Reducibilities among Equivalence Relations induced by Recursively Enumerable Structures

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1st December 2015

Abstract

In this paper we investigate the dependence of recursively enumerable structures on the equality relation which is fixed to a specific r.e. equivalence relation. We compare r.e. equivalence relations on the natural numbers with respect to the amount of structures they permit to represent from a given class of structures such as algebras, permutations and linear orders. In particular, we show that for various types of structures represented, there are minimal and maximal elements.

Keywords: recursively enumerable equivalence classes, algebraic structures, reducibilities, degrees, realisability.

1. Introduction

Recursively enumerable (r.e.) structures are given by a domain, usually fixed as the set ω of natural numbers, recursive functions representing the

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basic functions in the structure, plus some recursively enumerable predicates among which there is a predicate E representing the equality relation. When the equality relation E is fixed, the r.e. structures in which the equality relation coincide with E depend heavily on the nature of E. For example, Novikov constructed a finitely generated group with undecidable word-problem; in other words, there is a group which can be represented using an r.e. equivalence relation E but not using a recursive equivalence relation E. On the other hand, when dealing with Noetherian rings [28], Baur [3] showed that every r.e. Noetherian ring is already a recursive Noetherian ring, as the underlying equality E is a recursive relation. So only recursive equality relations E can be used to represent recursive Noetherian rings.

Our aim is to deepen the investigation of recursively enumerable structures with a special emphasis on the role of the equivalence relation E representing the equality in the structure. We want to study how these equivalence relations compare with each other, that is, which of them are more and which are less expressive in the amount of structures they permit to represent. Our main motivation comes from the basic homomorphism theorems in algebra. In the context of universal algebras, the theorem states that for every universal algebra $\mathcal{A}$ generated by some set S there exists a homomorphism $h : \mathcal{F}(S) \to \mathcal{A}$ from the absolutely free algebra $\mathcal{F}(S)$ with generator set S onto $\mathcal{A}$. The elements of $\mathcal{F}$ are called *terms* and they are defined inductively as follows. Each $s \in S$ is a term. If $t_1, \ldots, t_n$ are terms, and f is an n-ary operation symbol in the language of $\mathcal{A}$, then $f(t_1, \ldots, t_n)$ is also a term. The interpretation of each n-ary operation symbol f on the set of all terms is this. Given an n-tuple of terms $(t_1, \ldots, t_n)$ the value of f on $(t_1, \ldots, t_n)$ is the term $f(t_1, \ldots, t_n)$. On the set $F(S)$ of all terms define the following equivalence relation

$$E = \{(t, t') \mid h(t) = h(t')\}.$$

The relation E is often called the *word problem* of the algebra $\mathcal{A}$. All the operations f of the term algebra $\mathcal{F}(S)$ respect E, and induce well-defined operations (that we also denote by f) on the factor set $F(S)/E$. The homomorphism theorem then implies that the original algebra $\mathcal{A}$ is isomorphic to the algebra $\mathcal{F}(S)/E$. From the point of view of computable structures and recursion theory, the isomorphism between $\mathcal{A}$ and $\mathcal{F}(S)/E$ states that there exists an isomorphic copy of $\mathcal{A}$ whose elements are equivalence classes $F(S)/E$ such that the basic operations in this isomorphic copy are computable on the representatives of the equivalence classes. Hence, since the operations on $\mathcal{F}(S)/E$ are recursive, the recursion-theoretic complexity of $\mathcal{A}$ can be identified with the complexity of the equivalence relation E. This observation suggests the investigation of the class of those algebras whose elements are the E-equivalence classes and whose operations are induced by recursive functions that respect E. Formally, we define these classes below.

From now on all equivalence relations are recursively enumerable (r.e.) equivalence relations on the set of natural numbers ω; note that such equivalence relations are also called positive [10, 11, 30]. The intuition here is that we restrict ourselves to those structures for which the word problem is recursively enumerable. So, let E be an r.e. equivalence relation on ω. We say that a recursive function $f : \omega^n \to \omega$ *respects* E if for all natural numbers $x_1, y_1, \dots, x_n, y_n \in \omega$, where $(x_i, y_i) \in E$ for all $i = 1, \dots, n$, we have $(f(x_1, \dots, x_n), f(y_1, \dots, y_n)) \in E$. Similarly, a recursively enumerable predicate $P \subseteq \omega^n$ *respects* E if for all $x_1, y_1, \dots, x_n, y_n \in \omega$ such that $(x_i, y_i) \in E$, where $i = 1, \dots, n$, we have $P(x_1, \dots, x_n)$ if and only if $P(y_1, \dots, y_n)$.

If $f : \omega^n \to \omega$ respects E then f induces an n-ary operation on the quotient ω/E. We abuse notation and denote the induced map by f itself. Similarly, we use this convention for predicates. From now on we consider structures that contain only finitely many operations and predicates.

Definition 1. An *E-structure* is of the form $(\omega/E, f_1, \dots, f_k, P_1, \dots, P_m)$ where each f_i is a recursive function respecting E and each P_j is an r.e. predicate respecting E. We say that a structure is *recursively enumerable* if it is an E-structure for some r.e. equivalence relation E. An E-structure is an *E-algebra* if it contains no predicates.

We often identify structures up to isomorphisms. Due to this, we sometimes abuse the definition above and do not distinguish E-structures from their isomorphic copies.

We give two examples. Let G be a finitely presented group and E be the word problem for G, that is, $E = \{(x, y) \mid x \cdot y^{-1} = e\}$. Clearly, E is a r.e. equivalence relation. The group G is an E-algebra. For the second example, consider a first order consistent axiomatisable theory T. The theory T defines the following equivalence relation E on the set of all formulas $E = \{(\varphi, \psi) \mid T \vdash \varphi \leftrightarrow \psi\}$. The Lindenbaum Boolean algebra of T is an E-algebra.

Let C be a class of structures, where we identify structures up to isomorphism. The class can be the class of linear orders, the class of algebras or the class of all structures. Given a r.e. equivalence relation E we would like to single out those structures in the class C that are isomorphic to E-structures. We put this into the following definition:

Definition 2. Given a r.e. equivalence relation E, let $\mathcal{K}_C(E)$ be the class of all E-structures from C. In case C is the class of all structures or C is clear from the context we omit the index C.

We sometimes use the following terminology. If a structure $\mathcal{A}$ belongs to $\mathcal{K}(E)$ then we say that E *realises* $\mathcal{A}$. Otherwise, we say that E *omits* $\mathcal{A}$. Below we present several examples to give some intuition to the reader.

Example 3. Suppose that ω/E is finite. Then a structure $\mathcal{A}$ belongs to $\mathcal{K}(E)$ if and only if the cardinality of the domain equals the cardinality of ω/E. Note that here recursive enumerability of E is used essentially.

Example 4. Let E be the identity relation id_ω on ω. Then the class $\mathcal{K}(E)$ is the class of all infinite structures with domain ω and r.e. atomic predicates and recursive functions.

Example 5. Let $X \subseteq \omega$ be a r.e. set. Consider the following equivalence relation $E(X)$:

$$E(X) = \{(x, y) \mid x = y\} \cup \{(x, y) \mid x, y \in X\}.$$

Each equivalence class of $E(X)$ is either a singleton $\{i\}$ where $i \notin X$ or is the set X itself. A *permutation algebra* is an algebra of the form (A, f), where $f : A \to A$ is a bijection on A. If $f(x) = x$ then x is called a fixed point. It is not hard to see that every permutation algebra from $\mathcal{K}(E(X))$ has a fixed point if and only if X is a nonrecursive set.

We will be using the equivalence relation $E(X)$ presented in the example above throughout the paper, especially in the last section.

Example 6. Let E be an r.e. but not recursive equivalence relation. Then the class $\mathcal{K}(E)$ does not contain the successor structure (ω, S).

Here is one simple yet general fact that we use in the last section and that is interesting on its own.

Proposition 7. *Let C be a class of relational structures closed under substructures. Assume that there exists a recursive mapping $f : \omega \to \omega$ that induces an injection from ω/E_1 into ω/E_2. If E_1 omits every structure from C then so does E_2.*

Proof. Indeed, assume that there exists a structure $\mathcal{A}$ in C such that $\mathcal{A}$ is realised by E_2. Let $B \subseteq \omega/E_2$ be the image of the injection induced by f. The set B determines the substructure $\mathcal{B}$ of $\mathcal{A}$. Since f is an injection, we can "lift" the structure $\mathcal{B}$ back into the domain ω/E_1. Hence, $\mathcal{B}$ is realised by E_1. But, C is closed under substructures. Hence E_1 realises a structure from C contradicting the assumption. $\square$

All the examples above indicate that algebraic properties of structures in $\mathcal{K}(E)$ vary depending on the algorithmic properties of E. To indicate this even further, we introduce the notion of the transversal for equivalence relations E, and show how recursion-theoretic properties of the transversal might affect the algebraic properties of structures from $\mathcal{K}(E)$.

Definition 8. The *transversal* of a recursively enumerable equivalence relation E, denoted by $tr(E)$, is the set $\{n \mid \forall x\, [x < n \to (x, n) \notin E]\}$.

Thus, the transversal $tr(E)$ is the set of all minimal elements taken from the equivalence classes of E. It is not hard to see that E is Turing equivalent to $tr(E)$. Recall that a set X of natural numbers is *hyperimmune* if there does not exist a recursive function g such that $g(i) \geqslant x_i$ for all i, where $x_0 < x_1 < x_2 < \ldots$ and $X = \{x_0, x_1, \ldots\}$. We also say that a set X is *hypersimple* if X is recursively enumerable and its complement is hyperimmune.

Proposition 9 (Kasymov and Khoussainov [21])**.** *If the transversal of a r.e. equivalence relation E is hyperimmune then every E-algebra is locally finite, that is, all finitely generated subalgebras are finite.*

Proof. Let $\mathcal{A}$ be an E-algebra. Consider any finitely generated substructure of $\mathcal{A}$, and assume that the substructure is infinite. Let $n_0, \ldots, n_k$ be the E-representatives of the generators of the substructure. Define the following sequence: $X_0 = \{n_0, \ldots, n_k\}$, $X_{i+1} = X_i \cup \{f(\bar{x}) \mid \bar{x} \in X_i, f \in \sigma\}$, where $\bar{x}$ is an n-tuple of X_i and f is an n-ary operation of the language σ of the structure. Clearly, each X_i is a finite subset of natural numbers. Now let m_i be the maximal element of X_i. Note that for each i there exists an x_{i+1} in X_{i+1} such that $[x_{i+1}] \neq [y]$ for all $y \in X_i$ because the subalgebra is infinite. Hence, the function $m(i) = m_i$ is recursive and gives a counterexample for $tr(E)$ being hyperimmune. Thus, $\mathcal{A}$ is locally finite. $\square$

Corollary 10. *If X is a hypersimple set then the class $\mathcal{K}(E(X))$ contains no finitely generated algebra.*

We note that in [22] Khoussainov and Hirschfeldt construct a simple set X such that the class $\mathcal{K}(E(X))$ contains a finitely generated monoid.

To put all the examples and the proposition above in perspective, we would like to investigate and compare the classes $\mathcal{K}_C(E)$ by varying both C and E. This is formalised in the following definition.

