On Trees without Hyperimmune Branches*

Keng Meng Ng [1], Frank Stephan [2 3], Yue Yang [3] and Liang Yu [4]

[1] Division of Mathematical Sciences, School of Physical and Mathematical Sciences, Nanyang Technological University, 21 Nanyang Link, Singapore 637371, Republic of Singapore
kmng@ntu.edu.sg
[2] School of Computing, National University of Singapore, 13 Computing Drive, Block COM1, Singapore 117417, Republic of Singapore
fstephan@nus.edu.sg
[3] Department of Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, Block S17, Singapore 119076, Republic of Singapore
matyangy@nus.edu.sg
[4] Mathematical Department, Nanjing University, Nanjing City, Jiangsu Province 210093, People's Republic of China
yuliang.nju@gmail.com

Abstract. The current work includes a result announced in the year 2012 which was unproven for 10 years. The result shows that there is a co-r.e. tree with uncountably many infinite branches such that the nonisolated infinite branches of the constructed tree are all nonrecursive, generalised low, hyperimmune-free and form a perfect tree. This paper is the journal version of two conference articles from 2012 and 2022; the paper contains the main result proved in 2022 and also the other major results from 2012.

1 Introduction

The connections between trees and their branches are well-studied. A major result is Jockusch and Soare's Low Basis Theorem [8] which showed that every infinite binary recursive (or co-r.e.) tree has an infinite branch of low Turing degree; similarly they showed that it also has an infinite branch of hyperimmune-free Turing degree. This shows that trees (this word denotes from now on infinite binary trees) which are recursive or co-r.e. have hyperimmune-free infinite branches. Downey [3] provided a tree where all branches are hyperimmune-free and some are nonrecursive. Downey's construction left open the question whether such trees can be made to have uncountably many infinite branches.

Ng, Stephan, Yang and Yu constructed in 2011 and 2012 recursive trees with uncountably many infinite branches such that every infinite branch is of hyperimmune-free Turing degree. They furthermore studied various related results and announced in 2012 [13, Theorem 2.4] the result that there is a recursive binary tree with uncountably many infinite branches of Cantor-Bendixson rank 1 (thus removing the isolated branches makes it perfect) such that all infinite branches are hyperimmune-free and generalised low$_1$. They gave a proof sketch and announced

* Keng Meng Ng has been supported by the Singapore Ministry of Education grant RG23/19 at NTU. Frank Stephan and Yang Yue have been supported in part by the Singapore Ministry of Education Tier 2 grants AcRF MOE2019-T2-2-121 / R146-000-304-112 and MOE-000538-01 as well as by the NUS Tier 1 grants AcRF R146-000-337-114 and R252-000-C17-114. Liang Yu has been supported by NSFC grant No. 12025103 and the Natural Science Foundation of Jiangsu Province, grant No. BK20243060.

the full details of the proof in the journal version which never got ready due to the construction becoming more and more lengthy and remaining incomplete. The reason was that Ng, Stephan, Yang and Yu tried to follow the proof-sketch of the initial approach of their paper [13, Theorem 2.4] and intended to use the "full approximation method" which is a complicated infinite injury construction. The alternative proof in the present work uses a K-recursive guidance tree U whose infinite branches form a perfect tree of 2-generic inversions of the double jumps of sets in the cone above K'. This guidance tree brought down the complexity of the construction from infinite injury to finite injury in the translation of U to the final co-r.e. tree T.

The result is from the field of classical recursion theory and it connects two major notions of the field, namely that of hyperimmune-free Turing degrees (which are those Turing degrees where all functions computed are majorised by a recursive function) with that of recursive or co-r.e. trees having uncountably many infinite branches. Recursive functions are, in easy terms, those which can be computed by a abstract machine without any bounds on computation time and storage space; alternatively one can also say, due to a result of Matiyasevich [11], that a recursive function is one whose graph is Diophantine, that is, satisfies that there is a polynomial p with integer coefficients in variables $x, y, z_1, \ldots, z_k$ such that, for all $x, y \in \mathbb{N}$, $f(x) = y$ iff there are values for $z_1, \ldots, z_k \in \mathbb{N}$ with $p(x, y, z_1, \ldots, z_k) = 0$. This also works with partial-recursive functions and thus recursively enumerable sets. The classes of the infinite branches of such trees are also called Π^0_1 and subject to detailed studies. The authors also want to refer to standard text books on recursion theory and related fields for further reading of the background [1,10,17,18,19,21]. The main innovation of [13] was to show that Π^0_1-classes can be both uncountable and only have members of hyperimmune-free Turing degree. The add-on properties are introduced below one by one together with definitions or characterisations explaining them.

Theorem 1. *There is a co-r.e. tree T without finite branches which has $2^{\aleph_0}$ many infinite branches such that the nonisolated infinite branches are all hyperimmune-free, generalised low$_1$, Schnorr-trivial, of minimal Turing degree, jump traceable and form a perfect tree.*

Besides the above mentioned announcement [13, Theorem 2.4], the proof was never published and is also not in the follow-up works, that is, works citing the 2012 paper [6,7,15,25]. A result proven by Hirschfeldt, Jockusch and Schupp [7, Lemma 6.7] was also announced without proof [13] and states that every hyperimmune-free infinite branch A on a K-recursive tree is also on a recursive tree not containing any additional infinite branches; the tree might, however, omit some of the infinite branches of the original K-recursive tree. Further results, mainly from [13], are provided in the Section 4. This section and the questions in the conclusion should allow to get an updated picture of which items from the conference paper [13] can be considered as a proven fact and which are, due to incomplete evidence and problems with doing the proof, indeed to be left as an open question. Before this, here are some detailed explanations on the used notions.

Co-r.e. tree: This is a downward closed infinite subset of the binary strings such that the set of strings outside it is recursively enumerable. Branches are finite or infinite strings such that all finite prefixes of them are in the tree. The infinite branches of a co-r.e. tree form a Π^0_1-class and the concepts of recursive tree, co-r.e. tree and Π^0_1-class are interchangeable when talking about the class of infinite branches of a tree.

