary relation Edge $\subseteq \omega^2$ *respects* E if for all $x_1, y_1, x_2, y_2 \in \omega$ such that $(x_1, y_1) \in E$ and $(x_2, y_2) \in E$ we have $(x_1, x_2) \in$ Edge if and only if $(y_1, y_2) \in$ Edge. If Edge $\subseteq \omega^2$ respects E then Edge induces a binary relation on the quotient ω/E. We denote this edge relation by Edge$/E$. Note that, for Edge respecting E, $\mathrm{Edge}(x)/E = (\mathrm{Edge}/E)([x]_E) = \{[y]_E : (x, y) \in \mathrm{Edge}\}$. Note that the vertices of the graph (ω, Edge) are $x \in \omega$, but the vertices in the graph $(\omega/E; \mathrm{Edge}/E)$, for an r.e. set Edge respecting E, are the E-equivalence classes $[x]_E$, $x \in \omega$.

To ease the notation, we will be denoting graphs of the form $(\omega/E; \mathrm{Edge}/E)$ by simply $(\omega; \mathrm{Edge})/E$.

Definition 1. Let E be an r.e. equivalence relation.

1. A *directed E-graph* is a structure of the form $(\omega; \mathrm{Edge})/E$, where Edge is an r.e. binary relation respecting E.
2. An *E-graph* is a structure of the form $(\omega; \mathrm{Edge})/E$, where Edge is a symmetric, irreflexive and r.e. binary relation respecting E.
3. An *E-pseudograph* is a structure of the form $(\omega; \mathrm{Edge})/E$, where Edge is a symmetric and r.e. binary relation respecting E.

It is obvious that every directed E-graph (E-graph, E-pseudograph) is also a directed graph (graph, pseudograph). We say that a graph (directed graph, pseudograph) is *recursively enumerable* if it is isomorphic to an E-graph (directed E-graph, E-pseudograph) for some r.e. equivalence relation E.

Let C be a class of graphs (pseudographs, directed graphs), where we identify graphs up to isomorphism. Given an r.e. equivalence relation E we would like to single out those graphs in the class C that are isomorphic to E-graphs, as given in the following definition.

Definition 2. Given an r.e. equivalence relation E and a graph (directed graph, pseudograph) $\mathcal{G}$, we say that E *realises* $\mathcal{G}$ if and only if there is an r.e. relation Edge such that Edge respects E and $\mathcal{G}$ is isomorphic to $(\omega; \mathrm{Edge})/E$. If E does not realise $\mathcal{G}$, then we say that E *omits* $\mathcal{G}$. We let $\mathcal{K}_C(E)$ denote all those graphs from C which are realised by E.

In some proofs, we need to consider a join (also called disjoint union) of two equivalence relations. $E_0 \oplus E_1 = \{(2x, 2y) : (x, y) \in E_0\} \cup \{(2x+1, 2y+1) : (x, y) \in E_1\}$. Similarly, one can define a join of n equivalence relations: $E_0 \oplus E_1 \oplus \ldots \oplus E_{n-1} = \{(nx+i, ny+i) : i < n, (x, y) \in E_i\}$. (Note that, alternatively, one could have defined $E_0 \oplus E_1 \oplus \ldots \oplus E_{n-1}$ as $((\ldots((E_0 \oplus E_1) \oplus E_2)\ldots) \oplus E_{n-1})$. The mechanism used for the paper is just for ease of notation).

1.2 Examples

Now we present several examples that illustrate Definition 2 given above.

Example 3. Suppose E is an r.e. equivalence relation on ω. If ω/E is finite, then a graph $\mathcal{G} = (V; \mathrm{Edge})$ belongs to $\mathcal{K}_{\mathrm{Graph}}(E)$ if and only if the cardinality of V equals the cardinality of ω/E.

In the example above, recursive enumerability of E is used essentially. We note that the example above holds true also if E is a co-r.e. equivalence relation. In view of this example, from now on, *all our graphs will be countably infinite*, that is, graphs whose set of vertices is an infinite set. Thus, unless otherwise specified, we assume that all r.e. equivalence relations E considered in this paper are infinite (that is, ω/E is infinite).

Example 4. Let E be the identity relation id_ω on ω. Then the class $\mathcal{K}_{\mathrm{Graph}}(E)$ consists of all graphs $(\omega; \mathrm{Edge})$ where Edge is an r.e. set of unordered pairs. In particular, this class contains all recursive graphs.

Example 5. Let $X \subseteq \omega$ be a r.e. set. Consider the following relation $E(X)$:

$$E(X) = \{(x, y) : x = y\} \cup \{(x, y) : x, y \in X\}.$$

Each equivalence class of $E(X)$ is either a singleton $\{x\}$ where $x \notin X$ or is the set X itself. A *permutation directed graph* is a directed graph of the form $(A; \mathrm{Edge})$, where Edge determines a permutation on A. If $(a, a) \in \mathrm{Edge}$ for some $a \in A$, where Edge defines a permutation directed graph on A, then a is called a fixed point. For X being r.e. and coinfinite, [every permutation directed graph from $\mathcal{K}_{\mathrm{Dgraph}}(E(X))$ has a fixed point] if and only if [X is a nonrecursive set]. To see this, note that if X is recursive, then it is easy to construct a permutation directed graph in $\mathcal{K}_{\mathrm{Dgraph}}(E(X))$ which does not have a fixed point. This can be done by mapping X to y and y to X, for some fixed $y \in \overline{X}$, and having some recursive permutation of elements of $\overline{X} - \{y\}$ without a fixed point. On the other hand, if $(\omega; \mathrm{Edge})/E(X) \in \mathcal{K}_{\mathrm{Dgraph}}(E(X))$ does not have a fixed point, then for any fixed $x \in X$ and (the unique) y such that $(x, y) \in \mathrm{Edge}$, $\overline{X} = \{z : (\exists y' \neq y)[(z, y') \in \mathrm{Edge}]\}$, which is r.e., and thus X is recursive.

The paper [18] calls permutation directed graphs *permutation algebras* and the reader is referred to that paper for several properties of E-permutation directed graphs. We will be using the r.e. equivalence relation $E(X)$ of this example throughout the paper.

Example 6. Let E be an r.e. but not recursive equivalence relation. Then the class $\mathcal{K}_{\mathrm{Dgraph}}(E)$ does not contain the successor directed graph $(\mathbb{Z}, S)$, where

$$S = \{(x, y) : \ x \text{ and } y \text{ are integers and } x + 1 = y\}.$$

Indeed, if $(\mathbb{Z}, S) \in \mathcal{K}_{\mathrm{Dgraph}}(E)$ as witnessed by $(\omega; \mathrm{Edge})/E$ then E must be recursive. To see this, for distinct x, y, (i) $(x, y) \in E$ if (x, y) is enumerated in E, and (ii) $(x, y) \notin E$ if there exist $r \geqslant 2$, $z_1, z_2, \ldots, z_r$ such that $(z_i, z_{i+1}) \in \mathrm{Edge}$, for $1 \leqslant i < r$, and either $x = z_1$ and $y = z_r$ or $y = z_1$ and $x = z_r$ (that is, there exists a directed path from x to y or y to x in the directed graph represented by $(\omega; \mathrm{Edge})/E$).

Example 7. A *complete graph* is a graph that has edges between all pairs x, y of its vertices, where $x \neq y$. We call a pseudograph $\mathcal{G}$ an *n-complete pseudograph* if $\mathcal{G}$ has exactly n vertices with self-loops and has edges between all pairs x, y of its vertices, where $x \neq y$. Call $\mathcal{G}$ a *fully complete pseudograph* if there exists an edge between any pair of vertices of $\mathcal{G}$ (including self loops). We observe the following:

1. Every r.e. equivalence relation realises a fully complete pseudograph;
2. For each $n \in \omega$ there exists an r.e. equivalence relation E such that E realises a k-complete pseudograph if and only if $k \geqslant n$.

Part (1) is obvious. For part (2) we proceed as follows. Let $X_1, \ldots, X_n$ be pairwise disjoint r.e., but nonrecursive, sets such that $\omega - (X_1 \cup \ldots \cup X_n)$ is infinite. Consider the r.e. equivalence relation $E(X_1, \ldots, X_n)$:

$$(x, y) \in E(X_1, \ldots, X_n) \ \Leftrightarrow \ (x = y) \vee (\bigvee_{s=1}^{n} (x, y \in X_s)).$$

Assume that all the sets X_1, …, X_n are recursively enumerable but not recursive. Consider any k-complete pseudograph $\mathcal{G} = (\omega; \mathrm{Edge})/E(X_1, \ldots, X_n)$. Consider any $x_i \in X_i$, for $1 \leqslant i \leqslant n$. Then, $(x_i, x_i) \in \mathrm{Edge}$; otherwise, X_i would be co-r.e. and thus a recursive set as $x \notin X_i$ if and only if $(x, x_i) \in \mathrm{Edge}$. Hence, $\mathcal{G}$ must have at least n vertices with self-loops. Finally, it is easy to verify that the r.e. equivalence relation $E(X_1, \ldots, X_n)$ realises all k-complete pseudo graphs for all $k \geqslant n$: we can take the self loops for the vertices corresponding to $X_1, \ldots X_n$ and $(k - n)$ elements from $\omega - (X_1 \cup \ldots \cup X_n)$.

Example 8. If an infinite graph $\mathcal{G}$ has finitely many edges then every r.e. equivalence relation E realises $\mathcal{G}$.

The example above shows that for every E the class $\mathcal{K}_{\text{Graph}}(E)$ is not empty. Now we give the following definition through which we show, in Proposition 10 below, the influence of the r.e. equivalence relations E on algebraic properties of graphs realised by E.

Definition 9. The *transversal* of a r.e. equivalence relation E, denoted by $\mathrm{tr}(E)$, is the set $\{n : (\forall x)[x < n \Rightarrow (x, n) \notin E]\}$.

Thus, the transversal $\mathrm{tr}(E)$ is the set of all minimal elements taken from the equivalence classes of E. It is easy to see that for recursively enumerable E, E is Turing equivalent to $\mathrm{tr}(E)$.

Recall that a set X of natural numbers is *hyperimmune* if there does not exist a recursive function g such that $g(i) \geqslant x_i$ for all i, where $x_0 < x_1 < x_2 < \ldots$ and $X = \{x_0, x_1, \ldots\}$. Equivalently, if X is hyperimmune, then there does not exist a recursive function g such that $g(x) \geqslant \min\{y \in X : y > x\}$. We also say that a set X is *hypersimple* if X is recursively enumerable and its complement is hyperimmune [32]. We call a graph $\mathcal{G} = (V; \mathrm{Edge})$ *locally finite* if the set $\mathrm{Edge}(v)$ is finite for all $v \in V$. A locally finite E-graph $\mathcal{G} = (\omega; \mathrm{Edge})/E$ is *strongly locally finite* if there exists a recursive function, which we call a *witness function*, that given an $n \in \omega$ produces a tuple $m_1, \ldots, m_k$ such that the following properties hold:

1. $(n, m_i) \in \mathrm{Edge}$ for $i = 1, \ldots, k$;
2. For every y such that $(n, y) \in \mathrm{Edge}$, there exists an m_i for which $(y, m_i) \in E$.

A graph $\mathcal{G} = (V; \mathrm{Edge})$ is called *absolutely locally finite* if and only if every connected set of vertices is finite (here a set of vertices $V' \subseteq V$ is said to be connected (in $\mathcal{G}$) if and only if for all $u, v \in V'$, there exist $z_1, \ldots, z_k \in V'$ such that $(z_i, z_{i+1}) \in \mathrm{Edge}$, for $1 \leqslant i < k$, and $z_1 = u$, $z_k = v$). Now we present a simple proposition whose proof is essentially borrowed from [21], where E-algebras were considered (also see [18]).

Proposition 10. *Suppose E is an r.e. equivalence relation. If the transversal $tr(E)$ is hyperimmune then every strongly locally finite E-graph must be absolutely locally finite.*

Proof. Suppose Edge is r.e. and $\mathcal{G} = (\omega; \mathrm{Edge})/E$ is a strongly locally finite E-graph with a witness function f. Assume by way of contradiction that $\mathcal{G}$ has an infinite connected subset X of vertices with $e \in \mathrm{tr}(E)$ being least such that $[e]_E \in X$. Thus, for all $x \in \omega$, there exists a $y \in \mathrm{tr}(E)$ satisfying $y \leqslant x + e$ and $[y]_E \in X$, such that for some z, $[z \in \mathrm{tr}(E)$, $(y, z) \in \mathrm{Edge}$ and $z > x + e]$. Hence, $g(x) = \max\{f(y) : y \leqslant x + e\} > \min\{z \in \mathrm{tr}(E) : z > x + e\}$, contradicting $\mathrm{tr}(E)$ being hyperimmune. $\square$

1.3 Reducibilities

The definition of the class $\mathcal{K}_C(E)$ depends on two parameters: the class C of graphs (directed graphs, pseudographs) and the r.e. equivalence relation E. When we fix an r.e. equivalence relation E, the class $\mathcal{K}_C(E)$ calls for a description of those graphs from C that can be realised over E. From this point of view the class $\mathcal{K}_C(E)$ represents a graph-theoretic content of the universe ω/E. When we fix a class C of graphs, one considers those r.e. equivalence relations E

that realise graphs from C. The collection of these r.e. equivalence relations can be viewed as a computability-theoretic content of the class C. These observations call for the investigation of the relationship between r.e. equivalence relations in terms of graphs (from the class C) they realise. Formally, this is explained through the following definitions (also, see [18]).

Definition 11. Let C be a class of graphs (pseudographs or directed graphs) and let E_1 and E_2 be r.e. equivalence relations. We say E_1 is *C-reducible to* E_2, written $E_1 \leqslant_C E_2$, if and only if every graph in C realised by E_1 is also realised by E_2. In particular, we have the following reductions when C is the class of directed graphs, pseudographs or graphs.