Definition 11. Let C be a class of structures, E_1, E_2 be r.e. equivalence relations. We say E_1 is C-reducible to E_2, written $E_1 \leqslant_C E_2$ iff every structure in C realised by E_1 is also realised by E_2. In particular, we consider the following reducibilities:

1. $E_1 \leqslant_{alg} E_2$ iff every algebra realised by E_1 is also realised by E_2;
2. $E_1 \leqslant_{perm} E_2$ iff every permutation algebra realised by E_1 is also realised by E_2;
3. $E_1 \leqslant_{ord} E_2$ iff every linear order realised by E_1 is also realised by E_2;
4. $E_1 \leqslant_{struct} E_2$ iff every structure realised by E_1 is also realised by E_2.

Note that these relations are pre-orders on the set of all r.e. equivalence relations on ω. For $C \in \{alg, ord, perm, struct\}$, we say that $E_1 \equiv_C E_2$ iff $E_1 \leqslant_C E_2$ and $E_2 \leqslant_C E_1$. Thus $\leqslant_C$ determines a partial order on the $\equiv_C$-equivalence classes. We use the same symbol $\leqslant_C$ to denote this order. Example 4 shows that the identity relation id_ω represents a maximal element of the pre-orders $\leqslant_{alg}$ and $\leqslant_{struct}$. One of our aims is to study the relations $\leqslant_C$ more in detail.

We would like to make two important observations. The first is the following. By fixing the equivalence relation E, the class $K_C(E)$ calls for the description of those structures from C that can be realised over E. For instance, one can ask if there exists a linear order or a group or a Boolean algebra realised over E. If there is a structure (say, a group) in the class $K_C(E)$, one would like to describe isomorphism invariants of the structure. In this sense, the class $K_C(E)$ represents an algebraic content of the universe ω/E. The second observation is this. By fixing a class C of structures, one could consider those equivalence relations E that realise structures from C. This can be viewed as computability-theoretic content of the class C. In view of these observations, in this paper we mostly study the algebraic content of the universes ω/E.

2. Connections to related work

Orderings of equivalence relations have mainly been studied with respect to their complexity. Originally, mathematicians have been studying partial orders on the equivalence relations on reals, in particular the Borel reducibility among those, see the works by Fokina, Friedman, Harizanov, Knight, McCoy, Montalbán and Törnquist [13, 14, 15, 16] for a recent work in this direction. Bernardi and Sorbi [4, 5] as well as Ershov [10, 11] studied the m-reducibility between equivalence classes. Here, for r.e. equivalence relations E_1, E_2 on ω, one says that $E_1 \leqslant_m E_2$ iff there is a recursive function f with $\forall x, y\, [x\, E_1\, y \Leftrightarrow f(x)\, E_2\, f(y)]$. One can also consider a related equivalence relation $\sim_m$ where $E_1 \sim_m E_2$ iff there is a recursive function f witnessing $E_1 \leqslant_m E_2$ with the additional constraint that all equivalence classes of E_2 appear in the range of f. Note that $E_1 \sim_m E_2$ is actually an equivalence relation and also implies $E_2 \leqslant_m E_1$. When comparing it with $\equiv_m$ given as $E_1 \leqslant_m E_2 \wedge E_2 \leqslant_m E_1$, it turns out that $\sim_m$ is a more restrictive condition than $\equiv_m$, which stands in contrast to the one-one reducibility on sets. Note that if X_1, X_2 are two infinite r.e. sets then $E(X_1) \leqslant_m E(X_2)$ iff $X_1 \leqslant_1 X_2$ [9, 27]; hence $\leqslant_m$ is nearer to one-one reducibility than to many-one reducibility between r.e. sets. Coskey, Hamkins and Miller [8, 9] and Gao and Gerdes [17] also contributed to the study of r.e. equivalence relations and their partial order $\leqslant_m$. A recent paper by Andrews, Lempp, Miller, Ng, San Mauro and Sorbi [2] studies $\leqslant_m$-reducibility between equivalence relations, in particular, answering several questions posed in the work of Gao and Gerdes [17].

We also note that $\leqslant_m$ was studied by Fokina and Friedman in a more general context. In their study the domain of E_1 is not the full set ω but just a subset, the function f witnessing $E_1 \leqslant_m E_2$ is permitted to be partial-recursive as long as it is defined on the domain of E_1 and as it maps the domain of E_1 into the domain of E_2. Besides $\leqslant_m$, Friedman and Fokina also studied a hyperarithmetic reducibility where the function f can be hyperarithmetic; this reducibility has, however, not much relation to the ones we study and so we omit the details here.

3. Permutation algebras case

In this section we study the class *perm* of all permutation algebras as one of the simplest structures which can be defined and also investigate the relationship between the reducibilities $\leqslant_{perm}$ and $\leqslant_m$. The next example introduces various properties of permutation algebras realised over equivalence relations E.

Example 12. For this example, recall that *perm* stands for the class of all infinite permutation algebras. Below we explain which types of permutation algebras are realised over E as E varies.

There are permutation algebras which are only realisable over recursive equivalence relations E. Note that if $E \leqslant_{perm} F$ and E is recursive, then F is recursive. Indeed, the successor algebra (Z, S) is in $K_{perm}(F)$ since the algebra belongs to $K_{perm}(E)$. Any r.e. equivalence relation realising (Z, S) must be recursive. Hence, every recursive equivalence relation E represents a maximal element in the ordering $\leqslant_{perm}$ and all recursive equivalence relations E with infinitely many equivalence classes are equivalent under $\equiv_{perm}$.

Now consider a coinfinite r.e. set X and a permutation algebra (A, f) realised by $E(X)$. Note that X must represent a fixed point whenever X is not recursive.

If X is hyperhypersimple then there is an integer $n \geqslant 1$ such that (A, f) contains no cycle of length greater than n. In particular, the algebra contains no infinite cycle. The reason is that one can enumerate all the cycles of finite length and put the m-th element of each cycle along the transversion-order into the m-th member of a weak array intersecting the complement of X; hence if arbitrary long cycles existed then each member of the weak array would intersect the complement of X.

Furthermore, for hyperhypersimple X, $E(X) <_{perm} E(X \oplus \varnothing)$. Indeed, if $E(X)$ realises (A, f) then there is an n such that cycles of length n occur infinitely often in that structure and one could then extend this structure to $E(X \oplus \varnothing)$ by filling up the newly added singleton classes with copies of n-cycles. Furthermore, $E(X \oplus \varnothing)$ realises an algebra consisting of infinitely many fixed points plus one infinite cycle. Hence, we have the strict comparison: $E(X) <_{perm} E(X \oplus \varnothing)$.

If X is a maximal set (in the sense of recursion theory) then there are finitely many cycles of length greater than 1, and hence, all other members of A must be fixed points. The same condition holds when X is supersimple.

Furthermore, every equivalence relation of the form $E(X)$ with X being coinfinite realises all infinite permutation algebras where almost all members of A are fixed points. Thus, the class of all r.e. equivalence relations of the form $E(X)$ contains a smallest element with respect to $\leqslant_{perm}$.

If X is the halting problem then one can realise an algebra (A, f) such that there are infinitely many cycles of length $n + 2$ for $n \notin X$ and no cycles of length $n + 2$ for $n \in X$. Such an algebra cannot be realised when $E(X)$ is recursive. Since there exist permutation algebras realised only by recursive equivalence relations, the ordering of $\leqslant_{perm}$ does not have a greatest element. With this, we finished our explanation of Example 12.

The following lemma is clear; it is important for permutation algebras, but will also be used in various later proofs for other concepts.

Lemma 13. *If $E_1 \sim_m E_2$ then $E_1 \equiv_{perm} E_2$, $E_1 \equiv_{alg} E_2$ and $E_1 \equiv_{struct} E_2$.*

An important type of r.e. equivalence relations are the universal ones where E is *universal* iff $E' \leqslant_m E$ for every r.e. equivalence relation E'. Allthough universal equivalence relations E, E' satisfy by definition that $E \equiv_m E'$ (what abbreviates $E \leqslant_m E' \wedge E' \leqslant_m E$), they do not always satisfy $E \sim_m E'$ [25]: Lachlan [25] showed that all precomplete universal equivalence relations form one $\sim_m$ degree. Here Maltsev introduced the notion of precomplete equivalence relations [26] and defined that an r.e. equivalence relation E is precomplete iff for every partial-recursive function $\psi : \omega \to \omega$ there is a total-recursive function f such that for all $n \in dom(\psi)$, $\psi(n)\, E\, f(n)$. In the following it is shown that there are universal equivalence relations which realise some permutation algebras which a precomplete equivalence relation cannot realise and that therefore $\equiv_m$ does not imply $\equiv_{perm}$.

Example 14. *There exists a universal equivalence relation which realises all r.e. permutation algebras which have infinitely many n-cycles for some $n \in \{1, 2, 3, \ldots, \infty\}$.*

Proof. It is well-known that there is a uniform enumeration $E_0, E_1, \ldots$ of all r.e. equivalence relations; now let $\langle i, j, x\rangle\, E\, \langle i', j', y\rangle$ iff $i = i' \wedge j = j' \wedge x\, E_i\, y$. It is obvious and known that this is an universal equivalence relation [2].

Let an r.e. permutation algebra (A, f) be given which is realised by some equivalence relation, say E_0. Furthermore, assume that there is an n such that (A, f) has infinitely many n-cycles, $n \in \{1, 2, 3, \ldots, \infty\}$. Now one defines $g(\langle 0, 0, x\rangle) = \langle 0, 0, f(x)\rangle$. Furthermore, let h be a function which realises on ω infinitely many n cycles and no other cycle. Now let $g(\langle 0, j+1, x\rangle) = \langle 0, h(j)+1, x\rangle$ and $g(\langle i+1, j, x\rangle) = \langle i+1, h(j), x\rangle$. It is easy to see that g respects E and that g produces on the domain outside $\{0\} \times \{0\} \times \omega$ just infinitely many n-cycles. □

Note that this example is not a characterisation; the main purpose of it is to show that universal equivalence relations might realise non-trivial permutation algebras while, in contrast to this, they do not realise any linear order (as will be shown in the last section). There are of course permutation algebras not realised by universal equivalence relations, for example all those permutation algebras which are only realised by recursive equivalence relations. Now, using Lachlan's result that precomplete equivalence relations are universal [25] and Visser's result that precomplete equivalence relations do not have any pair of recursively separable equivalence classes [32], we get the following corollary.

Corollary 15. *Let E be the equivalence relation from Example 14 and E' a precomplete equivalence relation. Then E realises the permutation algebra (A, f) consisting of one infinite cycle and infinitely many cycles of fixed length $n < \infty$ while E' does not realise (A, f). In particular $E \equiv_m E'$, $E \not\equiv_{perm} E'$, $E \not\equiv_{alg} E'$ and $E \not\equiv_{struct} E'$.*

Proof. It follows from the criterion given in Example 14 that E realises (A, f). Furthermore, if E' realised (A, f) then one would enumerate in one set X all the x for which x belongs to an equivalence class in the infinite cycle of (A, f) and in another set Y all the y for which $f^n(y)\, E'\, y$, that is, y belongs to a cycle of length n. These two r.e. sets partition ω and respect E', hence ω has a recursive partition respecting E' in contradiction to the assumption that E' is precomplete and thus every two different equivalence classes of E' are recursively inseparable. □

Example 16. *Assume that X is maximal and E' is precomplete. Then $E(X)$ and E' form a minimal pair in the perm-degrees.*

Let (A, id) be the permutation algebra consisting of an infinite set A and the identity function. (A, id) is realised by every r.e. equivalence relation with infinitely many equivalence classes (what is assumed throughout this paper). The task is now to show that whenever $E(X)$ and E' both realise a permutation algebra (A, f) then $f = id$.