Remark 2. One can prune of the finite branches of an existing recursive tree S and receives a co-r.e. tree T in which every node has either one or two successors, but which does not have any deadends. The rule is this: Whenever $\sigma 0, \sigma 1$ are both in the complement of the tree T, then one also enumerates σ into the complement of T. The infinite branches of S and T are the same. This co-r.e. tree is notationally easier to handle, due to the direct correspondence between branches and the sequence of their branching bits for all nonrecursive infinite branches in T.

Hyperimmune-free: The easiest characterisation of a set A being hyperimmune-free is that every A-recursive function is majorised by a recursive function. Formally, it means that no set $B \leq_T A$ is hyperimmune, that is, for every infinite set $B \leq_T A$ there is a recursive array $I_0, I_1, \ldots$ of finite disjoint sets given by canonical indices with each I_k intersecting B. Hyperimmune sets were introduced by Post [23]. Dekker [2] showed that every nonrecursive r.e. Turing degree contains a hyperimmune set and Miller and Martin [12] showed that on one hand every nonrecursive Turing degree below the halting problem K contains a hyperimmune set while on the other hand there are also nonrecursive Turing degrees without hyperimmune sets; such degrees and the sets in them are called hyperimmune-free. Jockusch and Soare [8] showed that every co-r.e. tree with an infinite branch contains actually an infinite hyperimmune-free branch.

Schnorr-trivial: Franklin and Stephan [4] gave the following characterisation which serves as a definition: A set A is Schnorr-trivial if for every $f \leq_{tt} A$ there is a recursive function g such that $\forall n \, \exists m \leq 2^n \, [f(n) = g(n,m)]$.

Remark 3. The above notions have the following properties: Schnorr-trivial plus hyperimmune-free equals to a property called recursively traceable; however, the property Schnorr-trivial is easier to achieve in a construction, as one needs only to consider truth-table reductions and not to worry about Turing reductions. Furthermore, if φ_e^A is a truth-table reduction then it can be assumed that there is a recursive function g such that $\varphi_e^A(x)$ needs, independently of the choice of A, at most $g(x)$ steps and queries only below $g(x)$ and outputs a value y with $y < g(x)$. Similarly, if φ_e is a Turing reduction and $\varphi_e^A(x)$ needs s steps to converge then all places queried are below s and $\varphi_e^A(x) < s$. In particular, terminating computations $\varphi_e^\sigma(x)$ query σ only at values in the domain σ. Two partial functions φ_e^σ and φ_e^τ are consistent at stage s iff for all x where both $\varphi_e^\sigma(x)$ and $\varphi_e^\tau(x)$ output values y and z within s steps, it holds that $y = z$.

Jump Traceable: A set A is jump traceable if there are a recursive function h and a partial-recursive function g such that $\forall n \, \exists m \leq h(n) \, [\varphi_n^A(n) = g(n,m)]$. Here one uses the diagonal jump function $n \mapsto \varphi_n^A(n)$ in place of its domain A' and equality "=" of partial function values means that either both sides are undefined or both sides are defined and equal.

Generalised Low: A set A is generalised low$_1$ if $A' \leq_T A \oplus K$. This property was introduced to generalise the notion "low" in a senseful way beyond the classes of Turing degrees below the halting problem. Note that "generalised low$_1$" and "high" are not disjoint properties and some sets have both properties, for example the 1-generic set obtained by inverting the jump of K'. Nies showed the following proposition which allows to construct jump-traceable sets when one actually wants generalised low ones.

Proposition 4 (Nies [16]). *If A is jump traceable then $A' \leq_T A \oplus K$.*

Minimal Turing Degree: A set A is of minimal Turing degree iff A is not recursive and for every $B \leq_T A$, either B is recursive or $A \leq_T B$. Spector [22] constructed a minimal Turing degree which is below the jump K' of the halting problem K and Sacks [20] constructed a minimal Turing degree below K. A sufficient criterion to show that A is minimal is to show that for every Turing reduction φ_e there are a co-r.e. tree S such that A is an infinite branch of S and a node $\sigma \preceq A$ such that one of the following three conditions holds:

1. The function φ_e^A is total and every $B \succeq \sigma$ on S satisfies that $\varphi_e^A = \varphi_e^B$, thus φ_e^A is recursive;
2. Every $B \succeq \sigma$ on S with $B \neq A$ satisfies that there is an x with $\varphi_e^A(x) \neq \varphi_e^B(x)$ where both values are defined, thus $A \leq_T \varphi_e^A$;
3. There is an x with $\varphi_e^A(x)$ being undefined, thus φ_e^A is partial.

A tree S is e-splitting along A if it satisfies one of the three conditions and Spector [22] used this method to construct a minimal Turing degree.

Perfect Tree: A tree is perfect iff every infinite branch is the pointwise limit of other infinite branches which are different to it; in particular perfect trees have $2^{\aleph_0}$ many infinite branches and do not have isolated infinite branches. The tree T constructed in the main result will satisfy that its nonisolated branches form a perfect tree. Furthermore there are $\aleph_0$ many isolated branches added to this tree. These additional recursive branches cannot be avoided as T must have low branches by the Low Basis Theorem [8] and these branches are then recursive by being both hyperimmune-free and low. As all non-isolated infinite branches are non-recursive by the construction of T, the isolated infinite branches cannot be avoided.

