HIGHER KURTZ RANDOMNESS

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Abstract. A real x is Δ^1_1-Kurtz random (Π^1_1-Kurtz random) if it in no closed null Δ^1_1 set (Π^1_1 set). We show that there is a cone of Π^1_1-Kurtz random hyperdegrees. We characterize lowness for Δ^1_1-Kurtz randomness by being Δ^1_1-dominated and Δ^1_1-semitraceable.

1. Introduction

Traditionally one uses tools from recursion theory to obtain mathematical notions corresponding to our intuitive idea of randomness for reals. However, already Martin-Löf [10] suggested to use tools from higher recursion (or equivalently, effective descriptive set theory) when he introduced the notion of Δ^1_1-randomness. This approach was pursued to greater depths by Hjorth and Nies [8] and Chong, Nies and Yu [1]. Hjorth and Nies investigated a higher analog of the usual Martin-Löf-randomness, and a new notion with no analog in recursion theory: a real is Π^1_1-random if avoids each null Π^1_1 set. Chong, Nies and Yu [1] studied Δ^1_1-randomness in more detail, viewing it as a higher analog of both Schnorr and recursive randomness. By now a classical result is the characterization of lowness for Schnorr randomness by recursive traceability (see, for instance, [11]). Chong, Nies and Yu [1] proved a higher analog of this result, characterizing lowness for Δ^1_1 randomness by Δ^1_1 traceability.

Our goal is to carry out similar investigations for higher analogs of Kurtz randomness [3]. A real x is Kurtz random if avoids each Π^0_1 null class. The term is a bit of a misnomer as such a set need not be random in any intuitive sense. Each weakly 1-generic set is Kurtz random, so for instance the law of large numbers can fail badly. However, the term is by now commonly accepted.

It is essential for Kurtz randomness that the tests are *closed* null sets. Two higher analogs of Kurtz randomness make sense: one can require that these tests are Δ^1_1, or that they are Π^1_1.

Restrictions on the computational complexity of a real have been used successfully to analyze randomness notions. For instance, a Martin-Löf-random real is weakly 2-random iff it forms a minimal pair with $\emptyset'$ (see [11]). We prove a result of that kind in the present setting. Already [1] studied a property restricting the complexity of a real, being Δ^1_1 dominated. This is the higher analog of being recursively dominated (or of

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hyperimmune-free degree). We show that a Δ^1_1-Kurtz random Δ^1_1 dominated set is already Π^1_1-random. Thus Δ^1_1-Kurtz randomness is equivalent to a proper randomness notion on a conull set. We also study the distribution of higher Kurtz random reals in the hyperdegrees. For instance, there is a cone of Π^1_1-Kurtz random hyperdegrees. However, its base is very complex, having the largest hyperdegree among all Σ^1_2 reals.

Thereafter we turn to lowness for higher Kurtz randomness. Recursive traceability of a real x is easily seen to be equivalent to the condition that for each function $f \leq_T x$ there is a recursive function $\hat{f}$ that agrees with f on at least one input in each interval of the form $[2^n, 2^{n+1}-1)$. One says that x is recursively semitraceable if for each $f \leq_T x$ there is a recursive function $\hat{f}$ that agrees with f on infinitely many inputs. It is straightforward to define the higher analog of this notion, being Δ^1_1-semitraceable. Our main result is that lowness for Δ^1_1-Kurtz randomness is equivalent to being Δ^1_1-dominated and Δ^1_1-semitraceable. We also show using forcing that being Δ^1_1-dominated and Δ^1_1-semitraceable is strictly weaker than being Δ^1_1-traceable. Thus, lowness for Δ^1_1 Kurtz randomness is strictly weaker than lowness for Δ^1_1-randomness.

2. Preliminaries

We assume that the reader is familiar with elements of higher recursion theory, as presented, for instance, in Sacks [13]. A real is an element in 2^ω. Sometimes we write $n \in x$ to mean $x(n) = 1$. Fix a standard Π^0_2 set $H \subseteq \omega \times 2^\omega \times 2^\omega$ so that for all x and $n \in \mathcal{O}$, there is a unique real y satisfying $H(n, x, y)$. Moreover, if $\omega_1^x = \omega_1^{\mathrm{CK}}$, then each real $z \leq_h x$ is Turing reducible to some y so that $H(n, x, y)$ holds for some $n \in \mathcal{O}$. Roughly speaking, y is the $|n|$-th Turing jump of x. These y's are called H^x sets and denoted by H^x_n's. For each $n \in \mathcal{O}$, let $\mathcal{O}_n = \{m \in \mathcal{O} \mid |m| < |n|\}$. $\mathcal{O}_n$ is a Δ^1_1 set.

We use the Cantor pairing function, the bijection $p : \omega^2 \to \omega$ given by $p(n, s) = \frac{(n+s)^2+3n+s}{2}$, and write $\langle n, s\rangle = p(n, s)$. For a finite string σ, $[\sigma] = \{x \succ \sigma \mid x \in 2^\omega\}$. For an open set U, there is a presentation $\hat{U}$ so that $\sigma \in \hat{U}$ if and only if $[\sigma] \subseteq U$. We sometimes identify U with $\hat{U}$. For a recursive functional Φ, we use $\Phi^\sigma[s]$ to denote the computation state of Φ^σ at stage s. For a tree T, we use $[T]$ to denote the set of infinite paths in T. Some times we identify a finite string $\sigma \in \omega^{<\omega}$ with a natural number without confusion.