Note that X is the only nonrecursive equivalence class and hence f must map X to itself. If there are infinitely many $y \notin X$ which do not form a 1-cycle then there must be infinitely many $y \notin X$ with $y < f(y)$ and infinitely many $y \notin X$ with $y > f(y)$. Hence the set $\{y \mid f(y) < y\}$ would split the complement of X into two infinite sets what contradicts the maximality. Thus there are only finitely many $y \notin X$ with $f(y) \neq y$ and so (A, f) has only finitely many n-cycles with $n > 1$ and all these cycles are of finite length.

As E' realises (A, f), one can identify the finitely many equivalence classes $[x]$ such that $f([x]) \neq [x]$ and their union forms an r.e. set X. Furthermore, the complement $\{x \mid f([x]) = [x]\}$ is also r.e. and thus X must be recursive. As any two equivalence classes of E' are recursively inseparable, X must be empty. Thus $f = id$ and (A, id) is the only permutation algebra realised by both, $E(X)$ and E'. Hence $E(X)$ and E' form a minimal pair for the *perm*-degrees.

The next result shows that there is an infinite ascending chain such that each member of this chain realises only finitely many algebras.

Theorem 17. *There are r.e. equivalence relations $E_0, E_1, E_2, \ldots$ such that $E_n <_{perm} E_{n+1}$ and $E_n <_m E_{n+1}$ for all n.*

Proof. First one constructs an r.e. equivalence relation E_0 such that

- There is no partition of ω into two non-empty recursive sets which respects E_0;
- Every recursive function f respecting E_0 induces the identity on ω/E_0.

Such an E_0 can be built using a priority construction. One starts with $x \sim_0 y \Leftrightarrow x = y$ and defines inductively $\sim_{s+1}$ from $\sim_s$ in a recursive manner so that only finitely many equivalence classes have more than one member and that one knows these as an explicit table. Let $a_0, a_1, \ldots$ be at stage s the least members of all the equivalence classes in ascending order.

In the following, we consider all triples $\langle d, e, c\rangle \in \omega \times \omega \times \{0, 1\}$ which are identified with natural numbers in a straightforward way and check whether they satisfy the following conditions. If $c = 0$ then the conditions are the following:

- There is no $x \sim_s a_d$ such that $\varphi_{e,s}(x){\downarrow}\sim_s a_d$;

- There are $i, j \geqslant d + e$ and $x \sim_s a_i$ such that $\varphi_{e,s}(x)\downarrow \sim_s a_j$.

The goal is to make φ_e mapping the d-th equivalence class being mapped to itself whenever φ_e respects E_0; the first condition says that this is not yet enforced and the second condition says that the requirement needs attention, that is, can be enforced. If $c = 1$ then the conditions are the following:

- There is no $x, y \sim_s a_d$ such that $\varphi_{e,s}(x)\downarrow \neq \varphi_{e,s}(y)\downarrow$;

- There are $i, j \geqslant d + e$ and $x \sim_s a_i$ and ($y \sim_s a_j$ or $y \sim_s a_d$) such that $\varphi_{e,s}(x)\downarrow \neq \varphi_{e,s}(y)\downarrow$.

The goal here is to make φ_e taking two different values in the d-th equivalence class in order to enforce that φ_e is not a $\{0, 1\}$-valued function respecting E_0.

Now the activity for defining $\sim_{s+1}$ from $\sim_s$ consists just of the following step to be carried out using the above defined concepts:

If there are such triples $\langle d, e, c\rangle \leqslant s$ satisfying the above selection conditions

then one chooses the least of these triples with i, j denoting the corresponding parameters from the above conditions and one defines $x \sim_{s+1} y$ iff $x \sim_s y$ or there are $v, w \in \{a_d, a_i, a_j\}$ with $x \sim_s v \wedge y \sim_s w$

else $\sim_{s+1}$ is just the same as $\sim_s$.

Now E_0 given by $x \, E_0 \, y$ iff $x \sim_s y$ for some s. It is easy to verify that there are no finite recursive partitions respecting E_0. Furthermore, only the triples d, e, c with $d + e \leqslant k$ can cause some of the first k equivalence classes to be fusionated; these requirements act only finitely often and therefore there is a t_k such that for no $s \geqslant t_k$, two of the first k equivalence classes of $\sim_s$ will be fusionated into one for $\sim_{s+1}$. Hence E_0 has at least k equivalence classes. As this holds for each k, E_0 has infinitely many equivalence classes.

Furthermore, if φ_e is a recursive function inducing a permutation of ω/E_0 and d some number then there are i, j such that the i-th equivalence class is mapped to the j-th classes and $i, j > d+e$; hence, when s is sufficiently large, a_d, a_e, a_i, a_j have their final values and $\varphi_{e,s}(a_i) \sim_s a_j$; as the requirement $\langle d, e, 0\rangle$ does not get attention at stage s or later, it means that there is already an $x \, E_0 \, a_d$ with $\varphi_e(x) \, E_0 \, a_d$. Hence φ_e maps the d-th equivalence class to itself and φ_e realises on ω/E_0 the identity.

The idea is now to define E_n as "$\{0,1\}^n \times E_0$" which is realised as $(x \cdot 2^n + x') \, E_n \, (y \cdot 2^n + y')$ iff $x \, E_0 \, y \wedge x' = y'$ where $x', y' \in \{0, 1, \ldots, 2^n - 1\}$.

It is clear that $E_n \leqslant_m E_{n+1}$ via $u \mapsto u \cdot 2$. Now assume by way of contradiction that $E_{n+1} \leqslant_m E_n$ via a recursive function f. Let $x' \in \{0, 1, \ldots, 2^{n+1} - 1\}$. Let x'' be the remainder of $f(x')$ when divided by 2^n. The set $R = \{u \in \omega \mid f(u) \cdot 2^{n+1} + x'$ has remainder x'' modulo $2^n\}$ is a recursive subset of ω which respects E_0; as shown above, either $R = \varnothing$ or $R = \omega$. Hence f induces a function $f_{x'}$ which maps ω to ω and which satisfies $f(u \cdot 2^{n+1} + x') = f_{x'}(u) \cdot 2^n + x''$. The function f is an injection on ω respecting E_0 and the requirements $\langle d, e, 0 \rangle$ actually enforce that not only permutations but also injections on ω/E_0 are the identity. Hence each of the 2^{n+1} components of E_{n+1} is mapped surjectively to one of the 2^n component of E_n and therefore f is not injective. It follows that f cannot prove that $E_{n+1} \leqslant_m E_n$. Hence $E_n <_m E_{n+1}$.

By the arguments in the preceding paragraphs, one can show that every permutation respecting f is of the form that there is a permutation π of $\{0, 1, \ldots, 2^n - 1\}$ such that it maps the E_n-equivalence class of $x \cdot 2^n + x'$ (with $x' \in \{0, 1, \ldots, 2^n - 1\}$) to the E_n-equivalence class of $x \cdot 2^n + \pi(x')$. So E_n realises a permutation f iff there are finitely many numbers $m_1, m_2, \ldots, m_k$ with $2^n = m_1 + m_2 + \ldots + m_k$ such that f consists of infinitely many cycles of each of the lengths $m_1, m_2, \ldots, m_k$ and no other cycles. This says in particular that only finitely many permutation algebras are realised by E_n. Furthermore, it is clear that $E_n <_{perm} E_{n+1}$: By doubling the occurrences of each $m_1, m_2, \ldots, m_k$ one shows that an algebra (A, f) realised by E_n is also realised by E_{n+1}; furthermore, if f consists of infinitely many 2^{n+1}-cycles and $k = 1$ then this permutation algebra is realised by E_{n+1} but not by E_n. $\square$

4. The order $\leqslant_{alg}$ has a least element

In this section we assume that C is the class of all countably infinite algebras. Recall that a projection on a set A is a function $p : A^n \to A$ such that $p(x_1, \ldots, x_n) = x_i$ for all $x_1, \ldots, x_n \in A$. An algebra is *trivial* if all of its atomic operations are either projections or constants. The following is obvious.

Lemma 18. *All trivial algebras are E-algebras for all E.*

Let E be an equivalence relation. A set $A \subseteq \omega$ is called *E-closed* if A is a union of E-equivalence classes. For instance, consider the equivalence relation $E(X)$, then A is $E(X)$-closed if and only if either $X \subseteq A$ or $A \cap X = \varnothing$. For the equivalence relation E and $x \in \omega$, $E(x)$ denotes the E-equivalence class containing x; we might also use the notation $[x]_E$ instead of $E(x)$, or simply $[x]$ if E is clear from the context.

Our goal is to construct an equivalence relation E such that every E-algebra is trivial. This will show that $\leqslant_{alg}$ has a least element. The existence of the $\leqslant_{alg}$-minimal element had already been stated in [21]. To construct the minimal element, we provide the construction borrowed from Ershov [11]. The construction, given a r.e. and nonrecursive set S, produces an r.e. equivalence relation E_S. We assume $0 \notin S$. Let $s_1, s_2, \dots$ be a one-to-one enumeration of S. We construct E_S by stages. At stage $n > 0$ we construct $E^{[n]}$ such that $E^{[n-1]} \subseteq E^{[n]}$. The equivalence relation E_S is then defined to be $\cup_n E^{[n]}$.

Construction 19. At stage 0, $E^{[0]}$ is $id_\omega = \{(x, x) \mid x \in \omega\}$. Assume that by stage $n + 1$ we have the equivalence relation $E^{[n]}$ such that for all $x \in \omega$:

(A) The minimal element of $E^{[n]}(x)$ does not belong to $S_n = \{s_1, \dots, s_n\}$,

(B) $E^{[n]}(x) - \{\min E^{[n]}(x)\} \subseteq S_n$.

At stage $n+1$ we search for the first pair (i, e) (in a fixed effective enumeration of all pairs) with $e \leqslant n + 1$ such that

(1) $s_{n+1} \in W_{e,n+1}$,

(2) $E^{[n]}(i) \cap W_{e,n+1} = \varnothing$,

(3) $i < s_{n+1}$ and $i \notin S_{n+1}$.

If such i and e exist then we say that *e acts on i*. If e acts on i, then set $E^{[n+1]}$ to be the smallest equivalence relation that contains $E^{[n]}$ and (i, s_{n+1}). Otherwise, $E^{[n+1]}$ is the smallest equivalence relation containing $E^{[n]}$ and $(0, s_{n+1})$. Clearly, $E^{[n]} \subseteq E^{[n+1]}$. The relation $E^{[n+1]}$ preserves properties (A) and (B) stated above, but now for $n + 1$.

Set $E_S = \cup_n E^{[n]}$. We list several properties of the relation E_S.

Property 20. *The following hold for every $x \in \omega$:*

(a) The minimal element of $E_S(x)$ does not belong to S,

(b) $E_S(x) - \{\min E_S(x)\} \subseteq S$.