Proposition 5. *If V is a K'-recursive tree, then V is contained in a K-r.e. tree U such that every infinite branch A of U is also one of V and vice versa.*

Proof. This is a folklore result which holds relative to any oracle, not only relative K. However, it is needed here only relative to K. By the limit lemma, there is a K-recursive approximation V_s such that for all $\mu \in V$, $V_s(\mu) = V(\mu)$ for almost all s. Now one enumerates a node μ into U iff there is a stage $s > |\mu|$ such that all prefixes ν of μ are in V_s. If A is an infinite branch of V then clearly all prefixes of A are enumerated into U and A is an infinite branch of U. If A is not an infinite branch of V there is a prefix μ of A which is not in V and so there is a stage t such that $V_s(\mu) = 0$ for all $s \geq t$ and then no prefix of A longer than $|\mu| + t$ will ever be enumerated into U; thus A is not an infinite branch of U. □

Proposition 6 (Based on Friedberg Jump Inversion Theorem [5]). *There is a perfect K'-recursive tree V such that each infinite branch of it is 2-generic and that all Turing degrees above K' are double jumps of infinite branches of V.*

Proof. Given a set $B \subseteq \mathbb{N}$ and oracle K', one starts with the root $\sigma_0 = \varepsilon$ and puts it into the tree. Then one checks whether there is an extension η of σ_n in W_e^K and if so then one lets $\sigma_{n+1} = \eta B(n)$ for the first such η found else one lets $\sigma_{n+1} = \sigma_n B(n)$. This case-distinction is clearly K'-recursive. The resulting tree V is the union of this construction for all oracles B and $V[B]$ refers to the encoded infinite branch for B. For two oracles $B, \tilde{B}$ first differing at position n, the respective strings σ_{n+1} in the construction of $V[B]$ and $V[\tilde{B}]$ will differ exactly on the last bit and therefore $V[B]$ and $V[\tilde{B}]$ will be different infinite branches of V. The tree V will be uniformly K'-recursive.

The sets $V[B]$ are always 2-generic and $V[B] \leq_T B \oplus K'$, as the oracle K' can compute the tree V. Thus $V[B]'' \equiv_T V[B] \oplus K' \equiv_T B \oplus K'$. For every B there is the infinite branch $V[B]$ and the infinite branches $V[B]''$ stand in a one-one correspondence with $B \oplus K'$. For every possible doublejump $B \oplus K'$ of a set there is the set $V[B]$ with $V[B]'' \equiv_T B \oplus K'$, thus the Turing degrees of the cone above K' are covered by all sets of the form $B \oplus K'$ and all these sets are equal to a doublejump. $\square$

Corollary 7. *There is a K-r.e. tree U such that all its infinite branches are 2-generic and that the Turing degrees of the double jumps of these branches cover the whole cone above K'.*

2 The Main Result and its Construction

The following theorem implies Theorem 1 and uses the above constructed trees from Proposition 6 and Corollary 7. Recall that on a tree, a node σ is a branching node if there are infinite branches through both $\sigma 0$ and $\sigma 1$; one can follow a branch using an oracle which tells which nodes σ are branching nodes and the branching bits which are, in ascending sequence on the tree, the collection of first bits above each branching node.

Theorem 8. *Given U, V as in Proposition 6 and Corollary 7, there is a co-r.e. tree T with uncountably many infinite branches such that the nonisolated infinite branches of T correspond to infinite branches on U and V, respectively, in a one-one fashion and that these infinite branches of T are each of hyperimmune-free Turing degree, of minimal Turing degree, generalised low$_1$, jump traceable and Schnorr-trivial. Furthermore, the correspondence of each nonisolated infinite branch A of T to an infinite branch B of U satisfies the equality $A \oplus K \equiv_T B \oplus K$ and B consists of the branching bits of A in T.*

Construction. The construction works with markers c_n which are initialised as $c_{n,0} = n$ and move at stages t from $c_{n,t}$ to $c_{n,t+1}$; in parallel to the movement of markers, a co-r.e. tree will be constructed which has branching nodes only on levels on which a marker sits. This tree is called T and $T_0 = \{0,1\}^*$. Furthermore, $u_t(k)$ is the t-th approximation to $u(k)$ for the K-recursive enumeration $u(0), u(1), u(2), \ldots$ of the tree U; note that this approximation exists by the Limit Lemma and for each k and almost all t, $u_t(k) = u(k)$.

Recall that a finite tree up to a level can be described by the leaf nodes in that level; furthermore, one can for each leaf describe the string of branching bits which are those bits in the path to the leaf where one has to make a choice between two possibilities in order to select the leaf to which one wants to go; those where there is only one choice are omitted from the string.

At stage t, one first considers for each n with $c_{n,t} \leq t+1$ the finite tree L_n which contains all words $\sigma \in T_t$ with $dom(\sigma) \subseteq \{0, 1, \ldots, c_{n,t} - 1\}$. The branching bits of a $\sigma \in L_n$ are the last bits a of those prefixes $\eta a \prec \sigma$ where both $\eta 0, \eta 1 \in L_n$ and the branching string is the sequence of branching bits in the given order from the root to the node σ. One considers an extension H_n of L_n and calls it admissible when it satisfies the following conditions:

1. The leaves of H_n have the domain $\{0, 1, \ldots, t+1\}$ and $H_n \subseteq T_t$;

2. For each leaf σ of L_n there is exactly one $\tau \in H_n$ extending σ and having domain $\{0, 1, \ldots, t\}$; furthermore, for the branching bits μ of σ in L_n, if $\mu \in \{u_{t+1}(0), u_{t+1}(1), \ldots, u_{t+1}(n)\}$ then $\tau 0, \tau 1 \in H_n$ else only $\tau 0 \in H_n$;
3. There is progress from L_n to H_n in the way that certain enumeration, definability or splitting goals are obtained; more precisely there are $\sigma, \sigma' \in L_n$ with the corresponding τ, τ' such that $c_{n,t} \leq t+1$ and at least one of the following conditions is satisfied:
   (a) Some element $m \leq n$ became enumerated into K between time $c_{n,t}$ and $t+1$ or $u_{t+1}(m) \neq u_s(m)$ for some $s \in \{c_{n,t}, c_{n,t}+1, \ldots, t\}$;
   (b) There is an $e \leq n$ for which the function φ_e^σ is not defined on all of $0, 1, \ldots, n$ within $c_{n,t}$ steps while φ_e^τ is defined on all of $0, 1, \ldots, n$ within $t+1$ steps;
   (c) There is an $e \leq n$ for which $\varphi_e^\sigma(e)$ is not defined within $c_{n,t}$ steps while $\varphi_e^\tau(e)$ is defined within $t+1$ steps;
   (d) For some $e \leq n$ there is an x such that φ_e^σ and $\varphi_e^{\sigma'}$ are consistent at time $c_{n,t}$ while $\varphi_e^\tau(x), \varphi_e^{\tau'}(x)$ are defined and different for an $x < t+1$ within $t+1$ steps;
   (e) $|\sigma| = |\tau|$ and $c_{n,t} = t+1$.