The following results will be used in later sections.

Theorem 2.1 (Gandy)**.** *If $A \subseteq 2^\omega$ is a nonempty Σ^1_1 set, then there is a real $x \in A$ so that $\mathcal{O}^x \leq_h \mathcal{O}$.*

Theorem 2.2 (Spector [14] and Gandy [6])**.** *$A \subset 2^\omega$ is Π^1_1 if and only if there is an arithmetical predicate $P(x, y)$ such that*

$$y \in A \leftrightarrow \exists x \leq_h y P(x, y).$$

Theorem 2.3 (Sacks[12])**.** *If x is non-hyperarithmetical, then $\mu(\{y|y \geq_h x\}) = 0$.*

Theorem 2.4 (Sacks [13])**.** *The set $\{x|x \geq_h \mathcal{O}\}$ is Π^1_1. Moreover, $x \geq_h \mathcal{O}$ if and only if $\omega_1^x > \omega_1^{\mathrm{CK}}$.*

A consequence of the last two theorems above is that the set $\{x|\omega_1^x > \omega_1^{\mathrm{CK}}\}$ is a Π^1_1 null set.

A subset of 2^ω is $\mathbf{\Pi}^0_0$ if it is clopen. We can define $\mathbf{\Pi}^0_\gamma$ sets by a transfinite induction in an obvious way for all countable γ. Every such set can be coded by a real in an obvious way (more details can be found in [13]). Given a class $\mathbf{\Gamma}$ (for example, $\mathbf{\Gamma} = \Delta^1_1$) of subsets of 2^ω, a set A is $\mathbf{\Pi}^0_\gamma(\mathbf{\Gamma})$ if A is $\mathbf{\Pi}^0_\gamma$ and can be coded by a real in $\mathbf{\Gamma}$.

In the case $\gamma = 1$, every hyperarithmetic closed subset of reals is $\mathbf{\Pi^0_1}(\Delta^1_1)$. We also have the following result with an easy proof.

Proposition 2.5. *If $A \subseteq 2^\omega$ is Σ^1_1 and $\mathbf{\Pi}^0_1$, then $A \in \mathbf{\Pi}^0_1(\Sigma^1_1)$.*

Proof. Let $z = \{\sigma \mid \exists x(x \in A \wedge x \succ \sigma)\}$. Then $x \in A$ if and only if $\forall n(x \upharpoonright n \in z)$. So A is $\Pi^0_1(z)$. Obviously z is Σ^1_1. □

Note that Proposition 2.5 fails if we replace Σ^1_1 with Π^1_1 since $\mathcal{O}^{\mathcal{O}}$ is a Π^1_1 singleton of which the hyperdegree is greater than $\mathcal{O}$.

The ramified analytical hierarchy was introduced by Kleene, and applied by Fefferman [4] and Cohen [2] to study forcing, a tool that turns out to be powerful in the investigation of higher randomness theory. We recall some basic facts here following Sacks [13] whose notations we mostly follow, as given below:

The ramified analytic hierarchy language $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ contains the following symbols:

(1) Number variables: $j, k, m, n, ...$;
(2) Numerals: 0,1,2,...;
(3) Constant: $\dot{x}$;
(4) Ranked set variables: $x^\alpha, y^\alpha, ...$ where $\alpha < \omega_1^{\mathrm{CK}}$;
(5) Unranked set variables: $x, y, ...$;
(6) Others symbols include: $+$, $\cdot$ (times), $'$ (successor) and $\in$.

Formulas are built in the usual way. A formula φ is *ranked* if all of its set variables are ranked. Due to its complexity, the language is not codable in a recursive set but rather in the countable admissible set $L_{\omega_1^{\mathrm{CK}}}$.

To code the language in a uniform way, we fix a Π^1_1 path $\mathcal{O}_1$ through $\mathcal{O}$ (by [5] such a path exists). Then a ranked set variable x^α is coded by the number $(2, n)$ where $n \in \mathcal{O}_1$ and $|n| = \alpha$. Other symbols and formulas are coded recursively. With such a coding, the set of Gödel number of formulas is Π^1_1. Moreover, the set of Gödel numbers of ranked formulas of rank less than α is r.e. uniformly in the unique notation for α in $\mathcal{O}_1$. Hence there is a recursive function f so that $W_{f(n)}$ is the set of Gödel numbers of the ranked formula of rank less than $|n|$ when $n \in \mathcal{O}_1$ ($\{W_e\}_e$ is, as usual, an effective enumeration of r.e. sets).