Proof. These follow from (A) and (B) above. $\square$

Property 21. *Each E_S-closed recursive set is either $\varnothing$ or ω.*

Proof. Let X be an E_S-closed recursive set other than $\varnothing$ and ω. Let W_e be X. Let $i = \min \overline{X}$. Clearly $i \notin S$. We claim that there exists a stage n_0 such that for all $x > i$:

$$x \in X \cap \overline{S} \leftrightarrow \exists n \geqslant n_0\, [x \in W_{e,n} \;\&\; x \notin S_n]. \tag{$*$}$$

Set n_0 be such that $i < s_{n_0}$ and no $e' < e$ acts on i for all $n \geqslant n_0$. Now, it is not hard to see that $(*)$ is true. Indeed, if $x \in X \cap \overline{S}$ then obviously $x \notin S_n$ for all n and $x \in W_{e,n}$ for some $n \geqslant n_0$. Suppose there exists a stage $n \geqslant n_0$ such that at that stage we have $(x \in W_{e,n} \;\&\; x \notin S_n)$. Clearly, $x \in X$. We now prove that $x \in \overline{S}$. Assume that $x \in S$. Note $x > i$ and $x = s_{n_1}$ at some later stage $n_1 > n$. Hence e must act on i on stage n_1. At stage n_1 we put (i, x) into E_S. This is a contradiction.

It follows from $(*)$ that $X \cap \overline{S}$ is r.e. Since $\overline{X}$ is r.e. we similarly have that $\overline{X} \cap \overline{S}$ is r.e. So, $\overline{S} = (X \cap \overline{S}) \cup (\overline{X} \cap \overline{S})$ is r.e. contradicting our choice of S to be nonrecursive. $\square$

Lemma 22. *There exists a r.e. equivalence relation E for which every non-empty E-closed r.e. set is either ω or a union of finitely many E-equivalence classes.*

Proof. Take S, in our construction above, to be a maximal set. Then E_S is the desired equivalence relation. Indeed, suppose X is a r.e. E_S-closed set that is neither a union of finitely many E_S-equivalence classes nor ω. By Property 21 and the fact that all E_S-equivalence classes are r.e., $\overline{X}$ consists of infinitely many E_S-equivalence classes. But this means that a r.e. set X splits the cohesive set $\overline{S}$ into two infinite parts. This is a contradiction. $\square$

For the next theorem, recall that we are restricted to the class of algebras.

Theorem 23. *The partial order $\leqslant_{alg}$ has a least element.*

Proof. We show that E constructed in Lemma 22 gives us the $\leqslant_{alg}$-minimal element. For this it suffices to prove that every recursive function respecting E induces either a projection or a constant function on ω/E.

Claim. *If $f : \omega \to \omega$ respects E then either f is a constant or f is the identity on ω/E.*

Indeed, define the following equivalence relation E': $(x, y) \in E'$ if and only if $\exists n, m\, (f^n(x), f^m(y)) \in E$. The relation E' is a r.e. equivalence relation whose each class is a union of E-equivalence classes.

Assume that ω/E' has cardinality greater than 1. There does not exist an E'-equivalence class that contains infinitely many E-equivalence classes by the choice of E. Hence, each E'-equivalence class is finite. If there are infinitely many E'-equivalence classes that contain at least two E-equivalence classes, then the r.e. set $\{x \mid \exists y < x\, [(x, y) \in E']\}$ splits $\overline{S}$ into two infinite subsets. This contradicts the fact that S is maximal. Hence there are only finitely many E'-equivalence classes containing more than one E-equivalence class. But E'-equivalence classes that contain exactly one E-equivalence class are defined by $f(x) = x$ formula. And this gives us a r.e. E-closed set that is the union of infinitely many E-equivalence classes. By the choice of E, this set coincides with ω. Hence E' must coincide with E.

Assume that ω/E' has cardinality 1. For each $y \in \omega$ the set $f^{-1}(y) = \{x \mid (f(x), y) \in E\}$ is a r.e. set. Therefore there exists at most one element $a \in \omega/E$ with infinitely many f-pre-images. Assume that all elements $b \in \omega/E$ have finitely many pre-images. Then it is not hard to see that there exists a $b \in \omega/E$ such at least one of the sets $\cup_{i>0} f^{-i}(b)$, $\cup_{i>0} f^i(b)$ is a proper infinite subset of ω/E. In either case, this set determines a r.e. E-closed set which is the union of infinitely many E-equivalence classes. This is impossible. Hence there exists a unique a such that $f^{-1}(a)$ is infinite. If $f^{-1}(a)$ is proper subset of ω/E then again we have a r.e. E-closed set which is the union of infinitely many E-equivalence classes. Therefore, $f^{-1}(a)$ is the whole set ω/E. Hence f is a constant function.

Claim. *If $f : \omega^n \to \omega$ respects E then either f is a constant or f is a projection on $(\omega/E)^n$.*

We prove the claim for $n = 2$. The general case is similar. So, let $f(x, y)$ be a recursive operation that respects E. Consider the functions $f(x, y)$ of one variable y where x is fixed. These functions obviously respect E and hence either identities or constants on ω/E. Suppose ω/E is $\{[a_0], [a_1], \ldots\}$. We have the following cases to consider. In each case, the equalities $=$ are taken modulo E.

(1) Assume that the function $f(a_0, y)$ is identity. Consider the function $f(x, a_i)$ in one variable x, where $a_i \neq a_0$ is fixed. We have $f(a_0, a_i) = a_i$. Therefore, $f(x, a_i)$ can not be the identity function. Therefore $f(x, a_i) = a_i$. Hence $f(x, y) = y$ in this case.

(2) Assume that $f(a_0, y) = a_i$ and $f(a_1, y) = a_i$ are both constant functions (of y). Consider the function $f(x, a_j)$, where a_j is fixed. We have $f(a_0, a_j) = f(a_1, a_j) = a_i$. Hence, the function $f(x, a_j)$ is a constant function whose value is a_i. Thus, $f(x, y) = a_i$ for all x, y.

(3) Assume that $f(a_0, y) = a_0$ and $f(a_1, y) = a_1$ both the constant functions. Then $f(a_0, a_0) = a_0$ and $f(a_1, a_0) = a_1$. Hence, $f(x, a_0)$ is the identity function. We are in the case 1 above, where the roles of variables are switched.

(4) Assume that $f(a_0, y) = a_0$ and $f(a_1, y) = a_i$ for $i \geqslant 2$. This case is impossible because $f(x, a_0)$ is neither constant nor identical.

(5) Assume that $f(a_0, y) = a_i$ is the constant function, where $i > 0$. Then $f(a_0, a_j) = a_i$ for all j. So, the function $f(x, a_j)$ can not be the identity function. Hence, $f(x, y) = a_i$ for all x and y.

This finishes the proof of the theorem. □

We can apply the theorem above to build two equivalence relations E_1 and E_2 that are not Turing equivalent yet for which $\mathcal{K}(E_1) = \mathcal{K}(E_2)$.

Corollary 24. *There exist E_1 and E_2 such that $\mathcal{K}(E_1) = \mathcal{K}(E_2)$ and yet $E_1 \not\equiv_T E_2$. In fact, E_1 and E_2 are representatives of the least element with respect to $\leqslant_{alg}$.*

Proof. Let S_1 and S_2 be maximal sets such that $S_1 \not\equiv_T S_2$. We apply Construction 19 above to build the equivalence E_{S_1} and E_{S_2}. Note that the construction guarantees that E_S is Turing equivalent to S. By the theorem, these equivalence relations are the ones needed. □

5. Maximal elements

We have already observed that the recursive equivalence relations with infinitely many equivalence classes are the only ones which represent infinite recursive Noetherian rings [3], hence these form a $\leqslant_{alg}$-maximal equivalence class. Our goal is to show that there are infinitely many equivalence classes which are maximal with respect to $\leqslant_{alg}$.

An r.e. structure $\mathcal{A}$ is *computably categorical* if for any r.e. structure $\mathcal{B}$ isomorphic to $\mathcal{A}$ there exists a recursive function $f : \omega \to \omega$ that induces an isomorphism from $\mathcal{A}$ to $\mathcal{B}$. The next lemma says that any computably categorical E-structure $\mathcal{A}$ determines E up to a recursive isomorphism, which is a recursive permutation of natural numbers modulo the equivalence relation. Hence, E is a maximal element with respect to $\leqslant_{alg}$.

Lemma 25. *If $\mathcal{K}(E)$ contains a computably categorical structure then E is a maximal element in $\leqslant_{alg}$.*

Proof. Assume that $E \leqslant_{alg} E_1$. Let A be a computably categorical structure in $\mathcal{K}(E)$. Then $\mathcal{A} \in \mathcal{K}(E_1)$. Hence there exist an E-structure $\mathcal{B}$ and E_1-structure $\mathcal{C}$ such that $\mathcal{A} \cong \mathcal{B} \cong \mathcal{C}$. Hence there is a recursive isomorphism from $\mathcal{B}$ to $\mathcal{C}$. This isomorphism establishes a recursive isomorphism between E and E_1. By Lemma 13, $\mathcal{K}(E) = \mathcal{K}(E_1)$. $\square$

Examples of computably categorical structures are finitely generated structures. This can easily be seen in the next lemma.

Lemma 26. *Every finitely generated r.e. algebra $\mathcal{A}$ is computably categorical.*

Proof. Let $\mathcal{B}$ and $\mathcal{C}$ be r.e. presentations of $\mathcal{A}$. Let $[b_0], \ldots, [b_k]$ be generators of $\mathcal{B}$ and $[c_0], \ldots, [c_k]$ be the same generators in $\mathcal{C}$. The mapping $b_i \to c_i$, $i = 1, \ldots, k$, can be extended to be a recursive isomorphism. $\square$

Lemma 25 allows us to build maximal elements with respect to $\leqslant_{alg}$. Indeed, let $X \subseteq \omega$ be a r.e. set. Consider the equivalence relation $E(X)$.

Lemma 27. *If X is not simple and co-infinite then $\mathcal{K}(E(X))$ contains a finitely generated algebra. Hence, the equivalence relation $E(X)$ is a maximal element in $\leqslant_{alg}$.*

Proof. Let Y be an infinite recursive subset of $\overline{X}$. Our algebra contains two unary operations f and g. Let $y_0 < y_1 < \dots$ be a recursive enumeration of Y. The functions f, g are defined as follows:

$$f(x) = \begin{cases} x & \text{if } x \notin Y \\ y_{i+1} & \text{if } x = y_i, \end{cases}$$

$$g(x) = \begin{cases} x & \text{if } x \notin Y \\ i & \text{if } x = y_i. \end{cases}$$

The functions f, g respect $E(X)$. The algebra $(\omega/E(X); f, g)$ is generated by y_0. □

By $\mathcal{A}_X$ we denote the algebra constructed in the proof of the lemma above.

Lemma 28. *If $X \not\equiv_m Y$ then $\mathcal{A}_X \not\cong \mathcal{A}_Y$.*

Proof. Indeed, if $\mathcal{A}_X \cong \mathcal{A}_Y$ then $E(X) \sim_m E(Y)$. Hence $X \equiv_1 Y$. But this is impossible. □

Thus we have proven the following result.