1. $E_1 \leqslant_{\text{Dgraph}} E_2$ if and only if all directed graphs realised by E_1 are realised by E_2;
2. $E_1 \leqslant_{\text{Pgraph}} E_2$ if and only if all pseudographs realised by E_1 are realised by E_2;
3. $E_1 \leqslant_{\text{Graph}} E_2$ if and only if all graphs realised by E_1 are realised by E_2.

Sometimes we use a terminology borrowed from recursion theory. For instance, similar to m-degrees or Turing degrees, we say that E_1 and E_2 have the same *C-degree*, written $E_1 \equiv_C E_2$, if and only if $E_1 \leqslant_C E_2 \wedge E_2 \leqslant_C E_1$. The reducibility $\leqslant_C$ naturally induces the partial order on the set of all C-degrees. Without much confusion we use the same symbol $\leqslant_C$ to denote this partial order on C-degrees. Thus, there are two lines of investigation. One is to study basic properties of the partial order $\leqslant_C$ on the set of all C-degrees. The other is to investigate the graphs from $\mathcal{K}_C(E)$ by selecting various classes C of graphs. In this paper we initiate the study in both directions.

Most reducibilities (if not all) on r.e. equivalence relations and sets that have been studied aim to capture recursion-theoretic and set-theoretic complexities between r.e. equivalence relations. Typically a reduction from E to E' tells us that E' is a harder problem to solve than E. For instance, Turing reducibility from E to E' implies that by having an oracle with access to E' we can design an algorithm that decides E. In contrast, our reducibilities given in Definition 11 aim to compare r.e. equivalence relations E and E' in terms of their algebraic content — namely the classes $\mathcal{K}_C(E)$ and $\mathcal{K}_C(E')$ that they represent.

1.4 Connections to related work

The paper [18] initiated the study of E-structures in general setting. In particular, it investigated an important class $\mathcal{L}$ of pseudographs, namely the class of linearly ordered sets (where $(x, y) \in \text{Edge} \Leftrightarrow x \leqslant y$). It obtained some basic results about the partial order $\leqslant_{\mathcal{L}}$ and built certain r.e. equivalence relations E for which the classes $\mathcal{K}_{\mathcal{L}}(E)$ can easily be described. For instance, it constructed r.e. equivalence relations E that realise only n many linear orders, where n is fixed [18]. It also characterised some classes of linear orders that can be realised by r.e. equivalence relations of type $E(X)$. The paper [18] also studied the C-reducibility in the cases when C is chosen as some classes of universal algebras.

One of the reducibilities, the many-one reducibility, or m-reducibility, between equivalence relations, is a very well-studied notion. Originally introduced by Post to compare sets [31], it was later carried over to compare equivalence relations. In particular Maltsev [25,26] and subsequently Ershov [10,11] studied many-one reducibility in both the context of all equivalence relations on the set of natural numbers and in the restricted context of comparing r.e. equivalence relations; Bernardi and Sorbi [5] begun the first systematic study on r.e. equivalence relations with respect to many-one reducibility. The many-one reducibility between r.e. equivalence relations has later been revisited due to the works [12]-[15]. Friedman and Fokina [13] considered also reducibilities between equivalence relations in a more general context; however, in the context of this paper we need only the special case when the two r.e. equivalence relations compared have the domain ω.

Now we give the definition of m-reducibility in the context of comparing equivalence relations on the set of natural numbers. For r.e. equivalence relations E_1, E_2 on ω, we say that E_1 is *m-reducible to E_2*, written $E_1 \leqslant_m E_2$, if and only if there exists a recursive function f such that for all $x, y \in \omega$ we have $[(x, y) \in E_1 \Leftrightarrow (f(x), f(y)) \in E_2]$. This naturally induces the equivalence relation $\equiv_m$ between E_1 and E_2 given as $E_1 \leqslant_m E_2 \wedge E_2 \leqslant_m E_1$. In this case it is said that E_1 and E_2 have the same *m-degree*. It is not hard to see that $\leqslant_m$ has the largest element among all the m-degrees [2,5,10,16,24,27].

One can also consider the following equivalence relation $\sim_m$ between equivalence relations. We say that E_1 and E_2 are *$\sim_m$-equivalent*, written $E_1 \sim_m E_2$, if and only if there exists a recursive function f witnessing $E_1 \leqslant_m E_2$ such that all equivalence classes of E_2 appear in the range of f. Note that $\sim_m$ is an equivalence relation that implies $\equiv_m$. When comparing $\sim_m$ with $\equiv_m$, it turns out that $\sim_m$ is a more restrictive condition than $\equiv_m$, namely $\equiv_m$ does not always imply $\sim_m$. This stands in contrast to one-one reducibility between r.e. subsets of ω [32]. We also note that if X_1, X_2 are two infinite r.e. sets then for the r.e. equivalence relations $E(X_1)$ and $E(X_2)$ (defined in Example 5) we have the following: $E(X_1) \leqslant_m E(X_2)$ if and only if $X_1 \leqslant_1 X_2$ [9,11,28]. Coskey, Hamkins and Miller [8,9] as well as Gao and Gerdes [16] also contributed to the study of r.e. equivalence relations and m-reducibility. Finally, we mention a recent paper by Andrews, Lempp, Miller, Ng, San Mauro and Sorbi [2] that presents a comprehensive study of m-reducibility between r.e. equivalence relations, in particular, answering several questions posed in the work of Gao and Gerdes [16].

2 Isles

Let C be a class of pseudographs (directed graphs, graphs). Intuitively, there is not much connection between $\leqslant_C$-reducibility and m-reducibility on r.e. equivalence relations. In fact, we observe that the definitions of $\leqslant_m$ and $\leqslant_C$ imply that m-reducibility is an arithmetic definition while $\leqslant_C$-reducibility is a Σ^1_1-definition. However, in this section we show that for some natural classes of pseudographs, one might find connections between these two reducibilities. In this section we introduce such a class. We call pseudographs from this class *isles*. Here is the definition of isles.

Definition 12. An *isle* or an *island graph* is a pseudograph which has infinitely many isolated vertices. Formally, an isle is a pseudograph $(V; \mathrm{Edge})$ such that there are infinitely many vertices $x \in V$ satisfying $\mathrm{Edge}(x) = \varnothing$. Denote the class of all isles by Isle.

Now we can recast Definition 11 for the class Isle of all isles. Namely, we say that $E \leqslant_{\text{Isle}} E'$ if and only if every isle realised by E is also realised by E'. As we mentioned above, the importance of this class of pseudographs stems from the fact that Isle-reducibility can, in some ways, be related to the m-reducibility. This will be explained in this section. Furthermore, we give a characterisation of all the isles that can be realised by all infinite r.e. equivalence relations; the characterisation involves a graph-theoretic concept of clique graphs. In addition, we construct natural examples of r.e. equivalence relations which only realise these isles. We will also prove that there are the least and the greatest Isle-degrees.

2.1 Connection between m-reducibility and Isle-reducibility

In recursion theory, often for the partially ordered set $(P; \leqslant)$ given by some reducibility $\leqslant$, the *greatest* element is called *universal*. Intuitively, universal degrees represent the hardest problems to which other problems can be reduced. For instance, the m-reducibility has a universal degree. The main result of this subsection is the following theorem:

Theorem 13. *If $E \leqslant_m E'$ then $E \leqslant_{Isle} E'$. Moreover, E' is Isle-universal if and only if E' is m-universal.*

Proof. First we show that $E \leqslant_m E'$ implies $E \leqslant_{\text{Isle}} E'$. Let f be a recursive function witnessing $E \leqslant_m E'$, that is, $(x, y) \in E \Leftrightarrow (f(x), f(y)) \in E'$. Let Edge be an r.e. relation on ω respecting E such that the pseudo graph $(\omega; \text{Edge})/E$ is an isle. One defines

$$(x', y') \in \text{Edge}' \Leftrightarrow (\exists (x, y) \in \text{Edge})[(f(x), x') \in E' \wedge (f(y), y') \in E'].$$

It is clear that Edge' is an r.e. relation which respects E'. Furthermore, (x', y') is in Edge' if and only if $x' \in [f(x)]_{E'}$ and $y' \in [f(y)]_{E'}$ for some edge $(x, y) \in \text{Edge}$. In addition, $\{[z]_{E'} : z \notin \bigcup_{x\in\omega}[f(x)]_{E'}\} \cup \{[f(x)]_{E'} : \text{Edge}(x) = \varnothing\}$ is infinite (that is, the set of vertices in $(\omega; \text{Edge}')/E'$ which are not an image (for f) of any vertex in $(\omega; \text{Edge})/E$ or an image of an isolated vertex in $(\omega; \text{Edge})/E$ is infinite). Hence, $\{[z]_{E'} : \text{Edge}'(z) = \varnothing\}$ is infinite. Thus, $(\omega; \text{Edge}')/E'$ is an isle. The pseudograph $(\omega; \text{Edge}')/E'$ is isomorphic to $(\omega; \text{Edge})/E$ through the map defined as follows. For vertices $[x]_E$ with $\text{Edge}(x) \neq \varnothing$, the isomorphism is given by f. For the rest of the vertices, the isomorphism is given by a bijective function that maps $\{[x]_E : \text{Edge}(x) = \varnothing\}$ to $\{[z]_{E'} : \text{Edge}'(z) = \varnothing\}$. We note that we may not be able to use f by default to establish the isomorphism as the range of f may not intersect all E' equivalence classes.

It follows from above that every m-universal r.e. equivalence relation is Isle-universal. Now assume that E' is Isle-universal. Our goal is to show that E' is m-universal. So, let E be an r.e. equivalence relation defined on ω. We want to show that there is a recursive function f that witnesses $E \leqslant_m E'$.

We construct the following graph. The set of vertices of the graph is the disjoint union $\omega \cup \{a, b, c, d, s_0, s_1, s_2, \ldots, t_0, t_1, t_2, \ldots\}$. The set Edge of edges of the graph consist of the following edges (along with their symmetric versions) $(s_{3k}, a), (s_{3k+1}, b), (s_{3k+2}, c), (s_k, s_{k+1}), (n, d)$, for all $n \in \omega$ and (h, s_k), whenever $(k, h) \in E$. All the vertices t_i are completely isolated. Define $\tilde{E}$ on the set of vertices of the graph so that $\tilde{E}$ is E on ω and is the identity equivalence relation on the rest of the vertices. The relation $\tilde{E}$ is an r.e. equivalence relation. In addition, the set Edge of edges of the graph built respect $\tilde{E}$. Thus, the graph obtained is an $\tilde{E}$–graph. Moreover, the graph is an isle because of the completely isolated vertices t_i.

Since E' is Isle-universal, there exists an E'-respecting relation Edge' such that the graph $\mathcal{G}' = (\omega; \mathrm{Edge}')/E'$ is isomorphic to the graph constructed above. Hence there are a', b', c', d', s'_0 such that $[a']_{E'}$, $[b']_{E'}$, $[c']_{E'}$, $[d']_{E'}$, $[s'_0]_{E'}$ are the copies (in $\mathcal{G}'$) of the vertices a, b, c, d, s_0 (in $\mathcal{G}$). Now one can use these representatives to build an algorithm recovering the counterparts (in $\mathcal{G}'$) of the vertices s_k (in $\mathcal{G}$) as follows. To find s'_k such that $[s'_k]_{E'}$ corresponds to s_k, one starts from s'_0 and finds inductively for $\ell = 1, 2, \ldots, k$ that vertex s'_ℓ such that $(s'_\ell, s'_{\ell-1}) \in \mathrm{Edge}'$, and $(s'_\ell, a') \in \mathrm{Edge}'$ in the case that ℓ modulo 3 is 0, $(s'_\ell, b') \in \mathrm{Edge}'$ in the case that ℓ modulo 3 is 1, $(s'_\ell, c') \in \mathrm{Edge}'$ in the case that ℓ modulo 3 is 2. Having s'_k, one can find a vertex $f(k)$ such that $(f(k), s'_k) \in \mathrm{Edge}'$ and $(f(k), d') \in \mathrm{Edge}'$. Clearly this function f is recursive.

Now the claim is that f is an m-reduction from E to E': If $(x, y) \in E$, then $[x]_E = [y]_E$ and thus (s_x, x) and (s_y, x) (s_x, y) and (s_y, y) are all in Edge. Therefore, $(s'_x, f(x))$ and $(s'_y, f(x))$, $(s'_x, f(y))$ and $(s'_y, f(y))$ are all in Edge' and thus $(f(x), f(y)) \in E'$. If $(x, y) \notin E$, then $\{z \in \omega : (s_x, z) \in \mathrm{Edge}, (d, z) \in \mathrm{Edge}\}$ is disjoint from $\{z \in \omega : (s_y, z) \in \mathrm{Edge}, (d, z) \in \mathrm{Edge}\}$. Thus, $\{z \in \omega : (s'_x, z) \in \mathrm{Edge}', (d', z) \in \mathrm{Edge}'\}$ is disjoint from $\{z \in \omega : (s'_y, z) \in \mathrm{Edge}', (d', z) \in \mathrm{Edge}'\}$. This implies that, $[f(x)]_{E'} \neq [f(y)]_{E'}$. So f has the properties of an m-reduction. $\square$

Since m-reducibility has universal elements [2,5,10,16,24,27], we have the following corollary.

Corollary 14. *The Isle-reducibility has universal elements.*

2.2 The existence of the least element in the Isle-degrees

In this subsection we prove that Isle-reducibility has the least element. We also construct an example of an infinite chain in the set of Isle-degrees. We start with the following definition that singles out those isles for which the set of edges is finite.