One now defines a structure $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, where x is a real, analogous to the way Gödel's L is defined, by induction on the recursive ordinals. Only at successor stages are new sets defined in the structure. The reals constructed at a successor stage are arithmetically definable from the reals constructed at earlier stages. The details may be found in [13]. We define $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ for a formula φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ by allowing the unranked set variables to range over $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$, while the symbol x^α will be interpreted as the reals built before stage α. In fact, the domain of $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ is the set $\{y|y \leq_h x\}$ if and only if $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [13]).

A sentence φ of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$ is said to be Σ_1^1 if it is ranked, or of the form $\exists x_1, ..., \exists x_n \psi$ for some formula ψ with no unranked set variables bounded by a quantifier.

We have the following result which is a model-theoretic version of the Gandy-Spector Theorem.

Theorem 2.6 (Sacks [13])**.** *The set* $\{(n_\varphi, x)|\varphi \in \Sigma_1^1 \wedge \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ *is* Π_1^1*, where* n_φ *is the Gödel number of* φ*. Moreover, for each* Π_1^1 *set* $A \subseteq 2^\omega$*, there is a formula* $\varphi \in \Sigma_1^1$ *so that*

(1) $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \implies x \in A$*;*
(2) *if* $\omega_1^x = \omega_1^{\mathrm{CK}}$*, then* $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$*.*

Note that if φ is ranked, then both the sets $\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}$ (the Gödel number of φ is omitted) and $\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \neg\varphi\}$ are Π_1^1 and so Δ_1^1. Moreover, if $A \subseteq 2^\omega$ is Δ_1^1, then there is a ranked formula φ so that $x \in A \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi$ (see Sacks [13]).

Theorem 2.7 (Sacks [12])**.** *The set* $\{(n_\varphi, p)|\mu(\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi\}) > p \wedge \varphi \in \Sigma_1^1 \wedge$ *p is a rational number*$\}$ *is* Π_1^1 *where* n_φ *is the Gödel number of* φ*.*

Theorem 2.8 (Sacks [12])**.** *There is a recursive function* $f : \omega \times \omega \to \omega$ *so that for all* n *which is Gödel number of a ranked formula*

(1) $f(n, p)$ *is Gödel number of a ranked formula;*
(2) *The set* $\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} \supseteq \{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}$ *is open;*
(3) $\mu(\{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_{f(n,p)}\} - \{x|\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi_n\}) < \frac{1}{p}$*.*

Theorem 2.9 (Sacks [12] and Tanaka [15])**.** *If* A *is a* Π_1^1 *set of positive measure, then* A *contains a hyperarithmetical real.*

3. Higher Kurtz random reals and their distribution

Definition 3.1. *Given a point class* $\mathbf{\Gamma}$ *(i.e. a class of sets of reals). A real* x *is* $\mathbf{\Gamma}$-Kurtz random *if for every closed null set* $A \in \mathbf{\Gamma}$*,* $x \notin A$*.* x *is said to be Kurtz-random (y-Kurtz random) if* $\mathbf{\Gamma} = \Pi_1^0$ $(\mathbf{\Gamma} = \Pi_1^0(y))$*.*

We focus on Δ_1^1, Σ_1^1 and Π_1^1-Kurtz randomness. By the proof of Proposition 2.5, it is not difficult to see that a real x is Δ_1^1-Kurtz random if and only if x does not belong to any $\mathbf{\Pi}_1^0(\Delta_1^1)$ null set.

Theorem 3.2. Π_1^1*-Kurtz-randomness* $\subset \Sigma_1^1$*-Kurtz-randomness* $= \Delta_1^1$*-Kurtz-randomness.*

Proof. It is obvious that Π_1^1-Kurtz-randomness $\subseteq \Sigma_1^1$-Kurtz-randomness $\subseteq \Delta_1^1$-Kurtz-randomness.

Note that every Π_1^1-ML-random is Δ_1^1-Kurtz-random and there is a Π_1^1-ML-random real $x \equiv_h \mathcal{O}$ (see [8] and [1]). But $\{x\}$ is a Π_1^1 closed set. So x is not Π_1^1-Kurtz-random. Hence Π_1^1-Kurtz-randomness $\subset \Delta_1^1$-Kurtz-randomness.

Given a Π^1_1 open set A of measure 1. Define $x = \{\sigma \in 2^{<\omega} \mid \forall y(y \succ \sigma \implies x \in A)\}$. Then x is a Π^1_1 real coding A (i.e. $y \in A$ if and only if there is a $\sigma \in x$ for which $y \succ \sigma$, or $y \in [\sigma]$). So there is a recursive function $f : 2^{<\omega} \to \omega$ so that $\sigma \in x$ if and only if $f(\sigma) \in \mathcal{O}$. Define a Π^1_1 relation $R \subseteq \omega \times \omega$ so that $(k, n) \in R$ if and only if $n \in \mathcal{O}$ and $\mu(\bigcup\{[\sigma] \mid \exists m \in \mathcal{O}_n(f(\sigma) = m)\}) > 1 - \frac{1}{k}$. Obviously R is a Π^1_1 relation which can be uniformized by a Π^1_1 function f. Since $\mu(A) = 1$, f is a total function. So the range of f is bounded by a notation $n \in \mathcal{O}$. Define $B = \{y \mid \exists\sigma(y \succ \sigma \wedge f(\sigma) \in \mathcal{O}_n)\}$. Then $B \subseteq A$ is a Δ^1_1 open set with measure 1. So every Π^1_1 open conull set has a Δ^1_1 open conull subset. Hence Σ^1_1-Kurtz-randomness $= \Delta^1_1$-Kurtz-randomness. □

The following result clarifies the relation between Δ^1_1- and Π^1_1-Kurtz randomness.