Theorem 29. *There exist infinitely many maximal elements with respect to $\leqslant_{alg}$ in the class of all structures.*

The theorem above poses the following question. Does there exists a $\leqslant_{alg}$-maximal element E such that the class $\mathcal{K}(E)$ does not contain a finitely generated substructure. The answer is affirmative: Khoussainov, Lempp and Slaman [23] construct a computably categorical E-structure $\mathcal{A}$ such that the transversal $tr(E)$ of E is a hyperimmune set. Hence, by Proposition 9 $\mathcal{K}(E)$ does not contain a finitely generated substructure, and by Lemma 25, the equivalence relation E is maximal.

6. Recursively enumerable linear orders

In this section our class C of structures consists of linearly ordered sets, that is, sets with a reflexive, transitive, and antisymmetric relation. We consider the partial order $\leqslant_{ord}$ with $E \leqslant_{ord} E'$ iff every linear order realised by E is also realised by E'; we might often just write "order" when we mean "linear order". Furthermore, all classes $\mathcal{K}(E)$ will consist of E-linear orders only.

We start with some general properties of E-linear orders. An element x of L is an end point iff either $x < y$ for all $y \in L - \{x\}$ (then x is a lower end point) or $y < x$ for all $y \in L - \{x\}$ (then x is an upper end point). A subset I of L is called an *interval* iff for every $x, y \in I$ and every $z \in L$ it holds that $x < z < y$ implies $z \in I$. An interval is called *closed* if it has both end points and it is called *open* if the interval below I has an upper end point and if the interval above I has a lower end point. A point $x \in L$ is called isolated iff the interval $\{y \in L \mid y < x\}$ is either empty or has an upper end point and the interval $\{z \in L \mid x < z\}$ is either empty or has a lower end point. Furthermore, x is *an accumulation point* of L iff x is not an isolated point.

Remark 30. If E realises an order L and L is the union of two intervals I and J with I being below J and I having an upper end point and J a lower end point then $\{x \in \omega \mid [x] \in I\}$ and $\{x \in \omega \mid [x] \in J\}$ are recursive subsets of ω. Similarly, if E realises an order L and $x \in \omega$ satisfies that $[x]$ is isolated in L then $[x]$ is a recursive set. Hence, every nonrecursive equivalence class of E is an accumulation point of L. In particular, if E contains $n \leqslant \omega$ many nonrecursive equivalence classes then every linear order realised by L contains at least n accumulation points and every ordinal realised by L is at least $\omega \cdot n + 1$.

Any two different equivalence classes of E can be recursively separated. Indeed, consider x and y such that $x < y$. If there is a z with $x < z < y$ then one can make a recursive function which maps every $u < z$ to 0 and every $u > z$ to 1 and which takes either value of 0 and 1 on $[z]$. If there is no such z then one can map all u with $u \leqslant x$ to 0 and all u with $y \leqslant u$ to 1 and again has a recursive separation. In particular E cannot be a universal equivalence relation, as every universal equivalence relation has some pairs of equivalence classes which are recursively inseparable. In contrast to this, Example 14 provided a rich collection of permutation algebras realised by one universal equivalence relation.

An infinite linear order is *strongly discrete* if each element x of L satisfies one of the following conditions. Either x is the maximal element and has an immediate predecessor or x is the minimal element and has an immediate successor or x has both immediate successor and the immediate predecessor.

Corollary 31. *A set X is recursive and co-infinite if and only if the class*

$\mathcal{K}(E(X))$ *contains a strongly discrete order.*

Corollary 32. *If E contains a nonrecursive E-class then $id_\omega \not\leqslant_{ord} E$. Furthermore, id_ω is the maximal element with respect to $\leqslant_{ord}$.*

Proof. Indeed, the class $\mathcal{K}(id_\omega)$ contains the order of type ω that has no accumulation points. For the second part, we first note that since id_ω realises the orders of type ω and η (rationals), every equivalence relation that is $\leqslant_{ord}$-above id_ω realises these two orders as well. We prove that if an equivalence relation E realises both ω and η then E is recursive.

Let $\leqslant_\omega$ and $\leqslant_\eta$ be the orders of corresponding types realised by E. Uniformly for every natural number u, we can recursively enumerate all but finitely many classes $[x]$ such that $[u] <_\omega [x]$, particularity, $[u] \neq [x]$. To do this, we fix two numbers a and b such that $[a] <_\eta [b]$ and we enumerate all elements y such that $u \leqslant_\omega y$. Since there are only finitely many classes $\leqslant_\omega$-below $[u]$, we find y_1 and y_2 such that they are both $\leqslant_\omega$-greater than u and $[y_1] \leqslant_\eta [a] <_\eta [b] \leqslant_\eta [y_2]$. Hence, $[y_1] \neq [y_2]$ and $[u] \neq [y]$ for all $[y] \geqslant_\omega [y_2]$.

We now show that the complement of E is r.e. Let v and w be arbitrary natural numbers. Assume that $v \leqslant_\eta w$—otherwise we rename v and w. We now recursively list all elements from all but finitely many classes $[x]$ such that $[v] \neq [x]$ (as above). If $[v] \neq [w]$, we must discover en element $[x]$ such that $[v] \leqslant_\eta [x] \leqslant_\eta [w]$ and $[v] \neq [x]$. The existence of such an element implies that v and w are not E-equivalent and the complement of E is r.e. $\square$

Corollary 33. *If every equivalence class of E is not recursive then every point in a linear order realised by E must be an accumulation point.*

In relation to Corollary 32, we note that there are non-recursive equivalence relations E such that E realise the order ω yet each equivalence class of E is recursive. One simple example is the following. Let X be an r.e. and non-recursive set with $0 \notin X$. Let $0 = m_0 < m_1 < \ldots$ be the sequence of all elements in the complement of X. Consider the equivalence relation

$$E = \{(x, y) \mid \exists i(m_i \leqslant x < m_{i+1} \ \&\ m_i \leqslant y < m_{i+1})\}.$$

Each equivalence class of E is finite. The order $\leqslant$ induces the order on ω/E isomorphic to the order ω.

It turns out that E-linear orders can be identified with certain partitions in computable linearly ordered sets. We explain this below.

Definition 34. Let $(L, \leqslant_L)$ be a recursive linearly ordered set. A r.e. equivalence relation E on L is a *fine partition* if each equivalence class of E is an interval.

If E is a fine partition on L then $\leqslant_L$ naturally induces a r.e. linear order, also denoted by $\leqslant_L$, on the quotient set L/E: $[x]_E \leqslant [y]_E$ if and only if $x \leqslant_L y$ in L. Clearly, the induced linearly order set is a E-linear order. The next lemma shows that all E-linear orders can be realised through fine partitions.

Lemma 35. *For every linear order L from $\mathcal{K}(E)$ there exists a recursive linear order $(\omega, \leqslant_L)$ such that E forms a fine partition on $(\omega, \leqslant_L)$ and induces a r.e. linear order isomorphic to L.*

Proof. Assume that L is isomorphic to $(\omega/E; \sqsubseteq)$. We need to construct a recursive linear order $(\omega, \leqslant_L)$ with the desired properties. The construction is by stages. Assume that by stage $n+1$ we have constructed a finite linear order $(\{0, \dots, n\}, \leqslant_n)$ and its fine partition E_n such that $E_n \subset E$. At stage $n+1$ we proceed as follows. For the element $n+1$, we check which of the following two cases occurs first:

1. $n+1 \sqsubseteq i$ for all $i = 0, \dots, n$.
2. For all $i = 0, \dots, n$, either $i \sqsubseteq n+1$ or $n+1 \sqsubseteq i$, and there exists a $j \leqslant n$ such that $j \sqsubseteq n+1$.

If the first case occurs first, then $\leqslant_{n+1}$ is the linear order that contains $\leqslant_n$ and $\{(n+1, i) \mid i = 0, \dots, n\}$. If the second case occurs first, then we select the $\leqslant_n$-largest j such that $j \sqsubseteq n+1$, and set $\leqslant_{n+1}$ to extend $\leqslant_n$ so that $n+1$ is the immediate successor of j in $\leqslant_{n+1}$.

If there is an E_n-equivalence class A, $a, b \in A$ with $a \leqslant_{n+1} n+1 \leqslant_{n+1} b$, then we set E_{n+1} to be the minimal equivalence relation containing E_n and $(a, n+1)$. If there exists a pair (x, y) among the first $n+1$ pairs enumerated into E such that $x \leqslant n+1, y \leqslant n+1$, $(x, y) \notin E_n$ then we set E_{n+1} to be the minimal fine equivalence relation containing $E_n \cup \{(x, y)\}$. Otherwise, $E_{n+1} = E_n \cup \{(n+1, n+1)\}$. Thus, E_{n+1} is a fine partition and $E_n \subseteq E_{n+1}$. It is not hard to see that $E_{n+1} \subset E$.

We set $\leqslant_L = \cup_n \leqslant_n$. The equivalence relation E is a fine partition for the recursive linear order $(\omega, \leqslant_L)$. The linear order $(\omega/E, \leqslant_L)$ is isomorphic to the original linear order $(\omega/E, \sqsubseteq)$. □

Jockusch [20] introduced the notion of a semirecursive set and proved the characterisation that a set X is semirecursive iff there is a recursive linear ordering L such that X is closed downwards under L. We use this in the theorem below.

Theorem 36. *Let X be a coinfinite r.e. set. Then the following three statements apply:*

1. *$E(X)$ realises an order with X representing an isolated point iff X is recursive;*
2. *$E(X)$ realises an order with X being an end point iff X is semirecursive;*
3. *$E(X)$ realises a linear order iff X is one-one reducible to the join of two r.e. semirecursive sets.*

Proof. If X is an isolated point and not an end point of a linear order $\sqsubseteq$ then there are y, z such that $x \in X \Leftrightarrow x \notin \{y, z\} \wedge y \sqsubseteq x \wedge x \sqsubseteq z$, namely, y and z are taken to be the predecessor and successor of x. Hence X is recursive. If X is isolated and an end point then the formula can be adjusted in a straightforward way. On the other hand, if X is recursive then one can make a linear order with X being isolated by taking $x \sqsubseteq y \Leftrightarrow x \leqslant y$ for $x, y \notin X$ as well as $x \sqsubseteq y$ iff $x \in X$ for all x, y with $x \in X \vee y \in X$.

Jockusch [20] as well as Appel and McLaughlin showed that a semirecursive set is an initial segment of a linear ordering $\leqslant_X$; hence one can define $x \sqsubseteq y \Leftrightarrow x \leqslant_X y \vee x \in X$ in order to get that X is the lower end-point of an r.e. linear order $\sqsubseteq$ respecting $E(X)$. The converse is also true: If X is the lower end-point of an r.e. linear order $\sqsubseteq$ then by Lemma 35 X is also an initial segment of a recursive ordering and hence X is semirecursive.

So assume that $E(X)$ realises a linear order $\sqsubseteq$ and $x \in X$. Then one can define the sets $Y = \{y \mid y \sqsubseteq x\}$ and $Z = \{z \mid x \sqsubseteq z\}$. Both sets Y and Z are r.e. and the initial segment of $\sqsubseteq$ or its reverse, hence semirecurisve. Now one wants to show that $X \leqslant_1 Y \oplus Z$. Thus, for all u, define $f(u)$ according to the first of the two below cases which is found to apply:

- if $x \sqsubseteq u$ then let $f(u) = 2u$;
- if $u \sqsubseteq x$ then let $f(u) = 2u + 1$.