For the least n where one can find L_n, H_n as above, one does the following:

1. For $m < n$, $c_{m,t+1} = c_{m,t}$ and $c_{n,t+1} = t+1$ and for $m > n$, $c_{m,t+1} = t+1+m-n$;
2. Furthermore, one lets T_{t+1} consist of all the nodes which are comparable to one of the leaves of H_n, that is, one prunes off the tree all nodes which are extending some leaf σ of L_n but which are incomparable to all leaves of H_n.

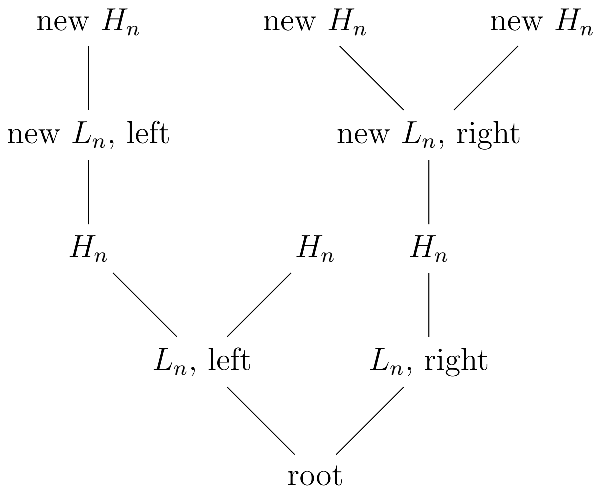
Note that such an n is always found as the n with $c_{n,t} = t+1$ can be selected when no smaller n qualifies.

Note that in the above, if H_n was defined at stage s, then the L_n at the next first stage t where $c_{n,t+1} > c_{n,s+1}$ satisfies that the leaves of that L_n at stage t are one bit shorter than the leaves of the H_n at stage s. This is to make sure that when $u_{t+1}(m) \neq u_{s+1}(m)$ for some $m \leq n$ then the sprouting of the leaves of the old H_n can be undone for the new H_n and is not incorporated into the new L_n.

Notation 9. In the following, the final values of L_n and H_n refer to the trees with leaves of level $c_{n,t} - 1$ and $c_{n,t}$, respectively, where t is so large that $c_{n,t}$ does not change at t or later.

The following proposition is true first due to the fact that when the last H_m with $m < n$ has been defined then $L_n = H_{m-1}$ and from then onwards, whenever L_n and H_n change, the extension of L_n extends each leaf σ of the old L_n uniquely to a leaf τ of the new L_n and furthermore puts two successor leaves into H_n iff the branching bits μ form a string in $\{u_{t+1}(0), u_{t+1}(1), \ldots, u_{t+1}(n)\}$.

Proposition 10. *Each leaf σ of L_n has two successors in H_n iff its branching string η is in $\{u(0), u(1), \ldots, u(n)\}$ and each leaf σa of H_n extends in a unique way to a leaf τ of L_{n+1}.*



The above picture shows how a tree L_n gets updated due to changes in $\{u_s(0), u_s(1), \ldots, u_s(n)\}$ where the updates from the old s to the larger new s removes 0 from the list (branching string for left) and adds 1 into the list (branching string for right). Nodes denoted with L_n and H_n denote the leaves of these trees, where the new left leaf of L_n is above the old left left leaf of H_n and the old left right leaf of H_n is enumerated into the complement of T. The new leaves of H_n have the branching strings 0, 10, 11. Left successors have branching bit 0 and right successors have branching bit 1 at each branching which survives.

3 Proof of the Properties of Infinite Branches

The following fact is useful. It is just based on the fact that given a K-recursive enumeration $v(0), v(1), v(2), \ldots$ of the above tree U, one can replace it by a new enumeration $u(\langle i, j\rangle)$ which has the following approximation: If $v_j(i) = v_s(i)$ for $s = j, j+1, \ldots, j+t$ then $u_t(\langle i, j\rangle) = v_j(i)$ else $u_t(\langle i, j\rangle)$ is some fixed element of U, say the root of the tree.

Fact 11. *Every nonempty K-r.e. set has a K-recursive enumeration in which the enumerating function can be approximated such that it makes at most one mind change in the computation of each number in the enumeration.*

Such type of enumerations are assumed for K-r.e. sets from now onwards for the rest of the proof.

Proposition 12. *Every infinite branch A of T is jump traceable and, by Proposition 4, also generalised low$_1$.*

Proof. Note that for the parameter n, it tracks only the behaviour of $\varphi_e^A(e)$ for $e \leq n$, thus one can in the following always assume that $e \leq n$. See the construction of Theorem 8 for more details about the activities tracked in item 3.

One can see by induction that L_n has always prior to step t at most 2^n leaves and that H_n (due to the branching nodes on the level of $c_{n,t+1}$ being there) can have at most 2^{n+1} leaves. Furthermore, if one enumerates the various possibilities for progress in item 3, one sees that the overall number of mind changes is bounded by $1 + (n+1) \cdot (2 + 2^n + 2^n + 4^n) \leq 8^{n+2}$. Now taking into account that activity of smaller c_m cause c_n to act again, one gets that c_n is modified at most 8^{n+2} times between any two changes of lower markers, so in total at most $8^{(n+1)(n+2)}$ times. Now one can for each leaf τ of each version of L_n enumerate at most one value of $\varphi_e^\sigma(e)$ and for each infinite branch A of T where $\varphi_e^A(e)$ is defined, $\varphi_e^A(e) = \varphi_e^\sigma(e)$ for one of the leaves of the final version of L_n after the last time that the marker $c_{n,t}$ moved, thus every infinite branch A is jump traceable with bound $2^{3n^2+10n+6}$. $\Box$

Proposition 13. *An infinite branch A of T is isolated iff there is an n such that the leaf of A restricted to final version of L_n has branching string μ with $\mu \notin U$.*

Proof. The reason for this is that if μ is not in U then $\mu \notin \{u(0), u(1), \ldots, u(n)\}$ for all n and the final versions of L_n and H_n will have that the leaf $\tau a \preceq A$ in H_n is of the form $\tau 0$ and $\tau 1 \notin H_n$. The same will be true for all $m \geq n$, as in them the restriction of A to $L_m = H_{m-1}$ will have the same branching string η. Thus the corresponding τ in H_m will have the unique extension and leaf $\tau 0$ in H_m and therefore the infinite branch A is isolated and thus recursive.