Definition 15. We call an r.e. isle *finitary* if and only if there are only finitely many vertices in the isle which have an edge to other vertices or themselves. That is, an isle $(\omega; \mathrm{Edge})/E$ is a *finitary isle* if and only if the set Edge/E is finite. If an isle is not finitary then we call it an *infinitary isle.*

The following lemma is easy to see.

Lemma 16. *Every r.e. equivalence relation E realises all finitary isles.*

Proof. If an isle is finitary and E is an r.e. equivalence relation with infinitely many equivalence classes then the set Edge can be represented by a finite list of pairs $(x_1, y_1), \ldots, (x_n, y_n)$ and therefore, one obtains that the relation

$$\mathrm{Edge} = \{(v, w) : (\exists m \in \{1, \ldots, n\})[(v, x_m) \in E \wedge (w, y_m) \in E]\}$$

respects E and represents the given isle in ω/E. Thus, the r.e. equivalence relation E realises the given isle. $\square$

The next result shows that the least element with respect to Isle-reducibility is determined by the class of all finitary isles. To prove this, we recall cohesive sets. Namely, an infinite set Z is *cohesive* if no r.e. set X exists that splits Z into two infinite subsets, that is, no r.e. X exists such that both $Z \cap X$ and $Z \cap (\omega - X)$ are infinite. *Maximal* sets are recursively enumerable sets whose complements are cohesive. Maximal sets are a well-known topic studied in recursion theory [30,32].

Theorem 17. *If the transversal tr(E) of an r.e. equivalence relation E is a cohesive set then the equivalence relation E realises only finitary isles. In particular, for every maximal set X, the r.e. equivalence relation $E(X)$ is the least element with respect to Isle-reducibility.*

Proof. By Lemma 16 above, E realises all finitary isles. For the rest of the proof assume that $\mathrm{tr}(E)$ is a cohesive set. Let Edge be an r.e. relation representing the edges of an isle realised by E. Now let $Y = \{y : (\exists x)[(x, y) \in \mathrm{Edge}]\}$. Since Edge represent edges of the isle, there are infinitely many pairwise non E-equivalent x which are not in Y. Note that Y respects E. Hence, the set Y must be a union of finitely many E-equivalence classes as otherwise Y would split $\mathrm{tr}(E)$ into two infinite subsets. Thus, there are only finitely many E-equivalence classes in Y. It follows that Edge represents a finite isle. Theorem now follows using Lemma 16. $\square$

Note that the Isle-least element contains infinitely many m-degrees. The reason is that if X and Y are maximal sets of different Turing degrees then $E(X)$ and $E(Y)$ are m-incomparable yet $E(X)$ and $E(Y)$ are Isle-equivalent. Hence, recursion-theoretically, two r.e. equivalence relations realising the same isles might be quite different.

Proposition 18. *The set of all Isle-degrees contains an infinite chain.*

Proof. Let X be a maximal set. For each $n \geqslant 3$, let E_n be defined as the join of $E(X)$ with itself n times, that is, $(m, m') \in E_n$ if and only if $m = m' \pmod{n}$ and $[\lfloor m/n \rfloor = \lfloor m'/n \rfloor$ or $[\lfloor m/n \rfloor \in X$ and $\lfloor m'/n \rfloor \in X]]$. It is clear that $E_n \leqslant_m E_{n+1}$ and thus $E_n \leqslant_{\mathrm{Isle}} E_{n+1}$.

Now let $G_n = (\omega; \mathrm{Edge}_n)/E_n$, where $\mathrm{Edge}_n = \{(m, m') : m \neq m' \pmod{n}$, and $m \neq 0 \pmod{n}$, $m' \neq 0 \pmod{n}\}$. Note that G_n is an isle graph with $(n-1)$ infinite subparts, each of whose vertices have an edge to the vertices of the other $(n-2)$ subparts.

Suppose by way of contradiction that G_n is realisable in E_{n-1}, for some $n \geqslant 3$. Suppose the realisation is via $(\omega; \mathrm{Edge})/E_{n-1}$. Then, let $\alpha_0, \alpha_1, \ldots, \alpha_{n-2}$ be representatives such that each $(\alpha_i, \alpha_j) \in \mathrm{Edge}$, for $i \neq j$, where $i, j < n-1$. Let $A_i = \{x : (\forall j < n-1, i \neq j)[(x, \alpha_j) \in \mathrm{Edge}]\}$. Then, the sets A_i are disjoint infinite sets respecting E_{n-1}. Thus, for each $k < n-1$, $\{(n-1)x + k : x \in X\} \subseteq A_i$ or $\{(n-1)x + k : x \in X\} \cap A_i = \varnothing$. By X being maximal we immediately have that, for each $i < n-1$, there exists a $k < n-1$ such that $\{(n-1)x + k : x \in X\} \cap A_i \neq \varnothing$; moreover by disjointness of different A_i, it follows that there exists a permutation $k_0, k_1, \ldots, k_{n-2}$ of $0, 1, 2, \ldots, n-2$ such that $\{(n-1)x + k_i : x \in X\} \subseteq A_i$ and $\{(n-1)x + k_j : x \in X\} \cap A_i = \varnothing$ for $j \neq i$, where $i, j < n-1$. Hence, as each A_i covers infinitely many E_{n-1} equivalence classes, we have by maximality of X that, for $i < n-1$, A_i is a finite variant of $\{(n-1)x + k_i : x \in \omega\}$. But then, $\omega - \bigcup_{i<n-1} A_i$ is finite and thus $(\omega; \mathrm{Edge})/E_{n-1}$ is not an isle and does not realise G_n. $\square$

2.3 Realising infinitary isles

In this subsection we characterise those r.e. equivalence relations that realise infinitary isles. The characterisation is given in terms of a well-known concept in graph theory, the notion of clique which is adapted to the concept of isles.

Definition 19. A *clique-isle* is an isle for which there is a set C such that (x, y) is an edge of the isle if and only if $x, y \in C$ and $x \neq y$. A *full-clique-isle* is an isle for which there is a set C such that (x, y) is an edge of the isle if and only if $x, y \in C$.

Note that both clique-isles and full-clique-isles (over a graph with countably many vertices) are uniquely determined by the cardinality of C and their first-order theories are $\aleph_0$-categorical. It turns out, as we prove in the theorem below, those r.e. equivalence relations that realise infinitary clique isles can be characterised in terms of m-reducibility.

Theorem 20. *An r.e. equivalence relation E realises an infinitary clique-isle if and only if $id_\omega \leqslant_m E$.*

Proof. $(\Rightarrow)$ Assume that E realises an infinitary clique-isle. Let Edge be the set of edges of this isle graph. Now we need to prove that $id_\omega \leqslant_m E$ by constructing a recursive function $f : \omega \to \omega$ such that $(f(i), f(j)) \notin E$ for all pairwise distinct i, j. Choose $f(0), f(1)$ as any pair of vertices satisfying $(f(0), f(1)) \in$ Edge. Now let, inductively, for $n \geqslant 2$, $f(n)$ be the first value x found such that $(f(m), x) \in$ Edge for all $m < n$. Note that this search terminates since the graph has an infinite clique and $f(0), \ldots, f(n-1)$ all belong to the infinite clique. Clearly, $f(n)$ also belongs to the infinite clique. Also note that $(f(n), f(i)) \notin E$ for all $i < n$ since the infinite clique-isle does not contain self-loops. Hence, $(f(i), f(n)) \notin E$ for all $i < n$. Therefore, the function f indeed witnesses that $id_\omega \leqslant_m E$.

$(\Leftarrow)$ Consider the following graph $(V; \text{Edge})$. The set V is ω and $(i, j) \in$ Edge if and only if $i \neq j$ and both i and j are even. The graph constructed is an infinitary clique-isle realised by id_ω. Therefore $id_\omega \leqslant_m E$ implies, by Theorem 13, that E realises this isle. $\square$

The theorem above allows us to easily prove the following statement:

Proposition 21. *There exists an r.e. equivalence relation E that realises an infinitary full-clique isle but omits all infinitary clique isles.*

Proof. Let S be a simple set and $g : \omega \to S$ be a recursive bijection. Consider the r.e. equivalence relation $E(g(S))$.

It is first shown that $id_\omega \not\leqslant_m E(g(S))$. Suppose by way of contradiction that f witnesses $id_\omega \leqslant_m E(g(S))$. Thus, as S is simple, range of f contains infinitely many elements of S. Hence, as f can map at most one element of ω to an element of $g(S)$, $g^{-1}(f(\omega) \cap S)$ is an infinite r.e. set containing at most one element of S, a contradiction to S being simple.

It follows using Theorem 20 that $E(g(S))$ does not realise an infinitary clique-isle. The binary relation Edge $= \{(i, j) : i, j \in S\}$ respects $E(g(S))$. Hence, $(\omega; \text{Edge})/E(g(S))$ is an infinitary full-clique-isle. $\square$

Now we characterise r.e. equivalence relations that realise infinitary full-clique-isle graphs.

Theorem 22. *For every r.e. equivalence relation E, the following statements are equivalent:*

(a) *E realises an infinitary isle;*
(b) *E realises an infinitary full-clique-isle;*
(c) *There is an r.e. set X respecting E such that X contains infinitely many E-equivalence classes and leaves out infinitely many E-equivalence classes.*

Proof. (a) $\Rightarrow$ (c): If E realises an infinitary isle with edge-relation Edge then one can define $X = \{x : (\exists y)[(x, y) \in \text{Edge}]\}$; the set X is r.e. and contains infinitely many equivalence classes of E. The set X is also disjoint from infinitely many equivalence classes of E. Furthermore, by definition, X respects E.

(c) $\Rightarrow$ (b): This can be done by defining that (x, y) is an edge if and only if $x, y \in X$. The edge-relation clearly respects E. It is also an r.e. relation; furthermore, it follows from the properties of X that the resulting graph is an infinitary full-clique-isle.

(b) $\Rightarrow$ (a): This follows directly from the definition, as every infinitary full-clique-isle is an infinitary isle. $\square$

2.4 The Isle-degrees have a unique atom

In Theorem 17 we proved that the partial order of the Isle-degrees has the least element. Our next theorem shows that there is an atom in this partial order. Recall that an *atom* in a partially ordered set $(X, \leqslant)$ with the least element z is a member a of X such that $a \neq z$ and no x exists strictly between z and a. Note that the theorem proves even more. Namely, not only the Isle-degrees have an atom, but also this atom is a lower bound of all non-zero Isle-degrees, which implies that the atom is unique. This in particular shows that the Isle-degrees are not dense.

Let $W_0, W_1, \ldots$ be a standard acceptable numbering of all r.e. sets. Let $edge_0, edge_1, \ldots$ be a standard acceptable numbering of all r.e. sets of (undirected) edges. $W_{i,t}$ denotes W_i enumerated in t time steps. We assume without loss of generality that $W_{i,t} \subseteq \{0, 1, \ldots, t-1\}$.

For a set A, we let $A(x) = 1$ if $x \in A$, and $A(x) = 0$ if $x \notin A$.

Theorem 23. *There exists an r.e. equivalence relation E such that E realises an infinitary isle and for every r.e. equivalence relation E', if E' realises an infinitary isle then $E \leqslant_{Isle} E'$.*

Proof. The idea is to construct an r.e. equivalence relation E and a maximal set X such that the following goals are met for any r.e. set Y and r.e. symmetric set Edge of pairs:

Goal 1 – X respects E and X/E is infinite;
Goal 2 – If Y respects E and Y/E is infinite, then $X \subseteq Y$;
Goal 3 – If Edge respects E and Edge$/E$ contains infinitely many disjoint non-self-loop edges (that is, if there exist infinitely many edges $(x_0, y_0), (x_1, y_1), \ldots$ in Edge such that $[x_0]_E, [y_0]_E, [x_1]_E, [y_1]_E, \ldots$ are all distinct), then $(x, y) \in \text{Edge}$ for all $x, y \in X$.

We will construct such r.e. equivalence relation E and the maximal set X later in the proof. For now, assume that we have constructed such E and X. Our goal is to show that every isle $\mathcal{G} = (\omega; \text{Edge})/E$ realised by E falls into one of the following five types:

Type 1 – The isle $\mathcal{G}$ is finitary;
Type 2 – Edge/E is a finite variant of $\{([x]_E, [x]_E) : x \in X\}$;
Type 3 – For some r.e. set Z respecting E with Z/E being finite, Edge/E is a finite variant of $\{([x]_E, [y]_E) : x, y \in X \text{ or } x \in X \wedge y \in Z \text{ or } x \in Z \wedge y \in X\}$;
Type 4 – For some r.e. set Z respecting E with Z/E being finite, Edge/E is a finite variant of $\{([x]_E, [y]_E) : x \in X \wedge y \in Z \text{ or } x \in Z \wedge y \in X\}$;
Type 5 – For some r.e. set Z respecting E with Z/E being finite, Edge/E is a finite variant of $\{([x]_E, [y]_E) : x \in X \wedge y \in Z \text{ or } x \in Z \wedge y \in X\} \cup \{([x]_E, [x]_E) : x \in X\}$.

It is clear that all of the isles of the above types can be realised by E using the fact that X is an r.e. set. Now we show that there are only these five types of isles realised by E. So, let $\mathcal{G} = (\omega; \text{Edge})/E$ be an isle realised by E.

Let $Y = \{x : \text{Edge}(x) \neq \varnothing\}$. If Y/E is finite, then the isle $\mathcal{G}$ is of the first type.