Note that in the case that $u \in X$ it does not matter which of the two options f choses and f always choses one of the options.

If $u \in X$ then $u \in Y$ and $u \in Z$, hence $f(u) \in Y \oplus Z$. If $u \sqsubset x$ then $f(u) = 2u+1$ and $u \notin Z$, hence $f(u) \notin Y \oplus Z$; If $x \sqsubset u$ then $f(u) = 2u$ and $u \notin Y$, hence $f(u) \notin Y \oplus Z$. Thus f is a one-one reduction to the join of two semirecursive sets.

If X is one-one reducible via a recursive function f to the join $Y \oplus Z$ of two r.e. semirecurisve sets then one can equip Y with an r.e. ordering $\sqsubseteq_Y$ where Y is the lower end point and Z with an r.e. ordering $\sqsubseteq_Z$ where Z is the upper end point of the ordering. Now one defines an r.e. linear ordering on $Y \oplus Z$ by taking $2v \sqsubseteq 2w \Leftrightarrow v \sqsubseteq_Y w$, $2v+1 \sqsubseteq 2w+1 \Leftrightarrow w \sqsubseteq_Z v$, $2v \sqsubseteq 2w+1 \Leftrightarrow v \in Y \wedge w \in Z$ and $2v+1 \sqsubseteq 2w$ for all v, w. Now let $x \sqsubseteq' y \Leftrightarrow f(x) \sqsubseteq f(y)$. As $\sqsubseteq$ is a linear order respecting $E(Y \oplus Z)$, $\sqsubseteq'$ is a linear order respecting $E(X)$. Thus $E(X)$ realises a linear ordering. $\square$

Assume now that X is semirecurisve or the join of two semirecursive sets. It is known that X is not hyperhypersimple [24, Proposition 2.4] and X is not creative [18, Theorem 23]. Furthermore, semirecursive simple sets are hypersimple [20, 24]: This generalises as whenever the join of two sets is hypersimple but not simple then the same applies to at least one half of the join. Note that there are semirecursive sets which are simple as well as sets X which are hypersimple and non-hyperhypersimple and not one-one equivalent to the join of finitely many semirecursive sets [24]. Hence one has the following result.

Corollary 37. *If X is r.e. and $E(X)$ realises a linear order then X is neither hyperhypersimple nor creative nor simple and non-hypersimple.*

Proof. We would like to give a self-contained proof in addition to above references of this corollary when X is simple and not hypersimple or when X is creative. So, suppose that a linear order $(\omega/E(X), \leqslant_L)$ belongs to $\mathcal{K}(E(X))$. Assume that X is simple but not hypersimple. Then the complement $\overline{X}$ of the set X possesses a strong array $\{D_i \mid i \in \omega\}$. Recall that $\{D_i \mid i \in \omega\}$ is a strong array for $\overline{X}$ if (1) the function $i \to |D_i|$ is recursive, (2) $D_i \cap \overline{X} \neq \varnothing$ for all $i \in \omega$, and (3) $D_i \cap D_j = \varnothing$ for all $i \neq j$. We denote by $\min_L D_i$ and $\max_L D_i$ the $\leqslant_L$-least and $\leqslant_L$-greatest elements of D_i, respectively. The set

$$\{\min_L D_i \mid \max_L D_i \in X, i \in \omega\} \cup \{\max_L D_i \mid \min_L D_i \in X, i \in \omega\}$$

is finite because otherwise we obtain an infinite r.e. subset of $\overline{X}$ contradicting simplicity of X. Hence, for all but finitely many $i \in \omega$, both $\min D_i$ and $\max D_i$ are in $\overline{X}$. Denote the set of $i \in \omega$ for which both $\min D_i$ and $\max D_i$ are in $\overline{X}$ by I. I is cofinite. Hence, the set

$$\{x \mid x = \min_L D_i \text{ for some } i \in I\} \cup \{y \mid y = \max_L D_j \text{ for some } j \in I\}$$

is an infinite r.e. subset of $\overline{X}$. This is again a contradiction.

Now assume that the set X is creative. Let Y be a r.e. set. Creative sets are universal with respect to 1-reductions [31]. Hence, there exists a recursive mapping $f : \omega \to \omega$ that induces a bijection from $\omega/E(Y)$ into $\omega/E(X)$. The class C of all linear orders is closed under substructures. Take Y such that $E(Y)$ omits all linear orders. Such Y exists as we have just proven above. By Proposition 7, we conclude that $E(X)$ also omits all linear orders. $\square$

We note that there are hypersimple and non-hyperhypersimple sets X, Y such that $E(X)$ realises a linear order and $E(Y)$ does not realise a linear order; the existence of X is just given by the existence of a simple and semirecursive set X as observed by Jockusch [20]. A set Y is called r-maximal iff Y is r.e. and there is no recursive set Z such that $Z \cap \overline{Y}$ and $\overline{Z} \cap \overline{Y}$ are both infinite. There are r-maximal sets which are not maximal [31, Section X.4] and such an r-maximal set is hypersimple but not hyperhypersimple. If Y would be 1-equivalent to the join of several semirecursive sets then one component of the join would be semirecursive and r-maximal. So assume now by way of contradiction that Y is r-maximal and semirecursive: Following an argument of Martin, Odifreddi [30, Proposition III.5.7] one shows that there is an infinite set Z retraceable disjoint to Y with a total retraceing function f; then the set of all even levels with respect to f would split Z recursively into two infinite halves, hence Y cannot be r-maximal. Hence the r-maximal set Y cannot be semirecursive and also not be 1-equivalent to the join of semirecursive sets. Thus Y is hypersimple and not hyperhypersimple. Thus, we have the next corollary.

Corollary 38. *If the set Y is r-maximal then the equivalence relation $E(X)$ does not realise a linear order.*

The next theorem characterises all linear orders realised over $E(X)$ in case X is a simple set. In particular, the last part of the theorem states that there exist equivalence relations E that realise exactly one linear order.

Theorem 39. *Assume that X is simple.*

1. *If X is not one-one reducible to the join of two semirecursive sets then $E(X)$ does not realise any linear order;*

2. *If X is semirecursive $E(X)$ realises the linear orders $\omega+n$, $n+\omega^*$ and $\omega+1+\omega^*$ for all n;*

3. *If X is one-one reducible to the join of two semirecursive sets but not semirecursive then $E(X)$ realises exactly the linear order $\omega+1+\omega^*$.*

Note that all three cases occur.

Proof. As mentioned in Corollary 37, there are simple sets which are semirecursive and simple sets which are not one-one reducible to the join of two semirecursive sets, for example maximal sets. Furthermore, given two simple semirecursive sets of different Turing degree (which exist), their join is simple but not semirecursive. Thus all three cases occur. It follows from Theorem 36 that $E(X)$ does not realise any order in the first case.

In the case that X is simple and semirecursive, $E(X)$ realises the orders $\omega+n$, $n+\omega^*$ and $\omega+1+\omega^*$. To see this first note that when $\sqsubseteq$ is a recursive order respecting $E(X)$ and y represents an element larger than X then there are only finitely many y with $x \sqsubseteq y$ due to simplicity; furthermore, if y represents an element below X then there are only finitely many y with $y \sqsubseteq x$. Hence only the orders listed above can be represented. Furthermore, there is a linear order represented by $E(X)$ and this linear order has X as its endpoint; hence $\omega+1$ and $1+\omega^*$ can be represented; by finite rearrangement the orders $\omega+n$ and $n+\omega^*$ can also be represented.

Furthermore, as each r.e. semirecursive coinfinite set is not r-maximal, there is a recursive infinite set R such that $\overline{X} \cap R$ and $\overline{X} \cap \overline{R}$ are both infinite. Now let $Y, Z = X$ and make $f(u) = 2u$ if $u \in R$ and $f(u) = 2u+1$ if $u \notin R$. This one-one reduces X to $Y \oplus Z$ where Y and Z are semirecursive and coinfinite and simple and Y is the lower end and Z the upper end of r.e. orderings $\sqsubseteq_Y$ and $\sqsubseteq_Z$. Now one defines $\sqsubseteq$ on $Y \oplus Z$ as in the last paragraph of Theorem 36 and obtains an ordering which is of the form $\omega+1+\omega^*$ as $Y \oplus Z$ is simple and there are infinitely many values below and infinitely many values above the equivalence class $Y \oplus Z$. Now let $x \sqsubseteq' y \Leftrightarrow f(x) \sqsubseteq f(y)$ and $\sqsubseteq'$ is a linear ordering respecting $E(X)$ which has infinitely many elements below and infinitely many elements above the equivalence class of X. Hence $E(X)$ realises $\omega + 1 + \omega^*$.

In the case that X is simple and one-one reducible to the join of two semirecursive sets and not semirecursive, then by Theorem 36, $E(X)$ realises an infinite order and X is not an end point of this order. As one can modify the ordering on finitely many elements and X cannot be made an end point of the ordering, the ordering cannot be of the form $\omega + n$ or $n + \omega^*$. Hence the order must be $\omega + 1 + \omega^*$. □

Corollary 40. *For each integer $n \geqslant 1$ there is an r.e. equivalence relation E which realises the linear order $(\omega + 1 + \omega^*) \cdot n$ and nothing else. Hence there are infinitely many equivalence relations which form in $\leqslant_{ord}$ a minimal cover of the least $\leqslant_{ord}$-element.*

Proof. Fix $n \geqslant 1$, and let S be a simple set such that $E(S)$ realises $\omega+1+\omega^*$ but no other linear order. Let $x\,E\,y$ iff $x = y$ or there are $m \in \{0, 1, \ldots, n-1\}$ and $x', y' \in S$ with $x = n \cdot x' + m$ and $y = n \cdot y' + m$.

It is easy to see that E realises $(\omega + 1 + \omega^*) \cdot n$ by taking an order $\sqsubseteq'$ realised by $E(S)$ and then defining $\sqsubseteq$ as follows: Given $x = x' \cdot n + m_x$ and $y = y' \cdot n + m_y$ with $x', y' \in \omega$ and $m_x, m_y \in \{0, 1, \ldots, n-1\}$, let $x \sqsubseteq y$ iff $m_x < m_y$ or $m_x = m_y$ and $x' \sqsubseteq' y'$.

Assume that $\sqsubseteq$ is an r.e. linear ordering realised by E. It is shown that $\sqsubseteq$ is essentially built in the same way as the previous ordering. Let $x' \in S$. For each m, let

$$T_m = \{y' \in S \mid \exists z' \exists m' \in \{0, 1, \ldots, n-1\} - \{m\}\,[n \cdot y' + m \sqsubseteq n \cdot z' + m' \sqsubseteq n \cdot x' + m \vee n \cdot x' + m \sqsubseteq n \cdot z' + m' \sqsubseteq n \cdot y' + m]\}.$$

Hence, $y' \in T_m$ iff it can be derived from the order $\sqsubseteq$ and $(n \cdot x' + m, n \cdot z' + m') \notin E$ that also $n \cdot y' + m$ belongs to a different E-equivalence class as $n \cdot x' + m$; hence $y \notin S$. Thus each set T_m is finite. Now it follows that $\sqsubseteq$ restricted to $\omega - \cup_m\{y \cdot n + m \mid y \in T_m\}$ consists of n $\sqsubseteq$-intervals $I_m = \{y \cdot n + m \mid y \in \omega - T_m\}$ on which $\sqsubseteq$ has the order $\omega + 1 + \omega^*$. The finitely many remaining members of ω sit somewhere between these intervals or below all or above all. As a result, the order realised is of the form $(\omega + 1 + \omega^*) \cdot n$. □

The next result provides three more equivalence relations E_0, E_1, E_2 such that E_k realises an ordering L iff L is a dense linear order with exactly k end points. So the corresponding equivalence relations are further examples different from the above of equivalence relations which form a minimal cover of the least ord-element.