Now consider the case that all branching strings of A are in U. Then, for each L_m there is a first $n \geq m$ such that the branching string μ of A in L_m is in $\{u(0), u(1), \ldots, u(n)\}$. It follows that the leaf τ of L_n extended by A will also have two extensions $\tau 0, \tau 1 \in H_n$ and so there is a branching node in T which is a leaf of L_n and a prefix of A. As this happens for infinitely many n, the infinite branch A is not isolated. $\Box$

Proposition 14. *Every infinite branch A of T is of hyperimmune-free Turing degree.*

Proof. Let A be an infinite branch of T which is not isolated and which therefore might be non-recursive; note that all isolated infinite branches are recursive. Now A corresponds to some 2-generic infinite branch B of U where the set B is then taken as the sequence of branching bits of A in T.

For a given Turing reduction φ_e, consider the following set: $W^K = \{\mu : \exists n \geq e$ [the final L_n has a leaf σ with branching string μ and $\varphi_e^\sigma(x)$ being undefined for some $x < n$]$\}$. This set is K-r.e.; note that when $\varphi_e^A(x)$ would be defined for some infinite branch A of T extending σ, then some progress would be possible and the L_n would not be the final one.

If an $A = T[B]$ satisfies that there is above every σ some infinite branch $\tilde{A} = T[\tilde{B}]$ of T such that $\tilde{A}$ also extends σ and $\varphi_e^{\tilde{A}}$ is not total, then there is an x with $\varphi_e^{\tilde{A}}(x)$ being undefined and any $n \geq x$ satisfies that some prefix $\tilde{\sigma}$ of $\tilde{A}$ is in L_n as a leaf and $\varphi_e^{\tilde{\sigma}}(m)$ is undefined on some $m \leq n$ and therefore its branching string ν is in W^K. Thus B has above every node some extension in W^K and so B has a prefix itself which is in W^K. This implies that φ_e^A is not total by the definition of W^K.

So one sees that there is a $\sigma \preceq A$ such that for all infinite branches $\tilde{A}$ of T which extend σ it holds that $\varphi_e^{\tilde{A}}$ is total. Thus one can, for every x, simulate the coenumeration of T and the pruning of the tree until a stage is found and a level ℓ such that for all $\tau \in \{0,1\}^\ell$ which extend σ and which are still in T it holds that $\varphi_e^\tau(x)$ converges within ℓ steps and then the maximum of these values is an upper bound for $\varphi_e^A(x)$; as this upper bound is computed by a recursive function, φ_e^A has a recursive upper bound. Thus, for each e, either φ_e^A is partial (as indicated in the preceding paragraph) or φ_e^A has a recursive upper bound and therefore every nonisolated infinite branch of T is of hyperimmune-free Turing degree. The isolated infinite branches are recursive and therefore of hyperimmune-free Turing degree as well. $\Box$

Proposition 15. *Every infinite branch A of T is Schnorr-trivial.*

Proof. If φ_e is a truth-table reduction with bound function g for the computation time then it will happen for almost all n that $c_{n,t}$ is eventually above $g(n)$; this is due to the fact that the time to enumerate the halting time of K up to n is a dominating function with respect to n. Thus one can, for all sufficiently large n, simulate the construction until the final value of the $c_{n,t}$ is above the use and then enumerate the 2^n values which are defined by the various branches of the co-r.e. tree which survive. The finitely many smaller values can be patched. Thus A is Schnorr-trivial. $\square$

Proposition 16. *The double jumps A'' of the infinite branches A of T form the cone above K'.*

Proof. Assume A is a non-isolated infinite branch of T; then $A = T[B]$ for some infinite branch B of U; furthermore, the bits of B are the branching bits of A in T. Note that the double jumps of the infinite branches B of V and thus U cover all Turing degrees above K' due to the B being obtained by uniform double jump inversion. Each B is 2-generic and satisfies $B'' \equiv_T B \oplus K'$. Furthermore, Marcus Triplett showed in his bachelor thesis [24] that $A'' \equiv_T A' \oplus K'$ for all hyperimmune-free sets A.

The halting problem K allows to reconstruct all branching nodes of T and these allow to recover the set B from $A \oplus K$. Thus $B \leq_T A \oplus K$. As B is 2-generic, $B'' \equiv_T B \oplus K' \leq_T A \oplus K' \leq_T A' \oplus K' \equiv_T A''$. Similarly one can recover A from $B \oplus K$, as the set B are the branching bits of A on T and K computes the branching nodes. It follows that $A' \leq_T (B \oplus K)' \leq_T B''$ and, as K' is the bottom element among all double-jumps, $K' \leq_T B''$. Thus $A'' \equiv_T B''$.

As the constructions of V and U included for every Turing degree above K' a 2-generic infinite branch B of U such that the double jump B'' is in the given Turing degree, one has that the double jumps of the infinite branches of T cover all Turing degrees above K'. $\square$

Proposition 17. *Every nonrecursive infinite branch A of T has minimal Turing degree.*

Proof. Let A be a non-recursive infinite branch of T. Consider a total function φ_e^A. By the proof of Proposition 14 there is a σ such that all infinite branches $\tilde{A}$ of T above σ, $\varphi_e^{\tilde{A}}$ is total.