If Y/E is infinite, then, by the second goal, $X \subseteq Y$. As the graph is an isle and X is maximal, it follows that almost all non-elements of X must also be outside Y. Let $Z' = \{x \in Y : \text{Edge}(x)/E \text{ is infinite}\}$. Consider the case when Z'/E is infinite. By the third goal, for all $x, y \in X$, $(x, y) \in \text{Edge}$. Thus, $X \subseteq Z'$. Note that if $x \in Z' - X$ then $\text{Edge}(x)/E$ is infinite, and thus by second goal $X \subseteq \text{Edge}(x)$. Hence $\{(x, y) : x \in X \wedge y \in Z' \text{ or } x \in Z' \wedge y \in X\} \subseteq \text{Edge}$. As X is maximal, we have that $Y - X$ and hence $Z' - X, Y - Z'$ are all finite. Thus, Edge/E contains only finitely many pairs $([x]_E, [y]_E)$ with both $x, y \notin X$ (as $Y - X$ is finite) and only finitely many pairs $([x]_E, [y]_E)$ such that $x \in X$ and $y \notin (X \cup Z')$ (as there are only finitely many such y, and for such y, $\text{Edge}(y)/E$ is finite). Thus, $\mathcal{G}$ is of the third type.

The remaining case is that Z'/E is finite. For each $x \in Z'$, $\text{Edge}(x)/E$ is infinite, thus by the second goal $X \subseteq \text{Edge}(x)$. It follows that $X \times Z' \cup Z' \times X \subseteq \text{Edge}$. Now, (a) Edge/E contains only finitely many pairs $([x]_E, [y]_E)$ with both $x, y \notin X$ (as $Y - X$ is finite), (b) Edge/E contains only finitely many pairs $([x]_E, [y]_E)$ such that $x \in X$ and $y \notin (X \cup Z')$ (as there are only finitely many such y, and for such y, $\text{Edge}(y)/E$ is finite), and (c) Edge/E contains only finitely many pairs $([x]_E, [y]_E)$ such that $x, y \in (X - Z')$ and $[x]_E \neq [y]_E$ (as otherwise, by the third Goal, $X \times X \subseteq \text{Edge}$ and thus Z'/E is infinite). Furthermore, if Edge/E contains infinitely many $([x]_E, [x]_E)$ then by the second goal, it contains $([x]_E, [x]_E)$ for all $x \in X$. It follows that $\mathcal{G}$ is of the fourth or the fifth type (where, it is of the fourth type if Edge/E contains finitely many $([x]_E, [x]_E)$, and is of the fifth type if Edge/E contains infinitely many $([x]_E, [x]_E)$). Note that if Z is empty, then the fourth and fifth types collapse to first and second types respectively.

Note that whenever a relation E' realises an infinitary isle then by Theorem 22 there is an r.e. set X' respecting E' such that X' contains infinitely many equivalence classes and is disjoint from infinitely many equivalence classes. Therefore it is easy to verify that E' realises all the isles of the first, the second, the third, the fourth and the fifth types. Hence, $E \leqslant_{\text{Isle}} E'$.

Thus it remains to show that the above mentioned E and X can be constructed. The construction will use movable markers $a_0, a_1, \ldots$ which are at the same time the least members of their respective equivalence classes; each marker a_n has a current position $a_{n,t}$ and a limit position $a_{n,\infty} = \lim_t a_{n,t}$. To each a_n, one assigns the e-state at stage t as

$$st(e,n,t) = \sum_{d<e} 2^{-d-1} \cdot W_{d,t}(a_{n,t}).$$

Furthermore, one maintains an enumeration X_t (at stage t) of X and an r.e. equivalence relation E_t approximating E where $(x,y) \in E_0 \Leftrightarrow x = y$, and $(x,y) \in E \Leftrightarrow (\exists t)[(x,y) \in E_t]$. Note that throughout the construction it is maintained that $(x,y) \in E_t \Rightarrow x = y \vee x, y \in X_t$ in order to enforce that the equivalence classes of numbers outside X will remain singletons. We also consider a number $pos(X,n,t)$ which is defined only for a_n with $a_{n,t} \notin X_t$ and which is the number of $m < n$ with $a_{m,t} \notin X_t$.

In the following, requirements are defined with actions assigned to them. Without explicitly giving the priority of the requirements, we just state the constraints needed: (a) the ordering of the requirements is linear such that for each requirement there are only finitely many requirements of higher priority, and (b) $R_{0,i}$ has higher priority than $R_{0,i+1}$ and $R_{1,i,j}$ has higher priority than $R_{1,i',j'}$ whenever $i+j < i'+j'$ and $R_{2,i,j,e}$ has higher priority than $R_{2,i',j',e'}$ whenever $i+j+e < i'+j'+e'$.

In addition, note that a requirement only receives attention at stage t if it needs attention at stage t, it is among the t highest priority requirements and it is the highest priority requirement needing attention (a void step can be done if none of the t highest priority requirements need attention at stage t).

1. Requirement $R_{0,i}$. This requirement aims at maximizing the e-state of non-members of X. The overall aim of requirements $R_{0,\cdot}$ is to achieve goal 1.
   The requirement $R_{0,i}$ needs attention at stage t if and only if $a_{i,t} \notin X_t$ and there is an $a_{j,t} \notin X_t$ with $j > i$ such that $st(e,i,t) < st(e,j,t)$ for $e = pos(X,i,t)$.
   If $R_{0,i}$ receives attention at stage t then $X_{t+1} = X_t \cup \{a_{k,t} : i \leqslant k < j\}$ and all other parameters remain unchanged. The effect of the requirement is that the e-state of the e-th element in the complement of X goes up from $st(e,i,t)$ at stage t, to $st(e,j,t+1) \geqslant st(e,j,t)$ at stage $t+1$.
2. Requirement $R_{1,i,j}$. This requirement intends to make sure that whenever W_j intersects more than $2 \cdot (i+j+1)$ equivalence classes and a_i belongs to an equivalence class of X then W_j intersects the one of a_i; here the factor 2 is due to the fact that we consider whether W_j contains $(i+j)$ distinct equivalence classes from either X or from the complement of X — separate bound is used for the complement of X_t to ensure that small enough elements of complement of X_t are not spoiled by this requirement (in order to make sure that complement of X is infinite).
   The overall aim of requirements $R_{1,\cdot,\cdot}$ is to achieve goal 2.
   The requirement $R_{1,i,j}$ needs attention at stage t if and only if (i) $a_{i,t} \in X_t$ and (ii) there is no $(x, a_{i,t}) \in E_t$ with $x \in W_{j,t}$, and (iii) there is a $k > i+j$ and $(y, a_{k,t}) \in E_t$ such that $y \in W_{j,t}$ and either $a_{k,t} \in X_t$ or $pos(X,k,t) > i+j$.

If $R_{1,i,j}$ receives attention at stage t with parameter k as in the previous paragraph then one defines that $(x, y) \in E_{t+1} \Leftrightarrow [(x, y) \in E_t \vee ((x, a_{i,t}) \in E_t \wedge (y, a_{k,t}) \in E_t) \vee ((x, a_{k,t}) \in E_t \wedge (y, a_{i,t}) \in E_t)]$ and $a_{h,t+1} = a_{h+1,t}$ for all $h \geqslant k$. All other parameters remain unchanged.

3. Requirement $R_{2,i,j,e}$. This requirement wants to make sure that $(a_{i,\infty}, a_{j,\infty}) \in edge_e$ whenever $edge_e$ respects E, $a_{i,\infty}, a_{j,\infty} \in X$ and there is an edge $(a_{k,\infty}, a_{k',\infty}) \in edge_e$ with k, k' being sufficiently large. The overall aim of requirements $R_{2,\cdot,\cdot,\cdot}$ is to achieve goal 3.

   The requirement $R_{2,i,j,e}$ needs attention at stage t if and only if (i) $a_{i,t}, a_{j,t} \in X_t$ and (ii) there are no $(x, a_i) \in E_t$ and $(y, a_j) \in E_t$ with $(x, y) \in edge_{e,t}$ and (iii) there are $k, k' > i + j + e$ with $(i \neq j \Rightarrow k \neq k')$ and $(a_{k,t}, a_{k',t}) \in edge_{e,t}$ and $a_{k,t} \notin X_t \Rightarrow pos(X, k, t) > i + j + e$ and $a_{k',t} \notin X_t \Rightarrow pos(X, k', t) > i + j + e$.

   If $R_{2,i,j,e}$ receives attention with parameters k, k' as in the previous paragraph then $X_{t+1} = X_t \cup \{a_{k,t}, a_{k',t}\}$. If $i \neq j$, then the E_{t+1} equivalence classes of $a_{i,t+1}$ is the union of the E_t equivalence classes of $a_{i,t}$ and $a_{k,t}$, the E_{t+1} equivalence classes of $a_{j,t+1}$ is the union of the E_t equivalence classes of $a_{j,t}$ and $a_{k',t}$. In case $i = j$, the E_{t+1} equivalence class of $a_{i,t}$ is the union of the E_t equivalence classes of $a_{i,t}$, $a_{k,t}$ and $a_{k',t}$. All other equivalence classes remain unchanged. Furthermore, $a_{h,t+1}$ is the h-th member of $\{a_{0,t}, a_{1,t}, \ldots\} - \{a_{k,t}, a_{k',t}\}$, that is, $a_{h,t+1}$ is defined as follows (where $(h \geqslant k)$ denotes the value of the characteristic function of $\geqslant$ on (h, k))

$$a_{h,t+1} = \begin{cases} a_{h+(h\geqslant k)+(h+1\geqslant k'),t} & \text{if } k < k'; \\ a_{h+(h+1\geqslant k)+(h\geqslant k'),t} & \text{if } k > k'; \\ a_{h+(h\geqslant k),t} & \text{if } k = k'. \end{cases}$$

The overall construction follows the pattern of a standard finite injury construction. Initially, $X_t = \varnothing$, and $a_{i,0} = i$. At stage t, if there is a requirement needing attention among the t requirements with highest priority then among the requirements needing attention that with the highest priority receives attention; otherwise, no requirement receives attention and all parameters are the same at stage $t + 1$ as at stage t.

It can be easily verified that each requirement, after it is no longer injured by higher priority requirements, needs attention and acts only finitely often. Furthermore, only finitely many requirements can enumerate the e-th non-element of X_t into X_{t+1}; hence, from some time on, the current e-th non-element will never be enumerated into X_t and therefore will remain outside X forever. This is necessary in order to enforce that X is coinfinite. In addition, it is guaranteed that the equivalence classes of a_i and a_j can only be merged by finitely many requirements; similarly there are only finitely many requirements which can make $a_{i,t+1}$ becoming $a_{k,t}$ for some $k > i$. If t is large enough, that is, if all these requirements do no longer act at stage t or beyond, then $a_{k,t} = a_{k,\infty}$ for all $k \leqslant \max\{i, j\}$ and the equivalence classes of $a_{i,\infty}, a_{j,\infty}$ will not merge but remain different throughout the construction. Another ingredient of the construction is that for every $t \in \omega \cup \{\infty\}$, $a_{0,t} < a_{1,t} < a_{2,t} < \ldots$, and $X_t \cup \{a_{0,t}, a_{1,t}, \ldots\} = \omega$; here X_∞ just denotes X.

Furthermore, the requirements of type $R_{0,i}$ make sure that X is maximal. This is done by ensuring that whenever $a_{n,\infty}$ is the e-th non-element of X then there is no $m > n$ with $a_{n,\infty} \notin X$ and $st(e, n, \infty) < st(e, m, \infty)$ where $st(d, k, \infty) = \lim_t st(d, k, t)$ for all d, k; this limit exists. It is also easy to see that whenever W_j intersects infinitely many equivalence classes then the requirements $R_{1,i,j}$ make sure that all the equivalence classes with $a_{i,\infty} \in X$ will intersect W_j. Similarly the requirements $R_{2,i,j,e}$ will make sure that in the limit $(a_{i,\infty}, a_{j,\infty}) \in edge_e$ whenever $edge_e$ respects X and $a_{i,\infty}, a_{j,\infty} \in X$ and one can find sufficiently many disjoint non-self-loop pairs in $edge_e$. This permits to conclude that all three goals are met by the construction. $\square$

The next result shows that there is no third-least Isle-degree, instead, the second-least Isle-degree is the meet of two larger degrees.

Theorem 24. *The second least Isle-degree is the meet of two incomparable degrees.*

Proof. The idea is to make modifications of the construction of E from Theorem 23 in order to construct r.e. equivalence relations E_1 and E_2 such that $E <_{\text{Isle}} E_1$, $E <_{\text{Isle}} E_2$ and all common lower bounds E' of E_1, E_2 satisfy $E' \leqslant_{\text{Isle}} E$. The modified goals for the relation E_1 are the following, for any r.e. set Y and r.e. symmetric set Edge of pairs:

Goal 1 – X respects E_1, X/E_1 is infinite, and elements of X/E_1 are "grouped into pairs" in an r.e. way;

Goal 2 – If Y respects E_1 and Y/E_1 is infinite then $X \subseteq Y$;

Goal 3 – If Edge respects E and Edge$/E$ contains infinitely many disjoint non-self-loop edges (that is, if there exist infinitely many edges $(x_0, y_0), (x_1, y_1), \ldots$ in Edge such that $[x_0]_{E_1}, [y_0]_{E_1}, [x_1]_{E_1}, [y_1]_{E_1}, \ldots$ are all distinct), then either Edge contains (x, y) for all $x, y \in X$ or Edge contains $\{([x]_{E_1}, [y]_{E_1}) : [x]_{E_1}, y_{E_1}$ belong to the same pair in the "grouping of X/E_1 into pairs" (including the case of $x = y$) $\}$ plus only finitely many pairs from $\{([x]_E, [y]_E) : [x]_{E_1}, [y]_{E_1}$ belong to different pairs in the "grouping of X/E_1 into pairs"$\}$.