Proposition 3.3. *If $\omega_1^x = \omega_1^{\mathrm{CK}}$, then x is Π^1_1-Kurtz random if and only if x is Δ^1_1-Kurtz random.*

Proof. Suppose that $\omega_1^x = \omega_1^{\mathrm{CK}}$ and x is Δ^1_1-Kurtz random. If A is a Π^1_1 closed null set so that $x \in A$. Then by Theorem 2.6, there is a formula $\varphi(z, y)$ whose only unranked set variables are z and y so that the formula $\exists z\varphi(z, y)$ defining A. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, $x \in B = \{y \mid \mathfrak{A}(\omega_1^{\mathrm{CK}}, y) \models \exists z^\alpha\varphi(z^\alpha, y)\} \subseteq A$ for some recursive ordinal α. Define $T = \{\sigma \in 2^{<\omega} \mid \exists y \in B(y \succ \sigma)\}$. Obviously $B \subseteq [T]$. Since B is Δ^1_1, $[T]$ is Σ^1_1. Since A is closed, so is $[T] \subseteq A$. A is null, so is $[T]$. By the proof of Theorem 3.2, there is a Δ^1_1 closed null set $C \supseteq [T]$. Hence $x \in C$, a contradiction. □

From the proof of Theorem 3.2, one sees that every hyperarithmetic degree above $\mathcal{O}$ contains a Δ^1_1-Kurtz random real. But this fails for Π^1_1-Kurtz random. We say that a hyperdegree $\mathbf{d}$ is a *base for a cone of* $\mathbf{\Gamma}$*-Kurtz randomness* if for every hyperarithmetic degree $\mathbf{h} \geq \mathbf{d}$, $\mathbf{h}$ contains a $\mathbf{\Gamma}$-Kurtz random real.

The hyperdegree of $\mathcal{O}$ is a base for a cone of Δ^1_1-Kurtz randomness as proved in Theorem 3.2. We shall prove that not every nonzero hyperdegree is a base of a cone for Δ^1_1-Kurtz random later.

Does there exist a base for a cone of Π^1_1-Kurtz randomness? If there exists such a base, say $\mathbf{b}$, then $\mathbf{b}$ is not hyperarithmetically reducible to any Π^1_1 singleton. This means that the bases must be rather complicated.

Lemma 3.4. *For any reals x and $z \geq_T x'$, there is an x-Kurtz-random real $y \equiv_T z$.*

Proof. Fix the reals x and z as the assumption. Fix an enumeration of x-r.e. open sets $\{U_n^x\}_{n\in\omega}$.

We construct an increasing sequence $\{\sigma_s\}_{s<\omega}$ step by step.

At stage 0. Let σ_0 be empty.

At stage $s+1$. Let $l_0 = 0$, $l_1 = |\sigma_s|$, and $l_{n+1} = 2^{l_n}$ for all $n > 1$. For every $n > 1$, let

$$A_n = \{\sigma \in 2^{l_n - 1} \mid \exists m < n \forall i \forall j (l_m \leq i, j < l_{m+1} \implies \sigma(i) = \sigma(j))\}.$$

Then

$$|A_n| \leq 2 \cdot 2^{l_{n-1}}.$$

In other words,

$$\mu(B_n = \bigcup\{[\sigma] \mid \sigma \succeq \sigma_s \wedge \sigma \notin A_n\}) \geq 2^{-l_1} \cdot (1 - 2^{l_n+1-l_{n+1}}).$$

Case(1): There is some $m > l_1 + 1$ so that $|\{\sigma \succeq \sigma_s \mid \sigma \in 2^m \wedge [\sigma] \subseteq U^x_s\}| > 2^{m-l_1-1}$. Let $n = m + 1$. Then $l_{n+1} - 1 - l_n > 2$ and $l_n > m$. So there must be some $\sigma \in 2^{l_n-1} - A_n$ so that there is a $\tau \preceq \sigma$ for which $[\tau] \subseteq U^x_s$ and $\tau \in 2^m$.

Let $\sigma_{s+1} = \sigma^\frown(z(s))^{l_n-1}$.

Case(2): Otherwise. Let $\sigma_{s+1} = \sigma_s^\frown(z(s))^{l_1-1}$.

This finishes the construction at stage $s + 1$.

Let $y = \bigcup_s \sigma_s$.

Obviously the construction is recursive in z. So $y \leq_T z$. Moreover, if U^x_n is of measure 1, then Case (1) happens at the stage $n + 1$. So y is x-Kurtz random.