If now there is such an infinite branch $\tilde{A}$ extending σ which is different from A but for which φ_e^A and $\varphi_e^{\tilde{A}}$ coincide, then it follows that above the branching node σ' of A and $\tilde{A}$, all infinite branches of S produce the same function φ_e^A and so this function is recursive. To see this, assume otherwise. Then $\sigma'0$ or $\sigma'1$ would be extended into a sufficiently long prefixes of the respective A and $\tilde{A}$ in order to achieve that these prefixes are mapped by φ_e on some value to different images and then one of the infinite branches A or $\tilde{A}$ would be cut off, so they would not be both on the tree.

So one has that there is a $\sigma'' \preceq A$ such that, for all infinite branches $\tilde{A}$ of S extending σ'', either the functions $\varphi_e^{\tilde{A}}$ are all different or are all the same.

If they are all different, one can Turing reduce A to φ_e^A. Whenever there are two different extensions above some $\sigma''' \succeq \sigma''$, one coenumerates T and simulates φ_e until either an e-splitting at some x above $\sigma'''0$ and $\sigma'''1$ is found so that $\varphi_e^A(x)$ says which branch to follow or one of the two nodes $\sigma'''0$ and $\sigma'''1$ has been enumerated out of T. Thus $A \equiv_T \varphi_e^A$ by the usual e-splitting analysis.

If the infinite branches of T above σ'' are mapped by φ_e all to the same image, then one can for each input x coenumerate T until a level $\ell > |\sigma''|$ and a time t are found such that all $\tau \in \{0,1\}^\ell \cap S_t$ which extend σ'' satisfy that $\varphi_{e,t}^{\tau}(x)$ is a unique value y and equals to $\varphi_e^A(x)$. Thus φ_e^A is recursive in this case. Thus the minimality of A is verified. $\square$

Proposition 18. *The nonisolated branches of T form a perfect tree.*

Proof. Recall that every infinite branch B of U is 2-generic and that the infinite branches of U form a perfect subtree V of U by Proposition 6 and Corollary 7.

There is a K-recursive operator which maps the infinite branch B of U to the infinite branch $A = T[B]$ where the "branching string" of A in T is exactly the set B. This mapping is one-one and continuous. Furthermore, if $B_0, B_1, \ldots$ converge pointwise to B and all $B_k \neq B$ then the infinite branches $T[B_k]$ converge pointwise to $A = T[B]$. Hence the nonrecursive infinite branches of U are mapped in a bijection to the nonisolated infinite branches of T and this bijection preserves pointwise limits; as the class of infinite branches of U and its image in T are both closed in the sense of topology, the limits of each convergent sequence is again an infinite branch of the corresponding tree. Thus the nonisolated infinite branches of T form a perfect subtree of T which is recursive in K'. $\square$

4 Further Results

This section provides certain results which were already proven in the conference version from 2012 but never published in a journal. The results are included for the sake of completeness of the exposition of this paper and some of the claimed results are left as open questions in the Conclusion. So this paper is based on both conference versions, that from 2012 and that from 2022.

Theorem 19. *If A is an infinite branch of a recursive tree T where every infinite branch is hyperimmune-free then A is also an infinite branch of a recursive tree S with uncountably many infinite branches such that all of these are hyperimmune-free.*

Proof. This can easily be seen. Let U be any tree with uncountably many infinite branches which are all of hyperimmune-free Turing degree and let $S = T \cup U$. Clearly S is also a recursive tree and a set B is an infinite branch of S if it is an infinite branch of one of these trees. Thus the number of these infinite branches is uncountable and all of these are of hyperimmune-free Turing degree. $\square$

Recall that a Π_2^0-singleton A is a set A for which there is an index e such that A is the only set for which φ_e^A is total. Without loss of generality one can select this index e such that whenever $\varphi_e^B(x)$ is defined for some oracle B and $y < x$ then $\varphi_e^B(y)$ is also defined.

Proposition 20. *Every hyperimmune-free Π_2^0-singleton is recursive.*

Proof. Assume now that A is hyperimmune-free and e is an index such that φ_e^B is total if and only if $B = A$. As A is hyperimmune-free, there is a recursive function which majorises the A-recursive function measuring for each input x how much time $\varphi_e^A(x)$ needs to compute the result and what the largest query is. In particular, there is a recursive function f such that, for all x, $\varphi^A_{e,f(x)}(x)$ is defined, that is, the computation converges within time $f(x)$ and does not query A beyond $f(x)$. Thus A is actually the singleton of a Π^0_1 class, as one can define the class as the class of all oracles B where, for all x, $\varphi^B_{e,f(x)}(x)$ converges within time $f(x)$ to its output and does not query values bigger than $f(x)$. So, A is an isolated infinite branch – indeed even the only infinite branch – of the recursive tree of all B where the use of $\varphi^B_e(x)$ is below $f(x)$ and thus A is recursive. $\square$

Proposition 21. *Only the recursive sets are both unrelativised hyperimmune-free and hyperimmune-free relative to some PA-complete set $B \leq_T K$.*

Proof. So assume that A is hyperimmune-free and undecidable. Then $A \not\leq_T K$. There is an uniformly co-r.e. sequence $\tilde{B}_0, \tilde{B}_1, \ldots$ of all co-r.e. trees and, as B is PA-complete, there is furthermore a sequence $B_0, B_1, \ldots$ of sets such that either B_e is an infinite branch of $\tilde{B}_e$ or $\tilde{B}_e$ does not have an infinite branch. Note that A is not in the list as $A \not\leq_T B$. Some $A \oplus B$ recursive function computes on input e the first x where $A(x)$ and $B_e(x)$ differ. Since A is hyperimmune-free relative to B, this function is majorised by a B-recursive function g. Now A is an infinite branch of the B-recursive tree T obtained by removing, for each e, all strings properly extending the prefix σ_e of B_e of length $g(e)+2$. Furthermore, as A is hyperimmune-free, there is an A-recursive function h such that $h(x)$ is the first time $t \geq x$ where all prefixes $A(0)A(1)\ldots A(y)$ with $y \leq x$ satisfy that they are in the t-th approximation of T. This function h has a recursive majorant ℓ as A is hyperimmune-free. Now one has that the following tree S is recursive and A is an infinite branch of it: S contains all binary strings σ such that for all prefixes τ of σ, τ is inside the t-th approximation of T for some $t \in \{|\sigma|, |\sigma|+1, \ldots, \ell(|\sigma|)\}$. Thus all prefixes of A are in S and A is an infinite branch of S. Furthermore, if $\sigma \notin T$, then $\sigma \notin T_t$ for all t from some s onwards and all $\tau \succeq \sigma$ with $|\tau| > s$ are not in S. Thus no B_e is an infinite branch of S. There is a co-r.e. index e of S and then B_e is an infinite branch of S, a contradiction. Thus A cannot have the following three properties at the same time: hyperimmune-free, hyperimmune-free relative to B and non-recursive. This completes the proof. $\square$