Here "grouping of X/E_1 into pairs" means that there is an r.e. equivalence relation E_1' such that if an equivalence class of E_1' is contained in X then it is the union of two equivalence classes of E_1 else it is identical with an equivalence class of E_1 outside X. The goals are realised by an adjustment of the priority construction from Theorem 23. The pseudographs realised by E_1 contain the first to fifth type from Theorem 23 (with E replaced by E_1 in the corresponding definition) plus the following two:

Type 6 – Edge$/E_1$ is a finite variant of $\{([x]_{E_1}, [y]_{E_1}) : x, y \in X \wedge (x, y) \in E_1'\}$;

Type 7 – For some r.e. set Z respecting E_1 with Z/E_1 being finite, Edge$/E$ is a finite variant of $\{([x]_{E_1}, [y]_{E_1}) : x \in Z \wedge y \in X$ or $x \in X \wedge y \in Z$ or $x, y \in X \wedge (x, y) \in E_1'\}$.

The goals for the relation E_2 are the following, where the maximal set X to be constructed will be the disjoint union of r.e. sets X' and X'' both respecting E_2 such that the following goals are met for any r.e. set Y and r.e. symmetric set Edge of pairs:

Goal 1 – X, X', X'' respect E_2 and X/E_2, X'/E_2 and X''/E_2 are all infinite;

Goal 2 – If Y respects E_2 then, (i) if Y contains infinitely many equivalence classes of E_2 outside X', then $X'' \subseteq Y$ and (ii) if Y which contains infinitely many equivalence classes of E_2 outside X'', then $X' \subseteq Y$;

Goal 3 – If Edge respects E_2, and there exists a $U \subseteq$ Edge respecting E_2 such that U/E_2 contains infinitely many disjoint non-self-loop edges (that is, if there exist infinitely many edges $(x_0, y_0), (x_1, y_1), \ldots$ in U such that $[x_0]_{E_2}, [y_0]_{E_2}, [x_1]_{E_2}, [y_1]_{E_2}, \ldots$ are all distinct), then (for each such U)
- if all pairs in U are from $X' \times X'' \cup X'' \times X'$ then Edge contains $\{(x, y) : x \in X' \wedge y \in X'' \text{ or } x \in X'' \wedge y \in X'\}$;
- if all pairs in U are from $X' \times X'$ then Edge contains $\{(x, y) : x, y \in X'\}$;
- if all pairs in U are from $X'' \times X''$ then Edge contains $\{(x, y) : x, y \in X''\}$.

Besides the five types of r.e. equivalence relations from Theorem 23 (with E replaced by E_2), the following additional types of graphs are realised by E_2 (where $b', b'' \in \{0, 1, 2\}$ and $b''' \in \{0, 1\}$):

Type 8 – For some r.e. sets Z', Z'' respecting E_2, with Z'/E_2 and Z''/E_2 being finite, $b', b'' \in \{0, 1, 2\}$ and $b''' \in \{0, 1\}$, Edge$/E$ is a finite variant of:
$\{([x]_{E_2}, [y]_{E_2}) : x, y \in X' \wedge b' = 1 \text{ or } x, y \in X' \wedge (x, y) \in E_2 \wedge b' = 2 \text{ or } x, y \in X'' \wedge b'' = 1 \text{ or } x, y \in X'' \wedge (x, y) \in E_2 \wedge b'' = 2 \text{ or } x \in X' \wedge y \in X'' \wedge b''' = 1 \text{ or } x \in X'' \wedge y \in X' \wedge b''' = 1 \text{ or } x \in Z' \wedge y \in X' \text{ or } x \in X' \wedge y \in Z' \text{ or } x \in Z'' \wedge y \in X'' \text{ or } x \in X'' \wedge y \in Z''\}$.

Note that this generic type 8 comprises some of the types one, two, three, four and five, in the case that $b' = b'' = b'''$ and $Z' = Z''$. One can see that no graph can happen to be at the same time of types six or seven and of type eight; hence every graph realised by both E_1 and E_2 is also of one of the first five types, thus realised by E. This shows that E_1 and E_2 form a pair with greatest lower bound E. $\square$

3 Partition graphs

In this section we consider another class of graphs that we call partition graphs. We denote this class by Part. As for isles we provide several characterisation results and examples. An importance of this class of graphs is that the Part-reducibility induced by Part behaves somewhat orthogonally to the Isle-reducibility induced by isles.

3.1 A general graph-theoretic construction

Let $\mathcal{G} = (V; \text{Edge})$ be a graph. We say that two vertices v_1 and v_2 are *anti-clique* equivalent if and only if for all vertices $w \in V$ we have $(v_1, w) \in$ Edge if and only if $(v_2, w) \in$ Edge. Note that the above indeed gives an equivalence relation. We call the equivalence classes *anti-clique components* of the given graph. If v_1 and v_2 are anti-clique equivalent then there is an automorphism of the graph sending v_1 and v_2 to each other and fixing all other vertices of the graph. Note that any two vertices adjacent via an edge can not be in the same anti-clique component. The definition allows us to collapse the equivalence classes and form the following graph $\mathcal{G}'$ (where, $[v]$ denotes the anti-clique component of G containing v).

1. Vertices of $\mathcal{G}'$ are the anti-clique components;

2. The edge relation Edge' is induced by the edge relation of $\mathcal{G}$, that is, $([v],[w]) \in \mathrm{Edge}'$ if and only if $(v,w) \in \mathrm{Edge}$.

Clearly, the definition of Edge' does not depend on the representatives of the anti-clique components. Moreover, the mapping $v \to [v]$ establishes a homomorphism from $\mathcal{G}$ onto $\mathcal{G}'$. For the next definition, recall that a *complete graph* is a graph that has edges between all pairs x, y of its vertices, where $x \neq y$.

Definition 25. Let $\mathcal{H}$ be a graph. We say that a graph $\mathcal{G}$ is an *$\mathcal{H}$-partition graph* if the reduced graph $\mathcal{G}'$ is isomorphic to $\mathcal{H}$. We say that $\mathcal{G}$ is a *partition graph* if it is a $\mathcal{H}$-partition graph for some complete graph $\mathcal{H}$.

Note that in the definition above we allow $\mathcal{H}$ to be finite. Every $\mathcal{H}$-partition graph can be built using the graph $\mathcal{H} = (V_H; \mathrm{Edge}_H)$ in the following manner. With every vertex $h \in V_h$ associate a non-empty set A_h such that $A_h \cap A_{h'} = \varnothing$ for all distinct $h, h' \in V_H$. Define the graph $\mathcal{G} = (V; \mathrm{Edge})$ as follows:

1. The set V is the union $\bigcup_h A_h$;
2. The set Edge contains (x,y) if and only if there exist $(u,v) \in \mathrm{Edge}_H$ such that $x \in A_u$ and $y \in A_v$.

It is clear that $\mathcal{G}$ is an $\mathcal{H}$-partition graph and the anti-clique components of $\mathcal{G}$ are the sets A_h. The family $\{A_h : h \in V_H\}$ forms a partition of the domain of $\mathcal{G}$. Using these arguments, partition graphs can be characterised as in the proposition below.

Proposition 26. *A graph $\mathcal{G} = (V; Edge)$ is a partition graph if and only if there is a partition $A_0, A_1, \dots$ of the set of vertices such that $(x,y) \in Edge$ if and only if no k exists for which $x, y \in A_k$.*

Note that in the proposition above, for the partition graph $\mathcal{G}$, the sets $A_0, A_1, \dots$ stated are the anti-clique components of $\mathcal{G}$. We denote the class of partition graphs with Part; this induces the Part-reducibility on r.e. equivalence relations.

3.2 Basic properties of the classes $\mathcal{K}_{\mathbf{Part}}(\boldsymbol{E})$

The graph $(V; \mathrm{Edge})$ in which all vertices are completely isolated, that is $\mathrm{Edge} = \varnothing$, is clearly a partition graph. The anti-clique component of this graph is V itself. An infinite complete graph is also an example of a partition graph. The anti-clique components of the complete graph are singletons. These are two examples of trivial partition graphs. However, these two trivial partition graphs are complete opposites of each other in terms of r.e. equivalence relations that realise them.

Theorem 27. *The following statements are true:*

1. *Every r.e. equivalence relation E realises the trivial partition graph in which all vertices are completely isolated;*

2. An r.e. equivalence relation E realises an infinite complete graph if and only if E is recursive.

Proof. The first part of the proposition is clearly true. For the second part, if E is recursive then it is clear that E realises a complete graph. In the other direction, assume that an r.e. equivalence relation E realises a complete graph $(\omega; \mathrm{Edge})/E$. Then for all x and y, we have $(x, y) \notin E$ if and only if $(x, y) \in \mathrm{Edge}$. Hence, E is a recursive relation. $\square$

In the study of r.e. equivalence relations, Maltsev [25] introduced the concept of precomplete equivalence relation and studied their properties. An r.e. equivalence relation E is *precomplete* if and only if for every partial-recursive function $\psi : \omega \to \omega$ there is a total-recursive function f such that for all $n \in dom(\psi)$, we have $(\psi(n), f(n)) \in E$. Lachlan [24] showed that all precomplete universal r.e. equivalence relations form one $\sim_m$ degree. The result below relates precomplete r.e. equivalence relations with partition graphs. We use the fact that no two distinct equivalence classes of precomplete r.e. equivalence relations are recursively separable [25]. Also, recall that we follow the convention that our r.e. equivalence relations have infinitely many equivalence classes.

Theorem 28. *Every precomplete r.e. equivalence relation realises only the trivial partition graph in which all vertices are isolated. Thus, every precomplete r.e. equivalence relation is Part-reducible to all other r.e. equivalence relations.*

Proof. Let E be a precomplete r.e. equivalence relation. We want to prove that E realises exactly one partition graph: the graph in which all vertices are isolated. So, assume that a partition graph $\mathcal{G} = (\omega; \mathrm{Edge})/E$ is realised by E and $\mathcal{G}$ has at least two anti-clique components. Let us select x and y from these two anti-clique components. Now for every z we define $f(z) = 0$ if (x, z) is enumerated into Edge before (y, z) and define $f(z) = 1$ if (y, z) is enumerated into Edge before (x, z). Thus defined recursive function f recursively separates the E-equivalence classes of x and y. This, as we mentioned above, contradicts with the fact that E is precomplete. $\square$

Corollary 29. *The partial order given by Part-reducibility has the least element.*

Note that there also exist non-precomplete equivalence relations E which realize only the trivial partition graph in which all vertices are isolated. For example there are universal r.e. equivalence relations that yield partitions which are recursively inseparable, but are not precomplete, see e.g. [2,4,24,27].

We would like to observe the way precomplete r.e. equivalence relations behave with respect to various reducibilities. From a recursion-theoretic point of view the precomplete relations are the most complex. This is attested by the fact that precomplete r.e. equivalence relations are universal with respect to m-reducibility [5]. From an algebraic point of view, however, precomplete r.e. equivalence relations behave quite unexpectedly. For instance, as we have already proven, for the class Isle of isles all m-universal r.e. equivalence relations (including the precomplete ones) form the largest Isle-degree. In contrast, for the class of partition graphs the precomplete relations belong to the least Part-degree. This also stands in contrast to the situation when we consider linear orders [18]; namely, no linear order can be realised over m-universal r.e. equivalence relations.

3.3 Partition graphs with infinite anti-clique components

Here we investigate r.e. equivalence relations that realise partition graphs with infinite components. For instance, we show that some m-universal r.e. equivalence relations are more powerful than precomplete ones in terms of partition graphs they realise. We also show some connections between m-reducibility and Part-reducibility. We start with two lemmas that are interesting on their own.

Lemma 30. *If one of the anti-clique components of an r.e. $\mathcal{H}$-partition graph $\mathcal{G} = (V; Edge)$ is infinite then the graph is realised by some m-universal r.e. equivalence relation.*

Proof. Let $E_0, E_1, \ldots$ be an effective enumeration of all r.e. equivalence relations. As in [26], consider the r.e. equivalence relation *univ* given by

$$(\langle x, y\rangle, \langle x', y'\rangle) \in \mathit{univ} \Leftrightarrow x = x' \wedge (y, y') \in E_x.$$

It is not hard to see that *univ* is a universal r.e. equivalence relation [10]. Now assume that E_z is an r.e. equivalence relation which realises an $\mathcal{H}$-graph $\mathcal{G} = (\omega; \mathrm{Edge}')/E_z$ such that one of the anti-clique components of $\mathcal{G}$ is infinite. Fix z', a member of one of the infinite anti-clique components of $\mathcal{G}$. Now we define the following Edge relation on ω/univ:

$$\begin{aligned}(\langle x, y\rangle, \langle x', y'\rangle) \in \mathrm{Edge} \Leftrightarrow & (x = z \wedge x' = z \wedge (y, y') \in \mathrm{Edge}') \\ & \vee\ (x = z \wedge x' \neq z \wedge (y, z') \in \mathrm{Edge}') \\ & \vee\ (x \neq z \wedge x' = z \wedge (y', z') \in \mathrm{Edge}').\end{aligned}$$

The definition of Edge says that Edge has on the slice indexed by z an image of Edge$'$ and that it extends the anti-clique component of $\langle z, z'\rangle$ to contain all the *univ* r.e. equivalence classes with a representative $\langle x', y\rangle$ where $x' \neq z$. This adds infinitely many vertices to the infinite anti-clique component of z'. Thus, the graph $(\omega; \mathrm{Edge})/\mathit{univ}$ is isomorphic to the graph $\mathcal{G}$. □

Lemma 31. *Let E be an r.e. equivalence relation with a non-recursive equivalence class $[x]_E$. Then in every partition graph realised over E the element $[x]_E$ belongs to an infinite anti-clique component.*

Proof. Assume that some partition graph $\mathcal{G} = (V; \mathrm{Edge})$ is realised by E and that the anti-clique component of $\mathcal{G}$ containing x is finite. Let $x, y_1, y_2, \ldots, y_n$ be representatives of the anti-clique component. We can decide the equivalence class of x as follows. On input z, search until either $(x, z) \in E$ or $(z, y_m) \in E$ for $m \in \{1, 2, \ldots, n\}$ or $(x, z) \in \mathrm{Edge}$ is satisfied. The search always terminates and the conditions in the search are disjoint; in the case that the first condition is found to be true then z is in the equivalence class of x; in the case that the second or third condition is found to be true then z is not in the equivalence class of x. □

There are several consequences of the lemmas above. We start with the following theorem.