Let $l_0 = 0, l_{n+1} = 2^{l_n}$ for all $n \in \omega$. To compute $z(n)$ from y, we y-recursively find the n-th l_m for which for all i, j with $l_m \leq i < j < l_{m+1}$, $y(i) = y(j)$. Then $z(n) = y(l_m)$. □

Let $\mathcal{A} \subseteq \omega \times 2^\omega$ be a universal Π^1_1 closed set. In other words, $\mathcal{A}$ is a Π^1_1 set so that for every n, $\mathcal{A}_n = \{x \mid (n, x) \in \mathcal{A}\}$ is a Π^1_1 closed set and every Π^1_1 closed set is some $\mathcal{A}_n$. By Theorem 2.2.3 in [9], the real $x_0 = \{n \mid \mu(\mathcal{A}_n) = 0\}$ is Σ^1_1. Let

$$\mathfrak{c} = \{(n, \sigma) \mid n \in x_0 \wedge \exists x((n, x) \in \mathcal{A} \wedge \sigma \prec x)\} \subseteq \omega \times 2^{<\omega}.$$

Then $\mathfrak{c}$ can be viewed as a Σ^1_2 real. Since every Π^1_1 null closed set is $\Pi^0_1(\mathfrak{c})$, every $\mathfrak{c}$-Kurtz-random real is Π^1_1-Kurtz random.

Theorem 3.5. *$\mathfrak{c}$ is a base for a cone of Π^1_1-Kurtz-randomness.*

Proof. By Lemma 3.4, for every $y \geq_T \mathfrak{c}'$, there is a real $z \equiv_T y$ for which z is $\mathfrak{c}$-Kurtz random and so Π^1_1-random. Thus $\mathfrak{c}$ is a base for Π^1_1-randomness. □

Let

$$\delta^1_2 = \text{ supremum of the } \Delta^1_2 \text{ wellorderings of } \omega,$$

and

$$\delta = \min\{\alpha \mid L \setminus L_\alpha \text{ contains no } \Pi^1_1 \text{ singleton}\}.$$

Proposition 3.6. *$\delta = \delta^1_2$.*

Proof. If $\alpha < \delta$, then there is a Π^1_1 singleton $x \in L_\delta \setminus L_\alpha$. Since $x \in L_{\omega^x_1}$ and ω^x_1 is a $\Pi^1_1(x)$-wellordering, it must be that $\alpha < \omega^x_1 < \delta^1_2$. So $\delta \leq \delta^1_2$.

If $\alpha < \delta^1_2$, there is a Δ^1_2 wellordering relation $R \subseteq \omega \times \omega$ of order type α. So there are two recursive relations $S, T \subseteq (\omega^\omega)^2 \times \omega^3$ so that

$$R(n, m) \Leftrightarrow \exists f \forall g \exists k S(f, g, n, m, k), \text{ and}$$
$$\neg R(n, m) \Leftrightarrow \exists f \forall g \exists k T(f, g, n, m, k).$$

Define a Π^1_1 set $R_0 = \{(f, n, m) \mid \forall g \exists k S(f, g, n, m, k)\}$. By Gandy-Spector theorem 2.2, there is an arithmetical relation S' so that $R_0 = \{(f, n, m) \mid \exists g \leq_h f(S'(f, g, n, m))\}$. Notice that every Π^1_1 nonempty set contains a Π^1_1-singleton. Then

$$R(n, m) \Leftrightarrow \exists f \in L_\delta \exists g \in L_{\omega^f_1}[f](S'(f, g, n, m)).$$

In other words, R is Σ_1-definable over L_δ. By the same method, $\neg R$ is Σ_1-definable over L_δ too. So R is Δ_1-definable over L_δ. It is clear that L_δ is admissible. So $R \in L_\delta$. Hence $\alpha < \delta$. Thus $\delta^1_2 = \delta$. $\square$

We analyze the complexity of $\mathfrak{c}$. Since every Π^1_1 singleton is recursive in $\mathfrak{c}$, $\mathfrak{c} \notin L_{\delta^1_2}$. Moreover, the set $\{x \mid x \text{ is a } \Pi^1_1\text{-singleton}\}$ is a $\Delta^1_1(\mathfrak{c})$ set. In other words, $\{x \mid x \text{ is a } \Pi^1_1\text{-singleton}\} \in L_{\omega_1^{\mathfrak{c}}}[\mathfrak{c}]$. So it is clear that $\omega_1^{\mathfrak{c}} > \delta^1_2$.

By the same argument in Proposition 3.6, the reals lying in $L_{\delta^1_2}$ are exactly those Δ^1_2-reals. So $\mathfrak{c}$ is not Δ^1_2. Moreover, since $\mathfrak{c}$ is Σ^1_2, it is Σ_1 definable over $L_{\delta^1_2}$. Hence $\mathfrak{c} \in L_{\delta^1_2+1}$. Actually all Σ^1_2 reals lies in $L_{\delta^1_2+1}$. This means:

$$\mathfrak{c} \text{ has the largest hyperdegree among all } \Sigma^1_2 \text{ reals.}$$

4. Δ^1_1-traceability and dominability

We begin with the characterization of Π^1_1-randomness within Δ^1_1-Kurtz randomness.