Theorem 41. *There are r.e. equivalence relations E_0, E_1, E_2 such that the following statements hold for every linear order L with countable domain:*

(1) E_0 realises L iff L is dense and has no end point;

(2) E_1 realises L iff L is dense and has one end point;

(3) E_2 realises L iff L is dense and has both end points.

Proof. Let I_0 be the set of all rational numbers, I_1 be the set of all rational numbers q with $q \geqslant 0$ and I_2 be the set of all rational numbers q with $0 \leqslant q \leqslant 10$. Fix $k \in \{0, 1, 2\}$ and a recursive bijection between I_k and ω; for the ease of notation, we work from now on with the domain I_k in place of ω. Let $q_0, q_1, \ldots$ be a recursive enumeration of the members of I_k.

For each $e = \langle d, i, j \rangle$, initialise $V_e = \varnothing$ and search for the first $x, y \in I_k$ found such that $q_i \leqslant x < y \leqslant q_j$ and the two values $\varphi_d(x), \varphi_d(y)$ are defined and different and $|x - y| < 2^{-e}$. If these x, y are found then update $V_e = \{q \in I_k \mid x \leqslant q \leqslant y\}$. The V_e form a family of uniformly r.e. sets.

Let $x \sim_0 y$ iff $x = y \vee \exists e\, [x, y \in V_e]$. Furthermore, for each n let $x \sim_{n+1} y$ iff there is a z such that $x \sim_n z \wedge z \sim_n y$. Let $x\, E_k\, y$ iff there is an n with $x \sim_n y$.

It is clear that E_k is an r.e. equivalence relation. Furthermore, if $x\, E_k\, y$ then $|y - x| < 2$ as every interval V_e has the length 2^{-e} and the path from x to y can be covered by going through finitely many of these intervals. In particular there are $x, y \in I_k$ with $\neg(x\, E_k\, y)$. Furthermore, the following claim holds.

Claim. *There are no rationals $q_i, q_j \in I_k$ with $q_i < q_j$ and no recursive function φ_d such that φ_d maps E_k-equivalent members of the interval $J = \{r \in I_k \mid q_i \leqslant r \leqslant q_j\}$ to the same value and takes on this interval at least two different values.*

To see this claim, assume that the claim fails as witnessed by the interval J and the function φ_d. Note that φ_d is defined on all members of J and maps two members $v, w \in I_k$ with $p \leqslant v < w \leqslant q$ to two different values. Let $e = \langle d, i, j \rangle$. Now one can choose an ascending sequence $r_0, r_1, r_2, \ldots, r_\ell$ of rationals for some ℓ such that $r_0 = v$, $r_\ell = w$ and $r_{h+1} - r_h < 2^{-e}$ for all $h < \ell$; there must be a h such that $\varphi_d(r_h) \neq \varphi_d(r_{h+1})$ and thus one could, for example, select V_e to be the set of all ratianals between r_h and r_{h+1}. Hence V_e is not empty and there are values which are equivalent with respect to E_k on which φ_d takes different values and therefore φ_d does not respect E_k on J.

Now assume that $\sqsubseteq$ is a linear order realised by E_k such that $\sqsubseteq$ is neither equal to $\leqslant$ nor to $\geqslant$ on I_k/E_k. The above claim directly gives that $\sqsubseteq$ is not discrete, that is, there are no $x, y \in I_k$ such that either $u \sqsubseteq x$ or $y \sqsubseteq u$ for all $u \in I_k$; otherwise one could define φ_d mapping the members below x to 0 and the members above y to 1. Hence $\sqsubseteq$ is a dense linear order.

Now assume that there are $x, y, z \in I_k$ in different equivalence classes such that $x < y < z$ and $\sqsubseteq$ induces an order on them which is neither compatible with $\leqslant$ nor with $\geqslant$, that is, which puts y to one of the ends. So assume, one would have $x \sqsubset z \sqsubset y$. Now one defines a recursive function φ_d on I_k with $\varphi_d(u) = 0$ if $u \sqsubseteq z$ and $\varphi_d(u) = 1$ if $z \sqsubseteq u$ where an arbitrary value of $0, 1$ is taken in the equivalence class of z. Now, on the set $J = \{r \in I_k \mid x \leqslant r \leqslant y\}$, every $u \in J$ is E_k-inequivalent to z and either satisfies $u \sqsubset z \wedge \varphi_d(u) = 0$ or satisfies $z \sqsubset u \wedge \varphi_d(u) = 1$, hence φ_d respects E_k on J and takes two different values ($\varphi_d(x) = 0, \varphi_d(y) = 1$) in contradiction to the statement in the above claim.

Thus E_k can only realise linear orders $\sqsubseteq$ with $\sqsubseteq$ being on I_k/E_k equal to $\leqslant$ or $\geqslant$. Furthermore, E_k just cuts out equivalence classes from $\leqslant$ which are intervals for $\leqslant$, thus $(I_k/E_k, \leqslant)$ is still a linearly ordered set and the orderings $x \sqsubseteq y \Leftrightarrow x \leqslant y \vee x \, E_k \, y$ as well as its reverse are realised.

In the case that $k = 0$, this is a linearly ordered set without end points; in the case that $k = 1$, this is a linearly ordered set with one end point; in the case that $k = 2$, this is a linearly ordered set with two end points. The linear order $\sqsubseteq$ derived from $\leqslant /E_k$ cannot have neighbours x, y with no equivalence class properly in between: If such would exist then one could define $\varphi_d(u) = 0$ if $u \sqsubseteq x$ and $\varphi_d(u) = 1$ if $y \sqsubseteq u$ in contradiction to the above claim. Hence the two linear orderings $\sqsubseteq$ defined by $\leqslant$ and by $\geqslant$ are dense. For $k = 0$ and $k = 2$, the corresponding ordered sets are unique. For $k = 1$, the ordering might either have a lower or an upper end point. As one can reverse the given ordering, either both orderings exist or none, hence the so obtained results are the best possible. $\square$

Remark 42. *It should be noted that in the above, the ordering E_k cannot be taken of the form $E(X)$ for an r.e. X. This is indeed also impossible, as the singleton equivalence classes in $E(X)$ are always recursive. Nevertheless, one* *can come near to this by constructing an X such that each linear ordering L reaslied by $E(X)$ has a dense interval, although L need not to be dense.*

Corollary 43. *There is an ascending chain of equivalence relations F_n with $F_0 <_{ord} F_1 <_{ord} F_2 <_{ord} \ldots$ where F_n realises all orderings $(\eta+2)\cdot m+\eta$ with $2m \leqslant n$, $1+(\eta+2)\cdot m+\eta$ with $2m+1 \leqslant n$, $(\eta+2)\cdot m+\eta+1$ with $2m+1 \leqslant n$ and $1+(\eta+2)\cdot m+\eta+1$ with $2m+2 \leqslant n$.*

Proof. Note that one can form the join $E \oplus E'$ of equivalence relations E and E' by saying that $2x\,(E \oplus E')\,2y$ iff $x\,E\,y$ and $2x+1\,(E \oplus E')2y+1$ iff $x\,E'\,y$ and $\neg 2x\,(E \oplus E')2y+1$ for all x, y. If one takes E, E' as E_0, E_1 or E_2 from the relations in Theorem 41, then any linear ordering $\sqsubseteq$ realised by $E \oplus E'$ satisfies that either $\forall x, y\,[2x \sqsubset 2y+1]$ or $\forall x, y\,[2x+1 \sqsubset 2y]$. Furthermore, define $E \oplus E' \oplus E''$ as $E \oplus (E' \oplus E'')$ and similarly for the join of any finite number of equivalence relations.

Let η denote the order type of a countable dense linear ordering without end points and $1+\eta$, $\eta+1$ and $1+\eta+1$ be the corresponding orders with one or two end points. Note that $\eta+1+\eta=\eta$ and $\eta+1+1+\eta$ is written as $\eta+2+\eta$. Now let F_n be the join of the equivalence relation E_0 representing η and of n copies of the equivalence relation E_1 representing $\eta+1$ where E_0, E_1 are taken (on the domains recoded to ω) as in Theorem 41. Let $\sqsubseteq$ be any ordering realised by such a join and let x, y be member of one component J and z be a member of another component. Then it cannot happen that $x \sqsubseteq z \sqsubseteq y$, as otherwise one could introduce a function φ_d with $\varphi_d(u)=0$ if $u \in J \wedge u \sqsubseteq z$, $\varphi_d(u)=1$ if $u \in J \wedge z \sqsubseteq u$ and $\varphi_d(u)=2$ if $u \notin J$. This φ_d respects the equivalence relation and partitions J into two recursive sets which is impossible as the equivalence relation restricted to J is isomorphic to E_0 or to E_1, thus one gets a contradiction to the claim inside Theorem 41.

Thus one has that every linear ordering realised by F_n is the sum of n components of the form $\eta+1$ or $1+\eta$ and one component η. Now using that $\eta+1+\eta=\eta$ and $\eta+\eta=\eta$, one obtains that F_n realises orders of the following types: $(\eta+2)\cdot m+\eta$ for all m with $2m \leqslant n$, $1+(\eta+2)\cdot m+\eta$ for all m with $2m+1 \leqslant n$, $(\eta+2)\cdot m+\eta+1$ for all m with $2m+1 \leqslant n$ and $1+(\eta+2)\cdot m+\eta+1$ for all m with $2m+2 \leqslant n$. Note that the orderings realised by F_n are also realised by F_{n+1} and that one finds for each n an ordering realised by F_{n+1} which is not realised by F_n. Thus one has the relation $F_0 <_{ord} F_1 <_{ord} F_2 <_{ord} F_3 <_{ord} \ldots$ forming an infinite ascending chain in the *ord*-degrees. $\square$

Corollary 44. *For every natural number k there is an equivalence relation which realises exactly k orders.*

Proof. There is an equivalence relation realising no linear orders. For $k = n + 1$, the idea is to consider the join of E_0 with a discrete equivalence relation having exactly n equivalence classes. Then m out of the n points can be below η and $n - m$ above η. Thus one gets all orders of the form $m + \eta + (n - m)$ so that there are k linear orders realised by this equivalence relation. □

The next theorem shows that $\leqslant_{ord}$ on the class of all equivalence relations of the form $E(X)$, id_ω forms a maximal element. Moreover, there exist $\leqslant_{ord}$-descending chains. For the theorem we will use the following result stated as a lemma and proven by Ambos-Spies, Cooper and Lempp [1].