Theorem 32. *An r.e. partition graph $\mathcal{G} = (V; Edge)$ is realised by some m-universal r.e. equivalence relation if and only if one of the anti-clique components of $\mathcal{G}$ is infinite.*

Proof. In one direction, assume that a partition graph $\mathcal{G}$ is realised by E and $\mathcal{G}$ has an infinite anti-clique component. By Lemma 30, $\mathcal{G}$ is realised by the universal r.e. equivalence relation *univ*. In the other direction, assume that some partition graph $\mathcal{G} = (V; \text{Edge})$ is realised by a universal r.e. equivalence relation E and that all anti-clique components of $\mathcal{G}$ are finite. Since E is universal, there must exist a non-recursive r.e. equivalence class $[x]_E$ of E. By Lemma 31, the element $[x]_E$ must belong to an infinite anti-clique component of $\mathcal{G}$. This contradicts with the choice of $\mathcal{G}$. $\square$

The next consequence of the two lemmas gives us a full description of the class $\mathcal{K}_{\text{Part}}(univ)$, where *univ* is as in Lemma 30. The equivalence relation *univ* was first introduced by Maltsev [26].

Corollary 33. *The class $\mathcal{K}_{Part}(univ)$ consists of all r.e. partition graphs that posses at least one infinite anti-clique component.*

It is worthwhile to mention that *univ* and precomplete r.e. equivalence relations are in the same m-degree. The corollary above puts these two prominent r.e. equivalence relations into completely opposite ends of Part-degrees. Finally, from the proof of Lemma 30 and Lemma 31, we can extract the following three facts about Part-reducibility.

Corollary 34. *Suppose E, E' are r.e. equivalence relations. Assume that $E \leqslant_m E'$ is witnessed by a recursive function f. If E has a non-recursive equivalence class and the set $Y = \{y : (\exists x)[(y, f(x)) \in E']\}$ is recursive, then $E \leqslant_{Part} E'$.*

Proof. Indeed, let $\mathcal{G}$ be a graph realised over E via $(\omega; \text{Edge})/E$ and let $[z']_E$ be a non-recursive equivalence class of E. By Lemma 31, $[z']_E$ belongs to an infinite anti-clique component of $\mathcal{G}$.

Now, let $(x, y) \in \text{Edge}'$ if and only if $(x', y') \in \text{Edge}$, where $f(x') = x$ and $f(y') = y$ or $x \notin Y$ and $(z', y) \in \text{Edge}$ or $y \notin Y$ and $(x, z') \in \text{Edge}$. Essentially, the above puts all members not in Y to be in the anti-clique component of $[z']_E$. It is easy to verify that $\mathcal{G}$ is isomorphic to $(\omega; \text{Edge}')/E'$. $\square$

Corollary 35. *If E and E' are r.e. equivalence relations and E contains some non-recursive equivalence class then $E \leqslant_{Part} E \oplus E'$.*

In this corollary it is important that E contains a non-recursive equivalence class. Indeed, consider id_ω as E and $E(X)$ as E' where X is not a recursive set. The identity relation id_ω realises the complete infinite graph. The r.e. equivalence relation $id_\omega \oplus E(X)$ is not a recursive relation. Hence, by Theorem 27, $id_\omega \oplus E(X)$ does not realise the complete graph. Thus, $id_\omega \not\leqslant_{\text{Part}} id_\omega \oplus E(X)$. However, note that $id_\omega \oplus E(X)$ realises non-trivial partition graphs.

Corollary 36. *If E realises a partition graph whose all anti-clique components are finite, then each equivalence class of E is recursive.* $\square$

3.4 Existence of Part-universal degree

A natural question about Part-degrees is whether Part-degrees contain the universal degree. The results above show that, as opposed to the case of Isle-degrees, m-universal r.e. equivalence relations are not Part-universal. It turns out Part-degrees have the universal degree witnessed by the identity r.e. equivalence relation id_ω. This is proved in the theorem below.

Theorem 37. *The identity r.e. equivalence relation id_ω constitutes the universal Part-degree.*

Proof. Suppose E, Edge are r.e. and $\mathcal{G} = (\omega; \mathrm{Edge})/E$ is an r.e. partition graph. We want to show that $\mathcal{G}$ can be realised over the r.e. equivalence relation id_ω. If $\mathcal{G}$ has finitely many anti-clique components, then it is clear that $\mathcal{G}$ can be realised over id_ω. So, assume that $\mathcal{G}$ has infinitely many anti-clique components.

Note that the set $\{y : (\forall x < y)[(x, y) \in \mathrm{Edge}]\}$ is an infinite r.e. set. Thus, one can effectively list an infinite increasing sequence $0 = k_0 < k_1 < k_2 \ldots$ of members of ω such that for all i, for all $x < k_i$, $(x, k_i) \in \mathrm{Edge}$.

Let E^s be recursive approximation to E such that each E^s is a recursive set (whose decision procedure can be obtained effectively from s), $E^0 = \{(x, x) : x \in \omega\}$, $E^s \subseteq E^{s+1}$, and $\bigcup_s E^s = E$.

Let $E^{s,n} = E^s \cap \{(x, y) : x, y \leqslant n\}$ (that is, $E^{s,n}$ is a restriction of E^s over the domain $\{0, 1, \ldots, n\}$).

Let $s_0 = 0$. For $n \geqslant 1$, inductively define s_n to be the least number greater than s_{n-1} such that E^{s_n,k_n} gives some equivalence relation for $\{0, 1, \ldots, k_n\}$.

Let $S = \{x : x = \min\{y : (x, y) \in E\}\}$. Note that $k_i \in S$ for each $i \in \omega$. Let $S_n = \{x \leqslant k_n : x = \min\{y : (x, y) \in E^{s_n,k_n}\}\}$. Note that $(S_n)_{n\in\omega}$ approximates S, and for all n, $S \cap S_n \subseteq S \cap S_{n+1}$.

Intuitively, our aim now is to define a partial function f (which may not be partial recursive) such that f gives a bijection from S to ω and — correspondingly — define a relation $\mathrm{Edge}' = \{(f(x), f(y)) : x, y \in S, (x, y) \in \mathrm{Edge}\}$ (which will turn out to be r.e.). This will give us that $\mathcal{G}$ is realised over id_ω.

For this purpose, approximations f_n, $n \in \omega$, whose programs are obtainable effectively from n, will be defined below such that the following properties are satisfied:

(A) Each f_n is 1–1 on its domain S_n.
(B) If for some $x \in S$, $f_n(x)$ is defined, then for all $m \geqslant n$, $f_m(x) = f_n(x)$;
(C) If for some $x \notin S$, $f_n(x)$ is defined, then for some $m > n$, $f_m(k_m) = f_n(x)$.

Let $f_0(0) = 0$ (note that $S_0 = \{0\}$). We now define f_{n+1} on the domain S_{n+1} as follows.

(i) For $x \in S_{n+1} \cap S_n$, $f_{n+1}(x) = f_n(x)$;
(ii) $f_{n+1}(k_{n+1}) = \min(\omega - \{f_{n+1}(x) : x \in S_n \cap S_{n+1}\})$;
(iii) For $z \in S_{n+1} - S_n - \{k_{n+1}\}$, define $f_{n+1}(z)$ such that f_{n+1} is 1–1, and for all $z \in S_{n+1} - (S_n \cup \{k_{n+1}\})$, $f_{n+1}(z)$ does not belong to $\{f_{n+1}(k_{n+1})\} \cup (\bigcup_{i\leqslant n}\{f_i(x) : x \in S_i\})$. Note that this can be easily achieved.

We now show that (A) to (C) hold. (A) is clearly ensured by the definition. (B) holds as $S_n \cap S \subseteq S \cap S_{n+1}$ for all n and part (i) in the definition above. To show (C), suppose $f_j(x)$ is defined for some $x \notin S$. Then, let $n > j$ be the least such that $x \notin S_n$. Now, for all $m \geqslant n$, f_m will not use $f_j(x)$ in its range via (iii) above. Thus, the first time $f_j(x)$ will be in the range of f_m (for some $m \geqslant n$), is due to (ii) and thus (C) holds (note that there will be first such time as the k_i's are distinct, and thus via (ii) every element in ω is used in the range).

Using (A) and (B) it also follows that $f(x) = \lim_{n\to\infty} f_n(x)$, is defined on its domain S.

Let $\mathrm{Edge}' = \{(f_n(x), f_n(y)) : n \in \omega, x, y \leqslant k_n \text{ and } (x, y) \in \mathrm{Edge}\}$. Note that Edge′ is clearly an r.e. set.

Claim: $\mathrm{Edge}' = \{(f(x), f(y)) : x, y \in S, (x, y) \in \mathrm{Edge}\}$.

Clearly, if $x, y \in S$ and $(x, y) \in \mathrm{Edge}$, then for large enough n, we have $x, y \leqslant k_n$, $x, y \in S_n$ and thus $(f(x), f(y)) = (f_n(x), f_n(y)) \in \mathrm{Edge}'$ (where $f(x) = f_n(x), f(y) = f_n(y)$ follow from (B)). On the other hand, suppose $(f_n(x), f_n(y))$ is in Edge′ due $x, y \leqslant k_n$ and $(x, y) \in \mathrm{Edge}$. Then we consider the following cases.

Case 1: $x, y \in S$.

Then, by (B) we have that $(f(x), f(y)) = (f_n(x), f_n(y))$, and hence it is okay to have $(f_n(x), f_n(y)) \in \mathrm{Edge}'$.

Case 2: $x \notin S$, $y \notin S$.

Then, by (C), for some $m > n, \ell > n$ $f_n(x) = f_m(k_m) = f(k_m)$ and $f_n(y) = f_\ell(k_\ell) = f(k_\ell)$. Furthermore, $k_n, k_\ell \in S$ and $(k_m, k_\ell) \in \mathrm{Edge}$. Hence it is okay to have $(f(k_m) = f_n(x), f(k_\ell) = f_n(y)) \in \mathrm{Edge}'$.

Case 3: $x \notin S$, $y \in S$.

Then, by (C), for some $m > n$ $f_n(x) = f_m(k_m) = f(k_m)$. Furthermore, $f_n(y) = f(y)$. Note also that $y \leqslant k_n < k_m$. Thus, $(y, k_m) \in \mathrm{Edge}$, and hence it is okay to have $(f(k_m) = f_n(x), f(y) = f_n(y)) \in \mathrm{Edge}'$.

Case 4: $x \in S$, $y \notin S$ is similar to Case 3.

This completes the proof of the claim and the theorem. ☐

4 On the Ideal of Finitary R.E. Equivalence Relations

In this section we consider partition graphs which have only finitely many anti-clique components, and correspondingly r.e. equivalence relations which only realise partition graphs which have finitely many anti-clique components.

Definition 38. We call an infinite partition graph $\mathcal{G}$ *finitary* if the number of its anti-clique components is finite.

Definition 39. We call an r.e. equivalence relation E *finitary* if all partition graphs realised by E are finitary partition graphs. We denote by $\mathcal{F}$ the class of all r.e. finitary equivalence relations.

It will be shown later that $\mathcal{F}$ forms an ideal with respect to $\leqslant_{\mathrm{Part}}$ (see Theorem 47 below).
It is easy to fully characterise isomorphism types of finitary partition graphs. This is done through the following definition.

Definition 40. Let $\mathcal{G} = (V; E)$ be a finitary partition graph. The *isomorphism invariant* of $\mathcal{G}$ is the tuple $(i, m, k_1, \ldots, k_m)$, where i is the number of infinite anti-clique components, m is the number of finite anti-clique components and $k_1, \ldots, k_m$ is the sequence, in non-decreasing order, of the cardinalities of all finite anti-clique components of the graph $\mathcal{G}$. Call the pair (i, m) the *type* of the graph $\mathcal{G}$.

Note that the sequence $k_1, \ldots, k_m$ might repeat the same number. It is easy to see that two finitary partition graphs are isomorphic if and only if they have the same isomorphism invariants. This implies that there are countably many isomorphism types of finitary partition graphs. Note that if (i, m) is a type of a finitary partition graph, then $i > 0$.

Now we prove two easy lemmas. The first implies a recursion-theoretic property, the second implies an algebraic property of finitary partition graphs.

Lemma 41. *Every anti-clique component of an r.e. finitary partition graph $\mathcal{G}$ is recursive.*

Proof. Suppose $\mathcal{G} = (\omega; \mathrm{Edge})/E$ where E and Edge are r.e. sets. Since $\mathcal{G}$ is finitary, let $x_1, \ldots, x_k$ be representatives of the different anti-clique components of $\mathcal{G}$. Now, the anti-clique component containing x_i is $A_i = \{y : (\forall j \neq i, 1 \leqslant j \leqslant k)[(y, x_j) \in \mathrm{Edge}]\}$. Thus, each A_i is an r.e. set. Since the A_i's partition ω, it immediately follows that A_i is recursive. $\square$

Thus, for any r.e. finitary partition graph $\mathcal{G}$, one can form another r.e. finitary partition graph $\mathcal{G}'$, by merging any given subset of its partitions.

Lemma 42. *If every equivalence class of an r.e. equivalence relation E is not recursive, then all anti-clique components of partition graphs realised over E are infinite.*

Proof. The proof is a simple consequence of Lemma 31. $\square$

Now we prove the following lemma which is useful in characterising which finitary partition graphs are realisable by $E \in \mathcal{F}$.