Definition 4.1. *A real x is Δ^1_1-dominated if for all functions $f : \omega \to \omega$ with $f \leq_h x$, there is a hyperarithmetic function g so that $g(n) > f(n)$ for all n (written as $g > f$).*

Recall that a real is Π^1_1-random if it does not belong to any Π^1_1-null set. The following result is a higher analog of the result that Kurtz randomness coincides with weak 2-randomness for reals of hyperimmune-free degree.

Proposition 4.2. *A real x is Π^1_1-random if and only if x is Δ^1_1-dominated and Δ^1_1-Kurtz-random.*

Proof. Every Π^1_1-random real is Δ^1_1-Kurtz random and also Δ^1_1-dominated (see [1]). We prove another direction.

Suppose x is Δ^1_1-dominated and Δ^1_1-Kurtz random. We show that x is Π^1_1-Martin-Löf random. If not, then fix a universal Π^1_1-Martin-Löf test $\{U_n\}_{n\in\omega}$ (see [8]). Since x is Π^1_1-dominated, $\omega_1^x = \omega_1^{\mathrm{CK}}$ (see [1]). Then by the same argument in the proof of Lemma 4.5, there is a Δ^1_1-sub-Martin-Löf test $\{\hat{U}_n\}_{n\in\omega}$ so that $x \in \bigcap_n \hat{U}_n$. Let $\hat{f}(n) = \min\{l \mid \exists \sigma \in 2^l(\sigma \in \hat{U}_n \wedge x \in [\sigma])\}$ be a $\Delta^1_1(x)$ function. Then there is a Δ^1_1-function f dominating $\hat{f}$. Define $V_n = \{\sigma \mid \sigma \in 2^{\leq f(n)} \wedge \sigma \in \hat{U}_n\}$ for every n. Then $P = \bigcap_n V_n$ is a Δ^1_1 closed set and $x \in P$. So x is not Δ^1_1-Kurtz random, a contradiction.

Hence x is Π^1_1-Martin-Löf random. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, x is also Π^1_1-random (see [1]). $\square$

Next we proceed to traceability.

Definition 4.3. (i) *Let $h : \omega \to \omega$ be a nondecreasing unbounded function that is hyperarithmetical. A Δ^1_1-trace with bound h is a uniformly Δ^1_1 sequence $(T_e)_{e\in\omega}$ such that $|T_e| \leq h(e)$ for each e.*

(ii) *$x \in 2^\omega$ is Δ^1_1-traceable [1] if there is $h \in \Delta^1_1$ such that, for each $f \leq_h x$, there is a Δ^1_1 -trace with bound h such that, for each e, $f(e) \in T_e$.*

(iii) *$x \in 2^\omega$ is Δ^1_1-semi-traceable if for each $f \leq_h x$, there is a Δ^1_1 -function g so that, for infinitely many n, $f(n) = g(n)$. We call that g semi-traces f.*

(iv) *$x \in 2^\omega$ is Π^1_1-semi-traceable if for each $f \leq_h x$, there is a partial Π^1_1 -function p so that, for infinitely many n's, $f(n) = p(n)$.*

Note that, if $(T_e)_{e\in\omega}$ is a uniformly Δ^1_1 sequence of finite sets, then there is $g \in \Delta^1_1$ such that for each e, $D_{g(e)} = T_e$ (where D_n is the nth finite set according to some recursive ordering). Thus

$$g(e) = \mu n \,\forall u\, [u \in D_n \leftrightarrow u \in T_e].$$

In this formulation, the definition of Δ^1_1 traceability is very close to that of recursive traceability.

Also notice that the choice of a bound as a witness for traceability is immaterial:

Proposition 4.4 (As in Terwijn and Zambella [16])**.** *Let A be a real that is Δ^1_1 traceable with bound h. Then for any monotone and unbounded Δ^1_1 function h', A is Δ^1_1 traceable with bound h'.*

Lemma 4.5. *x is Π^1_1-semi-traceable if and only if x is Δ^1_1-semi-traceable.*

Proof. It is not difficult to see that if x is Π^1_1-semi-traceable, then $\omega_1^x = \omega_1^{\mathrm{CK}}$ (otherwise, $x \geq_h \mathcal{O}$. But $\mathcal{O}$ cannot be Π^1_1-semi-traceable,).

Suppose that x is Π^1_1-semi-traceable and $\omega_1^x = \omega_1^{\mathrm{CK}}$, and $f \leq_h x$. Fix a Π^1_1 partial function p for f. Since p is a Π^1_1 function, there must be some recursive injection h so that $p(n) = m \Leftrightarrow h(n,m) \in \mathcal{O}$.

Let $R(n,m)$ be a $\Pi^1_1(x)$ relation so that $R(n,m)$ iff there exists $m > k \geq n$ for which $f(k) = p(k)$. Then there is a $\Pi^1_1(x)$ total, and so $\Delta^1_1(x)$, function g uniformizing R. So for every n, there is some $m \in [g(n), g(g(n)))$ so that $f(m) = p(m)$. Let $g'(0) = g(0)$, and $g'(n+1) = g(g'(n))$ for all $n \in \omega$. Define a $\Pi^1_1(x)$ relation $S(n,m)$ so that $S(n,m)$ if and only if $m \in [g'(n), g'(n+1))$ and $p(m) = f(m)$. Uniformizing S to be a $\Delta^1_1(x)$ function g''.