Lemma 45 (Ambos-Spies, Cooper and Lempp [1])**.** *Every Σ_2^0 initial segment of every recursive linear order has a recursive copy.*

Let $X_1, \dots, X_n$ be pairwise disjoint r.e. subsets of ω whose union is co-infinite. Consider the r.e. equivalence relations $E(X_1, \dots, X_i)$, $i = 1, \dots, n$. By assumption each of these equivalence relations has infinitely many equivalence classes. We now prove the following theorem.

Theorem 46. *For the equivalence relations $E(X_1, \dots, X_i)$, $i = 1, \dots, n$, we have the following inclusions (in the class of linear orders):*

$$\mathcal{K}(E(X_1, \dots, X_n)) \subseteq \mathcal{K}(E(X_1, \dots, X_{n-1})) \subseteq \dots \subseteq \mathcal{K}(E(X_1)) \subseteq \mathcal{K}(id_\omega).$$

In particular, every linear order in $\mathcal{K}(E(X_1))$ has a recursive copy.

Proof. We prove the theorem by induction on n. The case $n = 1$ is proven in the following lemma:

Lemma 47. *If a linear order L belongs to $\mathcal{K}(E(X))$ then L has a recursive copy.*

Indeed, let $\leqslant_L$ be a r.e. relation such that $(\omega/E(X), \leqslant_L) \cong L$. If X is recursive then there is nothing prove. So, let X be a nonrecursive set. The linear order L can be assumed to be of the form $(\omega/E(X), \leqslant_L)$. Define

$$A = \{[a] \mid [a] <_L X\} \text{ and } B = \{[b] \mid [b] >_L X\}.$$

So L is of the form $A + 1 + B$, where 1 represents X. By Lemma 35 there exists a recursive linear order $\sqsubseteq$ on ω such that $(\omega, \sqsubseteq)$ is of the form

$$L_1 + L_2 + L_3,$$

where $L_1 \cong A$, $L_3 \cong B$, and X coincides with the domain of L_2. Since L_1 is a Π^0_1-initial segment of a recursive linear order, we apply Lemma 45 and obtain that L_1 has a recursive copy. Looking at $L_1 + L_2 + L_3$ by reversing the order on it, we conclude that L_3 has a recursive presentation too. This clearly implies that L has a recursive copy which was required to be proven.

Let L be a linear order in $\mathcal{K}(E(X_1, X_2, \ldots, X_n))$. By Lemma 35 there exists a recursive linear order $(\omega, \sqsubseteq)$ for which $E(X_1, X_2, \ldots, X_n)$ is a fine partition such that $(\omega/E, \sqsubseteq)$ is isomorphic to L. Thus, by re-ordering the sets X_1, $\ldots$, X_n if need be, we can assume that the linear order $(\omega, \sqsubseteq)$ is represented as the sum

$$L_1 + X_1 + L_2 + X_2 + L_3 + \ldots + X_n + L_n$$

in which the sets X_1, X_2, $\ldots$, X_n are ordered according to $\sqsubseteq$. Note L is isomorphic to $L_1 + 1 + L_2 + 1 + L_3 + \ldots + 1 + L_n$.

If L_{n-1} has a maximal element a or $L_{n-1} = \varnothing$, then the set $X_n + L_n$ is a recursive linear order with domain D. Hence, $1 + L_n$ belongs to $\mathcal{K}(E(X_n))$, where $E(X_n)$ is taken on the domain D. By the base case, $1 + L_n$ has a recursive copy with domain D. Hence, we see that $L \in \mathcal{K}(E(X_1, \ldots, X_{n-1}))$.

So, we assume that $L_{n-1} \neq \varnothing$. Select elements $a \in L_{n-1}$ and $b \in L_n$. If $L_n = \varnothing$ then we just select a. Consider the recursive linear order $([a, b], \sqsubseteq)$. This can be represented as the sum of three linear orders:

$$A + X_n + B.$$

Let the domain of this order be D. Clearly

$$D \cap (L_1 + X_1 + L_2 + \ldots + X_{n-1}) = \varnothing.$$

The linearly order set $A + 1 + B$ belongs to $\mathcal{K}(E(X_n))$. Applying the base case to $A+1+B$, we see that $A+1+B$ has a recursive copy, call it L', over the domain D.

Now, by replacing the interval $[a, b]$ by L' in the order $(\omega, \sqsubseteq)$, it can easily be observed that L belongs to the class $\mathcal{K}(E(X_1, \ldots, X_{n-1}))$ which was required to be proven. $\square$

Corollary 48. *There are subsets X_1, …, X_n for which we have proper the inclusions:*

$$\mathcal{K}(E(X_1,\ldots,X_n)) \subset \mathcal{K}(E(X_1,\ldots,X_{n-1})) \subset \ldots \subset \mathcal{K}(E(X_1)) \subset \mathcal{K}(id_\omega).$$

Proof. Consider a linear order $\leqslant_L$ of the type

$$L_1 + X_1 + L_2 + X_2 + \ldots + X_n + L_n,$$

where each L_i has type ω and each X_j has the order type of integers. It is not hard to construct a recursive linear order isomorphic to L in which the sets X_1, …, X_n are r.e., semirecursive and non-recursive. These sets satisfy the corollary since, by Remark 30, the linear order $L_1 + 1 + \ldots + L_i + 1$ belongs to $\mathcal{K}(E(X_1,\ldots,X_i))$ but does not belong to $\mathcal{K}(E(X_1,\ldots,X_{i+1}))$. □

Corollary 49. *There is an infinite descending chain of equivalence relations with respect to $\leqslant_{ord}$.*

Finally, since this paper was submitted, this work has been continued in several papers [12, 19]. These papers study the relationships between linear orders, various types of graphs and r.e. equivalence relations.

References

[1] Klaus Ambos-Spies, S. Barry Cooper and Steffen Lempp. Initial segments of recursive linear orders. *Order*, 14(2):101–105, 1997.

[2] Uri Andrews, Steffen Lempp, Joseph S. Miller, Keng Meng Ng, Luca San Mauro and Andrea Sorbi. Universal computably enumerable equivalence relations. Manuscript, 2012.

[3] Walter Baur. Rekursive Algebren mit Kettenbedingungen. *Zeitschrift für mathematische Logik und Grundlagen der Mathematik*, 20:37–46, 1974.

[4] Claudio Bernardi. On the relation provable equivalence and on partitions in effectively inseparable sets. *Studia Logica*, 40:29–37, 1981.

[5] Claudio Bernardi and Andrea Sorbi. Classifying positive equivalence relations. *The Journal of Symbolic Logic*, 48(3):529–538, 1983.

[6] Richard Beigel, Martin Kummer and Frank Stephan. Quantifying the amount of verboseness. *Information and Computation*, 118:71–90, 1995.

[7] Samuel Buss, Yijia Chen, Jörg Flum, Sy-David Friedman and Moritz Müller. Strong isomorphism reductions in complexity theory. *The Journal of Symbolic Logic*, 76(4):1381–1402, 2011.

[8] Sam Coskey and Joel David Hamkins. Infinite time computable equivalence relations. *Notre Dame Journal of Formal Logic*, 52(2):203–228, 2011.

[9] Sam Coskey, Joel David Hamkins and Russell Miller. The hierarchy of equivalence relations on the natural numbers under computable reducibility. *http://www.arxiv.org/* Report number 1109.3375, 2012.

[10] Yuri L. Ershov. Positive equivalence. *Algebra and Logic*, 10(6):378–394, 1974.

[11] Yuri L. Ershov. *Theory of numberings.* Nauka, Moscow, 1977 (in Russian).

[12] Ekaterina Fokina, Bakhadyr Khoussainov, Pavel Semukhin and Dan Turesky. Linear orders realised by c.e. equivalence relations. *Journal of Symbolic logic*. Accepted.

[13] Ekaterina B. Fokina and Sy-David Friedman. Equivalence relations on classes of computable structures. *Proceedings of the Fifth Conference on Computability in Europe*, CiE 2009, Springer LNCS 7921:198–207, 2009.

[14] Ekaterina B. Fokina and Sy-David Friedman. On Σ^1_1 equivalence relations over the natural numbers. *Mathematical Logic Quarterly*, 58(1-2):113-124, 2012.

[15] Ekaterina B. Fokina, Sy-David Friedman, Valentina S. Harizanov, Julia F. Knight, Charles F. D. McCoy and Antonio Montalbán: Isomorphism relations on computable structures. *The Journal of Symbolic Logic*, 77(1):122–132, 2012.

[16] Ekaterina B. Fokina, Sy-David Friedman and Asger Törnquist. The effective theory of Borel equivalence relations. *Annals of Pure Applied Logic*, 161(7):837–850, 2010.

[17] Su Gao and Peter Gerdes. Computably enumerable equivalence relations. *Studia Logica*, 67(1):27–59, 2001.

[18] William I. Gasarch. Bounded queries in recursion theory: a survey. *Proceedings of the Sixth Annual Structure in Complexity Theory Conference.* IEEE Computer Society Press, 62–78, 1991.

[19] Alex Gavryushkin, Sanjay Jain, Bakhadyr Khoussainov and Frank Stephan. Graphs realised by r.e. equivalence relations. *Annals of Pure and Applied Logic,* 165(7–8):1263–1290, 2014.

[20] Carl Jockusch. Semirecursive sets and positive reducibility. *Transactions of the American Mathematical Society*, 131:420–436, 1968.

[21] Nadim Kasymov and Bakhadyr Khoussainov. Finitely generated enumerable and absolutely locally finite algebras. *Vychislitelnye Sistemy,* 116:3–15, 1986 (in Russian).

[22] Denis Hirschfeldt and Bakhadyr Khoussainov. On finitely presented expansions of computably enumerable semigroups. Algebra and Logic 51:435 - 444, 2012.

[23] Bakhadyr Khoussainov, Steffen Lempp and Theodore Slaman. Computably enumerable algebras, their expansions, and isomorphisms. *International Journal of Algebra and Computation*, 15(3):437–454, 2005.

[24] Martin Kummer and Frank Stephan. Recursion theoretic properties of frequency computation and bounded queries. *Information and Computation*, 120:59-77, 1995.

[25] Alistair H. Lachlan. A note on positive equivalence relations. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 33:43–46, 1987.

[26] Anatoly I. Maltsev. Towards a theory of computable families of objects. *Algebra i Logika*, 3(4):5–31, 1963.

[27] Luca San Mauro. *Forma e complessitá. Uno studio dei gradi delle relazioni di equivalenza ricorsivamente enumerabili.* Master's thesis, University of Siena, 2011.

[28] Emmy Noether. Idealtheorie in Ringbereichen. *Mathematische Annalen*, 83:24–66, 1921.

[29] Piotr S. Novikov. On the algorithmic unsolvability of the word problem in group theory. Proceedings of the Steklov Institute of Mathematics 44: 1–143, 1955 (in Russian).

[30] Piergiorgio Odifreddi. *Classical Recursion Theory.* North-Holland, 1989.

[31] Robert I. Soare. *Recursively Enumerable Sets and Degrees. A Study of Computable Functions and Computably Generated Sets.* Springer Verlag, Berlin–New York, 1987.

[32] Albert Visser. Numerations, λ-caclulus and arithmetic. *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus and Formalism*, pages 259–284, Academic Press, London, 1980.