Lemma 43. *Suppose r.e. equivalence relation E only realises finitary partition graphs. Let*

$$n(E) = \max\{i : E \textit{ realises a partition graph with } i \textit{ many infinite anti-clique components}\};$$

$$m(E) = \max\{i : E \textit{ realises a partition graph with } i \textit{ many finite anti-clique components}\}.$$

Then, for (n, m) satisfying $1 \leqslant n < 1 + n(E)$ and $m < 1 + m(E)$, E realises finitary partition graph of type (n, m) (here, we take $1 + \omega = \omega$).

Proof. We analyse several cases for the parameters $n(E)$ and $m(E)$. Note that, by Lemma 31 every equivalence class $[x]_E$ that belongs to a finite anti-clique component of some E-partition graph must be a recursive set. Hence, E has exactly $m(E)$ recursive equivalence classes.

Case 1: $n(E) = \omega$ and $m(E) = \omega$. We want to show that E realises a finitary partition graph of type (n, m) for $1 \leqslant n < \omega, m < \omega$. Indeed, take any isomorphism invariant $(n, m, k_1, \ldots, k_m)$.

Let $\mathcal{G}$ be any E-partition graph of type (n', m'), where $n' \geqslant n$. Select an infinite anti-clique component, say C, in $\mathcal{G}$. We change $\mathcal{G}$ to $\mathcal{G}'$ by (1) adding all $[x]_E$ that belong to finite anti-clique components of $\mathcal{G}$ to C, and (2) combining $n' - n$ many infinite anti-clique components all different from C with C. In this way, we changed $\mathcal{G}$ to $\mathcal{G}'$ in which the original C has enlarged to a new anti-clique component. The new graph $\mathcal{G}'$ has type $(n, 0)$. Now select $k_1 + \ldots + k_m$ recursive E-equivalence classes and change $\mathcal{G}'$ to $\mathcal{G}''$ by forming new m many anti-clique components (using these $k_1 + \ldots + k_m$ E-equivalence classes) of cardinalities k_1, …, k_m. The resulting graph $\mathcal{G}$ has type (n, m).

Case 2: $n(E) = \omega$ and $m(E) = r < \omega$. As noted above E has exactly r many recursive equivalence classes, say $[x_1]_E, \ldots, [x_r]_E$. By Lemma 41, no partition graph $\mathcal{G}$ over E has more than r many finite anti-clique components. Consider any (n, m) where $m \leqslant r$. We want to show that E realises a partition graph of type (n, m). Let $\mathcal{G}$ be an E-partition graph of type (n', m'), where $n \leqslant n'$. Select one infinite anti-clique component, say C. As in the previous case, we change $\mathcal{G}$ to $\mathcal{G}'$ by (1) adding all $[x]_E$ that belong to finite anti-clique components of $\mathcal{G}$ to C, and (2) combining $n' - n$ many infinite anti-clique components all different from C with C. In this way, we changed $\mathcal{G}$ to $\mathcal{G}'$ in which the original C has enlarged to a new anti-clique component. The new graph $\mathcal{G}'$ has type $(n, 0)$. Now select m many recursive equivalence classes and change $\mathcal{G}'$ to $\mathcal{G}''$ by making each of these recursive classes a singleton anti-clique component. The resulting graph $\mathcal{G}$ has type (n, m).

The other two cases when $n(E) < \omega$ and $m(E) = \omega$, and $n(E) < \omega$ and $m(E) < \omega$ are treated in a similar fashion. $\square$

The above lemma leads to the following definition.

Definition 44. An r.e. equivalence relation E *has type* (n, m) if n and m are the largest cardinalities with the following property. For all natural numbers i, j with $1 \leqslant i < n$ and $j < m$, the relation E realises finitary partition graphs of type (i, j).

We would like to construct r.e. equivalence relations E which realise only finitary partition graphs.

Theorem 45. *Let X be a simple set. Then $E(X)$ realises a partition graph $\mathcal{G}$ if and only if $\mathcal{G}$ is finitary whose type is of the form $(1, m)$, where $m \in \omega$.*

Proof. Suppose $\mathcal{G} = (\omega; \mathrm{Edge})/E(X)$ is a partition graph realised by $E(X)$. Then the anti-clique component of $\mathcal{G}$ containing X is infinite by Lemma 31. Let $x \in X$. The set $S = \{z : (z, x) \in \mathrm{Edge}\}$ is recursively enumerable, respects E, and is contained in the complement of X. Thus, S must be finite. Thus, $\mathcal{G}$ is of type $(1, m)$ for some m.
Now consider a graph $\mathcal{G}$ whose isomorphism invariant is given by $(1, m, k_1, \ldots, k_m)$. To construct an $E(X)$-graph isomorphic to $\mathcal{G}$, we select a set $T \subset \omega - X$ of size $k_1 + \ldots + k_m$, and build a partition graph whose anti-clique components are $\omega - T$, $T_1, \ldots, T_m$, where T is the disjoint union of T_1, …, T_m of cardinalities $k_1, \ldots, k_m$, respectively. $\square$

Theorem 45 above states that for simple sets X, the r.e. equivalence relation $E(X)$ has type $(2, \omega)$. By Theorem 28, a precomplete r.e. equivalence relation E has type $(2, 1)$. The next theorem gives us a full description of finitary r.e. equivalence relations.

Theorem 46. *For each pair (n, m) such that $1 < n \leqslant \omega$ and $1 \leqslant m \leqslant \omega$ there exists an r.e. equivalence relation of type (n, m).*

Proof. Let E be a precomplete r.e. equivalence relation. Let $E^{\oplus n}$ denote the disjoint union of E with itself n times. The r.e. equivalence relation E_1 defined as $(x, y) \in E_1$ if and only if $x = y$ or $x \geqslant m, y \geqslant m$ and $(x - m, y - m) \in E^{\oplus n}$. Then, E_1 has type $(n+1, m+1)$. Indeed, otherwise, as in the proof of Theorem 28, we would be able to recursively separate some pair of equivalence classes of the precomplete r.e. equivalence relation E.

For $n \geqslant 1$, consider the r.e. equivalence relation E_2 defined as the following disjoint union

$$E(X_1) \oplus \ldots \oplus E(X_n),$$

where each X_i is a simple set. The type of this r.e. equivalence relation is $(n + 1, \omega)$. Clearly, the above r.e. equivalence relation realises finitary partition graphs of type (i, k), for $i \leqslant n$ and $k \in \omega$. Now, assume that there is a finitary partition graph $(\omega; \mathrm{Edge})/E_2$ of type (i, k), where $i > n$. For $1 \leqslant j \leqslant n$, select $x_j \in X_j$. Consider the r.e. set $\{y : (y, x_1) \in \mathrm{Edge} \ \& \ \ldots \ \& \ (y, x_n) \in \mathrm{Edge}\}$. Since $i > n$, this set contains infinitely many E_2-equivalence classes. Hence, there exists an infinite r.e. subset in the complement of some X_j, $1 \leqslant j \leqslant n$. This contradicts X_j being simple. Similar argument also shows that all partition graphs realised by E_2 are finitary.

Now we show that there is an r.e. equivalence relation of type $(\omega, 1)$. Consider the r.e. equivalence relation E_3 defined as the product of $E(X)$, where X is simple, with a precomplete r.e. equivalence relation E'. Thus,

$$(\langle n_1, m_1 \rangle, \langle n_2, m_2 \rangle) \in E_3 \text{ if and only if } [(n_1, n_2) \in E(X) \ \wedge \ (m_1, m_2) \in E'].$$

Note that every equivalence class of E_3 is not recursive. This is because each equivalence class of E_3 is either of the form $\{i\} \times [n]_{E'}$ or of the form $X \times [n]_{E'}$ for some n. Hence, by Lemma 31, every anti-clique component of a partition graph $\mathcal{G}$ realised by E_3 is infinite.

Let $\mathcal{G} = (\omega; \mathrm{Edge})/E_3$ be a partition graph realised over E_3. We want to show that $\mathcal{G}$ has finitely many anti-clique components. Take two distinct equivalence classes $[x]_{E'}$ and $[y]_{E'}$ of the precomplete r.e. equivalence relation E'. Then for all i and j such that $(i, j) \in E(X)$ the E_3-equivalence classes $[\langle i, x \rangle]_{E_3}$ and $[\langle j, y \rangle]_{E_3}$ are in the same anti-clique component of $\mathcal{G}$. Otherwise, as proved in Theorem 28, it is easy to show that the equivalence classes $[x]_{E'}$ and $[y]_{E'}$ would be recursively separable. Hence, the following holds. For every $i \notin X$, all E_3-equivalence classes of the form $\{i\} \times [n]_{E'}$, where $n \in \omega$, belong to the same anti-clique component of $\mathcal{G}$. Similarly, all E_3-equivalence classes of the form $X \times [n]_{E'}$, where $n \in \omega$, are also in the same anti-clique component of $\mathcal{G}$. Now fix $a \in X$. Consider the following set $\{i : (\langle i, 0 \rangle, \langle a, 0 \rangle) \in \mathrm{Edge}\}$. This is an r.e. set in the complement of X. This set must be finite. Therefore, $\mathcal{G}$ must have finitely many anti-clique components. This proves that the type of E_3 is $(\omega, 1)$.

The r.e. equivalence relation E_3 allows us to build r.e. equivalence relations of type $(\omega, m+1)$ as follows. Given m consider the r.e. equivalence relation E_4 defined as $(x, y) \in E_4$ if and only if $x = y$ or $[x \geqslant m, y \geqslant m$, and $(x, y) \in E_3]$. It is not hard to see that the type of E_4 is $(\omega, m+1)$.

It remains to show that there is an r.e. equivalence relation E realising all finitary partitions only, that is, $\mathcal{K}_{\text{Part}}(E) = \{\mathcal{G} : \mathcal{G}$ has finitely many anti-clique components$\}$. The type of such r.e. equivalence relation is clearly (ω, ω). Consider the r.e. equivalence relation E_3 built above. Let E_5 be the disjoint union of E_3 and $E(X)$, where X is simple: $E_5 = E_3 \oplus E(X)$. Now, using the arguments above that E_3 has type $(\omega, 1)$ and that $E(X)$ has type $(2, \omega)$, it is not too hard to show that E_5 has the desired type (ω, ω). $\square$

Lemma 43 and Theorem 46 imply the following result that gives us a full characterisation of Part-reducibility in the class $\mathcal{F}$ of all finitary r.e. equivalence relations.

Theorem 47. *On the class $\mathcal{F}$ the order $\leqslant_{Part}$ satisfies the following properties:*

1. *The set $\mathcal{F}$ forms an ideal, that is, (i) for all $E \in \mathcal{F}$ if $E' \leqslant_{Part} E$ then $E' \in \mathcal{F}$, and (ii) for all $E, E' \in \mathcal{F}$ there is $E'' \in \mathcal{F}$ such that $E \leq_{Part} E''$ and $E' \leq_{Part} E''$.*
2. *For all E_1 and E_2 from $\mathcal{F}$ if E has type (n, m) and E' has type (n', m') then $E \leqslant_{Part} E' \Leftrightarrow n \leqslant n' \wedge m \leqslant m'$.*
3. *The degree-structure of $\mathcal{F}$ ordered by Part-reducibility is isomorphic to the two-dimensional grid-order $(\{(n, m) : n, m \in \omega \cup \{\omega\}\}; \leqslant)$, where $\leqslant$ is the component-wise order on the set of pairs.*

Proof. By Lemma 43, every $E \in \mathcal{F}$ has a unique type $(1 + n(E), 1 + m(E))$.

Part 1(i) follows by definition of $\mathcal{F}$. Part 1(ii) follows by using Theorem 46 to construct an E'' with type $(\max\{n, n'\}, \max\{m, m'\})$, where (n, m) and (n', m') are types of E and E' respectively.

Part 2 follows by definition of type and Lemma 43.

By Lemma 43 every $E \in \mathcal{F}$ has a unique type, and by Theorem 46, for every type (n, m), $1 < n \leqslant \omega$ and $1 \leqslant m \leqslant \omega$, there is an $E \in \mathcal{F}$ of type (n, m). Part 3 now follows from part 2. $\square$

5 Graphs in general

In this section we investigate the global structure of all r.e. equivalence relations with respect to Graph-reducibility. For instance, we prove that this partially ordered set possesses infinitely many maximal elements, the least element, and atoms. Our constructions of the least element and the atoms are based on the constructions of the least elements and atoms for the Isle-degrees. We denote by $\mathcal{E}_G$ the set of all Graph-degrees under Graph-reducibility and we start with the result that the partially ordered set $\mathcal{E}_G$ has infinitely many maximal elements.

Our proof is adapted from a similar proof in [18], where r.e. universal algebras are studied. An r.e. graph $\mathcal{G}$ is said to be *computably categorical* if for any r.e. graph $\mathcal{H}$ isomorphic to $\mathcal{G}$ there exists a recursive function $f : \omega \to \omega$ that induces an isomorphism from $\mathcal{G}$ to $\mathcal{H}$.