Define a $\Delta^1_1(x)$ set to be $H = \{h(m,k) \mid \exists n(g''(n) = m \wedge f(m) = k)\}$. Since $\omega_1^x = \omega_1^{\mathrm{CK}}$, $H \subseteq \mathcal{O}_n$ for some $n \in \mathcal{O}$. Since $\mathcal{O}_n$ is a Δ^1_1 set, we can define a Δ^1_1 function $\hat{f}$ to be: $\hat{f}(i) = j$ if $h(i,j) \in \mathcal{O}_n$; $\hat{f}(i) = 1$, otherwise. Then there are infinitely many i's so that $f(i) = \hat{f}(i)$. □

The following result gives another characterization of Δ^1_1-semi-traceability.

Proposition 4.6. *x is Δ^1_1-semi-traceable if and only if there is an increasing Δ^1_1 function g so that for every function $f \leq_h x$, there is a function $F : \omega \to \omega^{<\omega}$ with $\underline{\lim}_n \frac{|F(n)|}{g(n)} < 1$ so that there are infinitely many n's with $f(n) \in F(n)$.*

Proof. It suffices to show the if direction. Given $f \leq_h x$. Let $g(n) = \langle f(g(n)), f(g(n)+1), ..., f(g(n+1)-1)\rangle$ for all $n \in \omega$. Then there is a Δ^1_1 function F as described. Then for all $g(n) \leq m < g(n+1)$, let $\hat{f}(m) =$ the $(m-g(n))$-th column of the $(m-g(n))$-th element of $F(n)$ if there exists such an m; otherwise, let $\hat{f}(m) = 1$. Since $\underline{\lim}_n \frac{|F(n)|}{g(n)} < 1$, it is not difficult to see that for infinitely many n's, there is a number $m \in [g(n), g(n+1))$ so that $f(m) = \hat{f}(m)$. □

Note that the Δ^1_1 dominated reals form a measure 1 set [1] but the set of Δ^1_1-semi-traceable reals is null. Chong, Nies and Yu [1] constructed a non-hyperarithmetic Δ^1_1-traceable real.

Proposition 4.7. *Every Δ^1_1-traceable real is Δ^1_1-dominated and -semi-traceable.*

Proof. Obviously every Δ^1_1-traceable real is Δ^1_1-dominated.

Given a Δ^1_1-traceable real x and $\Delta^1_1(x)$ function f. Let $g(n) = \langle f(2^n), f(2^n+2), \dots, f(2^{n+1}-1)\rangle$ for all $n \in \omega$. Then there is a Δ^1_1 trace T for g so that $|T_n| \leq n$ for all n.

Then for all $2^n+1 \leq m \leq 2^{n+1}$, let $\hat{f}(m) =$ the $(m-2^n)$-th column of the $(m-2^n)$-th element of T_n if there exists such an m; otherwise, let $\hat{f}(m) = 1$. It is not difficult to see that for every n there is at least one $m \in [2^n, 2^{n+1})$ so that $f(m) = \hat{f}(m)$. □

From the proof above, one can see the following corollary.

Corollary 4.8. *A real x is Δ^1_1-traceable if and only if for every x-hyperarithmetic $\hat{f}$, there is a hyperarithmetic function f so that for every n, there is some $m \in [2^n, 2^{n+1})$ so that $f(m) = \hat{f}(m)$.*

The following proposition is useful.

Proposition 4.9. *For any real x, the following are equivalent.*

1. *x is Δ^1_1-semi-traceable and Δ^1_1-dominated.*
2. *For every function $g \leq_h x$, there are an increasing Δ^1_1 function f and a Δ^1_1-function $F : \omega \to \omega^{<\omega}$ with $|F(n)| \leq n$ so that for every n, there exists some $m \in [f(n), f(n+1))$ so that $g(n) \in F(n)$.*

Proof. (1) $\Longrightarrow$ (2): Obviously.

(2) $\Longrightarrow$ (1). Give a function $\hat{g} \leq_h x$. Without loss of generality, $\hat{g}$ is non-decreasing. Let Δ^1_1 functions f and F be the corresponding functions. Define $j(n) = \sum_{i \leq f(n+1)} \sum_{k \in F(i)} k$. Then j is a Δ^1_1 function dominating $\hat{g}$.

To show that x is Δ^1_1-traceable. Give a function $\hat{g} \leq_h x$. Let $h(n) = \langle g(2^n+1), g(2^n+2), ..., g(2^{n+1}-1)\rangle$. Then there are the corresponding Δ^1_1 functions f_h and F_h. For every n and $m \in [2^n, 2^{n+1})$, let $g(m) =$ the $(m-2^n)$-th column of the $m-2^n$-th element in $F_h(n)$ if such an m exists; let $g(m) = 1$ otherwise. Then g is a Δ^1_1 function semi-tracing $\hat{g}$. □

To separate Δ^1_1-traceability from the conjunction of Δ^1_1-semi-traceability and Δ^1_1-dominability, we have to modify Sacks' perfect set forcing.