The next result shows some type of highness-preservation when relativising relative to a hyperimmune-free oracle A. The main reason is Martin's Theorem that highness and domination of all recursive / A-recursive functions is the same. That is, if A is hyperimmune-free and B is high, then B computes some function f which dominates all recursive functions – thus f also dominates all A-recursive functions and furthermore $f \leq_T A \oplus B$, as everything computable relative to B is also computable relative to $A \oplus B$. By Martin's Theorem, relativised to A, it then follows that $(A \oplus B)' \geq A''$. This is stated formally in the next proposition.

Proposition 22. *If A is hyperimmune-free and B is high then $A \oplus B$ is high relative to A, that is, $(A \oplus B)' \geq_T A''$.*

Kučera and Nies [9] proved that if a Σ^0_2-set is strictly above K then it computes the jump of a nonrecursive hyperimmune-free set. The next result looks at the converse of this, what can be said about a Σ^0_2-set B computed by the jump of any hyperimmune-free set.

Theorem 23. *If A is hyperimmune-free and A' computes a Σ^0_2-set B then $B \leq_T K$.*

Proof. The set B can be approximated by a recursive function g in the sense that $g(x, s) = 1$ for almost all x if and only if $x \in B$; furthermore, there is an A-recursive function f which converges for all x to $B(x)$, that is, $B(x) = \lim_s f(x, s)$. Now there is an A-recursive function h such that, for all x, $h(x)$ is the first stage s such that, for all $y \leq x$, there are at least x stages $t \leq s$ with $f(x, t) = g(x, t)$; note that for each x there are infinitely many such t: if $f(x, t)$ converges to 0 then $g(x, t) = 0$ for infinitely many t and if $f(x, t)$ converges to 1 then $g(x, t)$ also converges to 1. There is now a recursive majorant k of h, that is, $k(x) \geq h(x)$ for all x.

Now one defines $\tilde{g}(x, s)$ as follows: If there are at least $x + s$ many $t \leq k(x + 2s)$ with $g(x, t) = 0$ then $g(x, s) = 0$ else $g(x, s) = 1$.

If $x \in B$ this can happen only finitely often as $g(x, s)$ converges to 1; if $x \notin B$ then f makes only finitely often a mistake and there are at most u many t with $f(x, t) = 0$. It follows that there are, for $s \geq u$, at least $x + 2u$ many $t \leq k(x + 2s)$ where $f(x, t) = g(x, t)$; so there are $x + s$ many $t \leq k(x + 2s)$ with $g(x, t) = 0$. Thus $\tilde{g}$ witnesses that B is limitrecursive, that is, $B \leq_T K$. $\square$

Theorem 24. *If B is 2-generic and $B \leq_T K'$ then there exists a hyperimmune-free A such that $A' \equiv_T B' \equiv_T B \oplus K$; furthermore, A can be chosen to be Schnorr trivial.*

The idea is that one takes the K'-recursive tree V from Proposition 6 and forms the union with the single tree having the set B – which is also K'-recursive – as its only and infinite branch. So one obtains again a K'-recursive tree having only 2-generic infinite branches. Then one uses Corollary 7 and the main result Theorem 8 to get the corresponding trees U and T. Now there is a hyperimmune-free set A of T such that $A \oplus K \equiv_T B \oplus K$ where B is the given infinite branch of V and, thus, U. The A constructed is also Schnorr-trivial and of minimal Turing degree. While most properties of the constructions also apply to the A mentioned here, it cannot be guaranteed that A is not isolated on the tree. This would require further tweeking, that is, if B is isolated on V one would have to create further infinite branches of V in order to make B part of a dense tree consisting entirely of 2-generic sets. The old conference version from 2012 contains a relativised version of the above theorem with an alternative proof [13, Theorem 4.4].

Furthermore, one can also modify V such that it only contains uncountably many infinite branches B such that each B is 2-generic and hyperimmune-free relative to K'. Then one constructs from V the K-r.e. tree U (with the same infinite branches) and the recursive tree T such that the branches A of T derived from infinite branches B of V are hyperimmune-free and minimal and Schnorr-trivial. The so obtained tree T has only hyperimmune-free branches which are also hyperimmune-free relative to K'. Thus one gets the following improved version of [13, Proposition 3.5].

Theorem 25. *There is a recursive tree T with uncountably many infinite branches such that all of these are hyperimmune-free unrelativised as well as hyperimmune-free relative to K'.*

5 Concluding Remarks

The presented proof shows that one can construct a co-r.e. tree such that every branch is infinite and that every infinite branch is either isolated or has all of the following properties: nonrecursive, hyperimmune-free, generalised low_1, minimal Turing degree, Schnorr-trivial and jump traceable.

Furthermore, the nonisolated infinite branches form a perfect tree and its double jumps cover the cone above K' and each of its jumps is also the jump of a 2-generic set. The key idea was to use the guidance tree U and to transform it into the final co-r.e. tree T so that every non-isolated infinite branch A of T has the branching bits from an infinite branch B of U, which guides the building of the tree. The construction could be a bit varied to get the following related results.

First, instead of taking V to be the union of all $V[\tilde{B}]$ in Proposition 6, one could take a K'-recursive tree with uncountably many K'-hyperimmune-free and 2-generic branches and then get the result that there is a tree T with uncountably many infinite branches such that every infinite branch is hyperimmune-free, hyperimmune-free relative to K', jump-traceable and jump-traceable relative to K'. It is possible to iterate this.