Lemma 48. *If $\mathcal{K}_{Graph}(E)$ contains a computably categorical graph then E determines a maximal element in the partially ordered set $\mathcal{E}_G$.*

Proof. Assume that $E \leqslant_{\text{Graph}} E'$. Let $\mathcal{H}$ be a computably categorical graph in $\mathcal{K}_{\text{Graph}}(E)$. Then $\mathcal{H} \in \mathcal{K}_{\text{Graph}}(E')$. Hence there exist an E-graph $\mathcal{G}$ and an E'-graph $\mathcal{G}'$ such that $\mathcal{G} \cong \mathcal{G}' \cong \mathcal{H}$. Since $\mathcal{H}$ is computably categorical, there is a recursive $f : \omega \to \omega$ that induces an isomorphism from $\mathcal{G}$ to $\mathcal{G}'$. This recursive isomorphism establishes a bijection between ω/E and ω/E'. Using f, it is easy to show that every E-graph can be isomorphically mapped to an E'-graph, and every E'-graph can be isomorphically mapped to an E-graph. Hence, $\mathcal{K}_{\text{Graph}}(E) = \mathcal{K}_{\text{Graph}}(E')$, and therefore $E \equiv_{\text{Graph}} E'$. $\square$

We now provide a simple construction that builds computably categorical graphs. Let E be an r.e. equivalence relation. We construct the following graph $\mathcal{G}$. The set of vertices of the graph is ω. The set Edge of edges of the graph are as given below (along with their symmetric versions):

$$(0,1),(0,2),(0,3),(1,2),(1,3),(2,3),(6k+5,0),(6k+7,1),(6k+9,2),(2k+5,2k+7),$$

$$(2k+5,2k+6),(4,2k+6) \text{ for all } k \in \omega \text{ and } (2n+5,2m+6) \text{ for all } (m,n) \in E.$$

Then Edge respects the relation E' consisting of $id_\omega \cup \{(2n+6,2m+6) : (n,m) \in E\}$.

The idea behind the construction is that $[2n+6]_{E'}$ represents $[n]_E$ and that $n \mapsto 2n+6$ is an m-reduction from E to E'. Furthermore, the vertices $\{0,1,2\}$ are used to make a direction of modulo counting on the vertices $5,7,9,\ldots$ so that we can determine the vertices $7,9,11,\ldots$ starting from vertex 5 (see more on this below).

Note that in the graph the set $\{0,1,2,3\}$ is unique as it is the only 4-clique in the graph. In this 4-clique, 3 is not linked to any other vertex and 0 is the only vertex which is linked to 5 where 5 is the unique vertex which neighbours a member of the clique and has exactly one neighbour (outside the clique) which neighbours a member of the clique. The vertices $7,9,11,\ldots$ are neighbours of one of the members of the 4-clique and have two neighbours (outside the clique) which are neighbours of one of the members of the 4-clique. Furthermore, $7,9,11,\ldots$ are at a distance two from vertex 4. Node 4 is the unique vertex which has distance 3 from the nearest member of the clique.

Now, suppose $\mathcal{G}$ is isomorphic to r.e. graph $\mathcal{H}$. Then, knowing representatives of the vertices $0,1,2,3,4,5$ in $\mathcal{H}$, one can inductively find representatives in $\mathcal{H}$ for all the other vertices from $\mathcal{G}$ as follows. Knowing a representative of $6k+5$ (which is a neighbour of 0) in $\mathcal{H}$, one can find a representative for $6k+7$ in $\mathcal{H}$ by searching for a vertex in $\mathcal{H}$ which is (i) a neighbour of both, the representative of $6k+5$ and the representative of 1, and (ii) is at a distance 2 from the representative of 4. A representative for $6k+9$ is found by searching for a vertex in $\mathcal{H}$ which is (i) a neighbour of both, representative of $6k+7$ and the representative of 2, and (ii) is at a distance 2 from the representative of 4. A representative for $6k+11$ is found by searching for a vertex in $\mathcal{H}$ which is (i) a neighbour of both, representative of $6k+9$ and the representative of 0, and (ii) is at a distance 2 from the representative of 4. Representative of $6k+13$ is found by considering it as $6(k+1)+7$ and using the above method, and so on. Furthermore, having a representative of a vertex of the form $2k+5$, one can find a representative of a vertex of the form $2k+6$ by searching for a vertex in $\mathcal{H}$ which is a neighbour of both, representative of 4 and representative of $2k+5$. This permits to find representatives in $\mathcal{H}$ for all vertices in $\mathcal{G}$, thus giving us an isomorphism from $\mathcal{G}$ to $\mathcal{H}$. Thus the graph $(\omega; \mathrm{Edge})/E'$ is computably categorical.

Theorem 49. *There exist infinitely many maximal elements in the partially ordered set $\mathcal{E}_G$.*

Proof. Note that if we obtain, based on the method given before this theorem, from two r.e. equivalence relations E_1 and E_2 consisting only of non-recursive equivalence classes, two r.e. equivalence relations E_1', E_2', then $\mathcal{K}_{\mathrm{Graph}}(E_1') = \mathcal{K}_{\mathrm{Graph}}(E_2')$ would imply that E_1' and E_2', and hence E_1 and E_2 are m-equivalent (as the isomorphism between E_1' and E_2' cannot map non-recursive equivalence classes to recursive equivalence classes).

As there are infinitely many different m-degrees formed using r.e. equivalence relations consisting only of non-recursive equivalence classes, and r.e. equivalence relations E' with a computably categorical graph represent a maximal degree in the graph-reduction, theorem follows. □

Thus, the theorem shows how different C-degrees can be for various classes of graphs. For instance, in the cases of Isle-degrees and Part-degrees, there are universal elements in each. In contrast, in the case of Graph-degrees there are infinitely many maximal elements.

In view of the theorem above, a natural question arises if the set of all Graph-degrees has the least element. For instance, we have already proven that the Isle-degrees and Part-degrees possess the least element. It turns out that we can use, in the theorem below, the r.e. equivalence relation E and the maximal set X constructed in the proof of Theorem 23.

Theorem 50. *The set $\mathcal{E}_G$ of all Graph-degrees possesses the least degree.*

Proof. Consider the set X and the r.e. equivalence relation E constructed in Theorem 23. Let $f : \omega \to X$ be a recursive bijective mapping. Define the following r.e. equivalence relation on ω: $E' = \{(n,m) : (f(n), f(m)) \in E\}$. Our goal is to show that every graph realised by E' must have finitely many edges. This will prove that E' is the least element of Graph-degrees.

To obtain a contradiction, assume that E' realises a graph $\mathcal{G}' = (\omega; \mathrm{Edge}')/E'$ with infinitely many edges. Using the r.e. relation Edge' and the function f, define the following set of edges respecting E: $\mathrm{Edge} = \{(x', y') : (x', f(x)) \in E, (y, f(y)) \in E, (x, y) \in \mathrm{Edge}'\}$.

For each x, the set $\mathrm{Edge}'(x)$ must be a union of finitely many E'-equivalence classes. Otherwise, the set $\mathrm{Edge}(f(x))$ is a union of infinitely many E-equivalence classes, and does not contain $[f(x)]_E$ — however, by construction of X, every r.e. set $Y \subseteq \omega$ that is a union of infinitely many E-equivalence classes contains X. Hence, each $\mathrm{Edge}'(x)$ is a union of finitely many E'-equivalence classes. Thus, each $\mathrm{Edge}(f(x))$ is a union of finitely many E-equivalence classes. Therefore, since Edge'/E' (and hence Edge/E) is infinite and each $\mathrm{Edge}(f(x))$ is a union of finitely many E-equivalence classes, there are infinitely many disjoint pairs $([f(x)]_E, [f(y)]_E) \in \mathrm{Edge}/E$ (that is, for any distinct $([f(x')]_E, [f(y')]_E)$, $([f(x'')]_E, [f(y'')]_E)$ in this infinite list of edges, $[f(x')]_E, [f(y')]_E, [f(x'')]_E$ and $[f(y'')]_E$ are all distinct). But in this case, by the properties of X and E, it must be the case that $(x, y) \in \mathrm{Edge}$ for all $x, y \in X$, and thus E (and hence E') has self-loops. Hence, every graph realised by E must have finitely many edges. $\square$

Recall that a graph is called *locally finite* if and only if every vertex has an edge to only finitely many vertices. It turns out locally finiteness can be used to characterise certain atoms in the partially ordered set $\mathcal{E}_G$.

Theorem 51. *Exactly two atoms E_1 and E_2 in the Graph-degrees $\mathcal{E}_G$ realise non-locally finite graphs.*

Proof. The first atom E_1 is obtained as follows. Consider the r.e. equivalence relation E' from Theorem 50. Define E_1 as follows: $(x, y) \in E_1 \Leftrightarrow (x = 0 \wedge y = 0) \vee (x > 0 \wedge y > 0 \wedge (x-1, y-1) \in E')$. Now a graph realised by E_1 has either finitely many edges or its set of edges is the union of finitely many edges with all the edges defined by the relation $\{0\} \times \{1, 2, \ldots\}/E_1$. Any $E \leqslant_{graph} E_1$ either realises only graphs with finitely many edges or realises all graphs realised by E_1; hence E_1 represents an atom.

Now we show how to construct the second atom E_2. One can modify the construction of Theorem 23 such that one obtains an r.e. equivalence relation E_2 and an r.e. set X containing infinitely many equivalence classes of E_2 and being disjoint from another infinite number of equivalence classes of E_2 such that the following properties hold:

- If an r.e. set Y respects E_2 and contains infinitely many equivalence classes of X then $X \subseteq Y$;
- If an r.e. set Y respects E_2 and contains infinitely many equivalence classes not contained in X, then $Y = \omega$;
- If an r.e. edge-relation Edge respects E_2 and contains infinitely many pairs $(x_1, y_1), (x_2, y_2), \ldots$ such that all $[x_i]_{E_2}$ and $[y_j]_{E_2}$ are pairwise different then Edge contains a self-loop.

Clearly, E_2 realises every graph with finitely many edges and also realises every graph whose set of edges is determined by a union of a set with finitely many edges and a set of the form $(X \times F \cup F \times X)/E_2$, where F is a union of finitely many equivalence classes not contained in X.

Now one has to show that every graph $\mathcal{G} = (\omega; \mathrm{Edge})/E_2$ realised by E_2 is of the form described above. For all $x \in X$, $\mathrm{Edge}(x)/E_2$ is finite (otherwise, $\mathrm{Edge}(x)$ contains X and thus there is a self-loop). For $y \notin X$, $\mathrm{Edge}(y)/E_2$ is either finite or a finite variant of X/E_2 (otherwise, $\mathrm{Edge}(y) = \omega$ and thus there is a self-loop). Furthermore, the r.e. set $Y = \{y : (\exists x \in X)[(x, y) \in \mathrm{Edge}]\}$ is different from ω (otherwise, one could find, for every finite set F, an edge $(x, y) \in \mathrm{Edge}$ with both $[x]_{E_2}, [y]_{E_2}$ not in F; this would imply, by the third condition in the construction of E_2 above, that the graph would have a self-loop). Hence Y can contain only finitely many equivalence classes besides the members of X. Now, iterating the above argument (as $Y - X$ is finite), one can show that the set Y itself is a union of finitely many equivalence classes. Similarly, one can show that Edge/E does not contain infinitely many edges $([x]_{E_2}, [y]_{E_2})$, with both $x, y \notin X$ (otherwise, as for $x \notin X$, $\mathrm{Edge}(x)/E_2$ contains only finitely many equivalence classes outside X, by third clause in the construction of E_2, $\mathcal{G}$ will have a self-loop). Thus, for some finite union of equivalence classes F disjoint to X, Edge/E_2 is a finite variant of $\{([x]_{E_2},[y]_{E_2}) : x \in X \wedge y \in F$ or $x \in F \wedge y \in X\}$.

Assume now that $E \leqslant_{graph} E_2$ and that E realises a graph $(\omega; \mathrm{Edge})/E$ with infinitely many edges. As the graph is also realised by E_2, Edge/E is a finite variant of $\{(x,y) : x \in X' \wedge y \in F'$ or $x \in F' \wedge y \in X'\}$, for F' being a union of finitely many E-equivalence classes and X' being a union of infinitely many E-equivalence classes which leaves out infinitely many E-equivalence classes and $X' \cap F' = \varnothing$. Let $y_1, y_2, \ldots, y_k$ be a set of representatives of F'. Then, X' is a finite variant of $\{x : (x,y_1),(x,y_2),\ldots,(x,y_k) \in \mathrm{Edge}\}$. Thus, X' is recursively enumerable. It follows that all the graphs realised by E_2 are also realised by E and hence $E \equiv_{\mathrm{Graph}} E_2$.

Now, for the last part, consider an r.e. equivalence relation E that realises a graph $(\omega; \mathrm{Edge})/E$ such that for some x, $\mathrm{Edge}(x)/E$ is infinite. Then the set $X = \{y : (x,y) \in \mathrm{Edge}\}$ is an r.e. set which respects E. If $\omega/E - X/E$ is finite, say for example $\{[x]_E, [x']_E, [x'']_E\}$, then E realises every graph with finitely many edges and every graph whose edges are a union of a finite set and the set $[x]_E \times (\{[x']_E, [x'']_E\} \cup X/E)$. Hence the first atom discussed above is Graph-reducible to E. If $\omega/E - X/E$ is infinite, then E realises all the graphs of the second atom. Hence every E which realises a graph which is not locally finite is indeed above at least one of the atoms. $\square$

Corollary 52. *The partial order $\mathcal{E}_G$ contains the sequence $E_0 <_{Graph} E_1 <_{Graph} E_2 <_{Graph} \ldots$ of type ω such that, for all $i \geqslant 1$, no r.e. equivalence relation E exists with $E_i <_{Graph} E <_{Graph} E_{i+1}$.*

Proof. Consider the first atom E constructed in Theorem 51 (in the theorem we denoted it by E_1). Define E_i such that, $(x,y) \in E_i$ if and only if $[x = y$ or $[x > i$ and $y > i$ and $(x-i, y-i) \in E]]$ Now, it is not too hard to show that the sequence defined satisfies the corollary. $\square$

It is currently an open question whether there are infinitely many atoms in the Graph-degrees. Note that all but the two atoms given by Theorem 51 must be given by r.e. equivalence relations which only realise locally finite graphs.

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