Second, if $B \leq_T K'$ is 2-generic then one can construct a K-r.e. tree U with B being its unique infinite branch. The resulting tree T has then exactly one nonisolated and nonrecursive branch which is $A = T[B]$. This gives a co-r.e. tree of Cantor-Bendixson rank 2 with exactly one nonisolated branch $A = T[B]$ which is nonrecursive, hyperimmune-free, generalised low$_1$ and satisfies $A \oplus K = B \oplus K$.

Third, note that a perfect co-r.e. tree T without deadends has always an infinite branch which is high, thus one cannot avoid the isolated branches in the construction of T which make the tree T to be a nonperfect enlargement of a perfect tree. If T would not have isolated infinite branches then all infinite branches would be hyperimmune-free and minimal, thus they would not be below the halting problem in contradiction to the Low Basis Theorem of Jockusch and Soare [8]. So the Cantor-Bendixson rank of the tree T constructed in Theorem 1 must be at least 1, as was also stated without proof by Ng, Stephan, Yang and Yu [13]. The same applies to trees with uncountably many infinite branches where the jumps of all infinite branches avoid some upper cone.

The following questions are left open. For the first, it was claimed by Ng, Stephan, Yang and Yu [13, Theorem 3.6] that a modification of the proof of the main result of this paper would give the result; however, this proof is now a new proof by other means and the authors are not aware how to incorporate the Robinson guessing into it. The original proof did not work out and required more and more corrective items to make the construction work without the whole construction ever getting completed.

Open Problem 26. *1. Are there any hyperimmune-free sets which are hyperimmune-free relative to every low recursively enumerable set? If so, are there uncountably many of those?*

2. Is every jump of a nonrecursive hyperimmune-free set also the jump of a 2-generic set? The converse is known [13, Theorem 4.4].

3. Is the jump of every hyperimmune-free set also the jump of a hyperimmune-free Schnorr-trivial set?

References

1. Cristian S. Calude. *Information and Randomness – An Algorithmic Perspective.* Springer, Berlin and Heidelberg, Second Edition, 2002.
2. James C. E. Dekker. A theorem on hypersimple sets. *Proceedings of the American Mathematical Society*, 5:791-–796, 1954.

3. Rodney G. Downey. On Π^0_1-classes and their ranked points. *Notre Dame Journal of Formal Logic*, 32(4):499–512, 1991.
4. Johanna Franklin and Frank Stephan. Schnorr-trivial sets and truth-table reducibility. *The Journal of Symbolic Logic*, 75(2):501–521, 2010.
5. Richard Friedberg. A criterion for completeness of degrees of unsolvability. *The Journal of Symbolic Logic*, 22:159–160, 1957.
6. Rupert Hölzl and Christopher P. Porter. Randomness for computable measures and initial segment complexity. *Annals of Pure and Applied Logic*, 168(4):860–886, 2017.
7. Denis R. Hirschfeldt, Carl G. Jockusch Jr. and Paul E. Schupp. Coarse computability, the density metric, Hausdorff distances between Turing degrees, perfect trees, and reverse mathematics. Technical report on `http://arxiv.org/abs/2106.13118`, 2021.
8. Carl G. Jockusch and Robert I. Soare. Π^0_1 classes and degrees of theories. *Transactions of the American Mathematical Society*, 173:33–56, 1972.
9. Antonín Kučera and André Nies. Demuth randomness and computational complexity. *Annals of Pure and Applied Logic*, 162(7):504–513, 2011.
10. Ming Li and Paul M. B. Vitányi. *An Introduction to Kolmogorov Complexity and its Applications.* Springer Verlag, Fourth Edition, New York, 2019.
11. Yuri V. Matiyasevich. Diofantovost' perechislimykh mnozhestv. *Doklady Akademii Nauk SSSR*, 191:297-282, 1970 (Russian). English translation: Enumerable sets are Diophantine, *Soviet Mathematics Doklady*, 11:354-358, 1970.
12. Webb Miller and Donald Martin. The degrees of hyperimmune sets. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 14:159–166, 1968.
13. Keng Meng Ng, Frank Stephan, Yue Yang and Liang Yu. The computational aspects of hyperimmunefree degrees. *Proceedings of the Twelfth Asian Logic Conference*, pages 271–284, World Scientific, 2013.
14. Keng Meng Ng, Frank Stephan, Yue Yang and Liang Yu. On Trees Without Hyperimmune Branches. *Conference on Computability in Europe* – CiE 2022, Springer Lecture Notes in Computer Science, 13359:234–245, 2022.
15. Keng Meng Ng and Hongyuan Yu. Effective domination and the bounded jump. *Notre Dame Journal of Formal Logic*, 61(2):203–225, 2020.
16. André Nies. Reals which compute little. *Proceedings of Logic Colloquium 2002, Lecture Notes in Logic*, 27:261–275, 2002.
17. André Nies. *Computability and Randomness.* Oxford University Press, Oxford, 2009.
18. Piergiorgio Odifreddi. *Classical Recursion Theory.* North-Holland, Amsterdam, 1989.
19. Piergiorgio Odifreddi. *Classical Recursion Theory, Volume II.* Elsevier, Amsterdam, 1999.
20. Gerald E. Sacks. On the degrees less than $\mathbf{0}'$. *Annals of Mathematics*, 77:211–231, 1963.
21. Robert Soare. *Recursively Enumerable Sets and Degrees. A Study of Computable Functions and Computably Generated Sets.* Springer-Verlag, Heidelberg, 1987.
22. Clifford Spector. On degrees of unsolvability. *Annals of Mathematics*, 64:581–592, 1956.
23. Emil Leon Post. Recursively enumerable sets of positive integers and their decision problems. *Bulletin of the American Mathematical Society*, 50:284–316, 1944.
24. Marcus Anthony Triplett. *Algorithmic Complexity and Triviality.* Bachelor Thesis, The University of Auckland, New Zealand, 2014.
25. Hongyuan Yu. *On equivalence relations and bounded Turing degrees.* PhD dissertation. Nanyang Technological University, Singapore, 2018.