Definition 4.10.
1. *A Δ^1_1 perfect tree $T \subseteq 2^{<\omega}$ is* fat *at n if for every $\sigma \in T$ and $|\sigma| \in [2^n, 2^{n+1})$, then $\sigma^\frown 0 \in T$ and $\sigma^\frown 1 \in T$.*
2. *A Δ^1_1 perfect tree $T \subseteq 2^{<\omega}$ is* fat *if there are infinitely many n's so that T is fat at n.*
3. *Fat forcing $\mathbb{F} = (\mathcal{F}, \subseteq)$ is a partial order of which the domain $\mathcal{F}$ is the collection of fat trees.*

Let φ be a sentence of $\mathfrak{L}(\omega_1^{\mathrm{CK}}, \dot{x})$. Then we can define the forcing relation, $T \Vdash \varphi$, as done by Sacks in Section 4, IV [13].

(1) φ is ranked and $\forall x \in T(\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi)$, then $T \Vdash \varphi$.
(2) If $\varphi(y)$ is unranked and $T \Vdash \varphi(\psi(n))$ for some $\psi(n)$ of rank at most α, then $T \Vdash \exists y^\alpha \varphi(y^\alpha)$.
(3) If $T \Vdash \exists y^\alpha \varphi(y^\alpha)$, then $T \Vdash \exists y \varphi(y)$.
(4) If $\varphi(n)$ is unranked and $T \Vdash \varphi(m)$ for some number m, then $T \Vdash \exists n \varphi(n)$.
(5) If φ and ψ are unranked, $T \Vdash \varphi$ and $T \Vdash \psi$, then $T \Vdash \varphi \wedge \psi$.
(6) If φ is unranked and $\forall P(P \subseteq T \implies P \not\Vdash \varphi)$, then $T \Vdash \neg\varphi$.

The following lemma can be deduced as done in [13].

Lemma 4.11. *The relation $T \Vdash \varphi$, restricted to Σ_1^1 φ's, is Π_1^1.*

Lemma 4.12. (1) *Let $\{\varphi_i\}_{i\in\omega}$ be a hyperarithmetic sequence of Σ_1^1 sentences. Suppose for every i and $Q \subseteq T$, there exists some $R \subseteq Q$ so that $R \Vdash \varphi_i$. Then there exists some $Q \subseteq T$ so that for every i, $Q \Vdash \varphi_i$.*
(2) *$\forall\varphi\forall T\exists Q \subseteq T(Q \Vdash \varphi \vee Q \Vdash \neg\varphi)$.*

Proof. Note that $\mathcal{R}(R, i, \sigma, P)$ if and only if

$\sigma \in R \wedge \log|\sigma| - 1$ is the i-th fat number of $R \wedge P \restriction |\sigma| = \{\tau \mid \tau \prec \sigma\} \wedge P \subseteq R \wedge P \Vdash \varphi_i$

is a Π_1^1 relation where $P \restriction n = \{\tau \in 2^{\leq n} \mid \tau \in P\}$. Then $\mathcal{R}$ can be uniformized by a partial Π_1^1 function $F : \mathcal{F} \times \omega \times 2^{<\omega} \to \mathcal{F}$. A hyperarithmetic family $\{P_\sigma \mid \sigma \in 2^{<\omega}\}$ can be defined by recursion on σ.

$P_\emptyset = T$.
If $\log|\sigma| - 1$ is not a fat number of P_σ, then $P_{\sigma^\frown 0}, P_{\sigma^\frown 1} = P_\sigma$.
Otherwise: If $\sigma \notin P_\sigma$, then $P_{\sigma^\frown 0} = P_{\sigma^\frown 1} = \emptyset$.
Otherwise: $P_{\sigma^\frown 0} \cap P_{\sigma^\frown 1} = \emptyset, P_{\sigma^\frown 0} \cap P_{\sigma^\frown 1} \subseteq P_\sigma$,
$P_{\sigma^\frown 0} \restriction |\sigma|, P_{\sigma^\frown 1} \restriction |\sigma| = \{\tau \mid \tau \prec \sigma\}$ and
$P_{\sigma^\frown 0}, P_{\sigma^\frown 1} \Vdash \wedge_{j \leq i} \varphi_j$ where
i is the number so that $\log|\sigma| - 1$ is the i-th fat number of T.

Let $Q = \bigcap_n \bigcup_{|\sigma|=n} P_\sigma$. Then $Q \in \mathcal{F}$. It is routine to check that for every i, $Q \Vdash \varphi_i$.

The proof of (2) is the same as the proof of Lemma 4.4 IV [13]. □

x is a generic if for each Σ_1^1 sentence φ, there is a condition T such that $x \in T$ and either $T \Vdash \varphi$ or $T \Vdash \neg\varphi$. One can check that for every Σ_1^1-sentence φ (Lemma 4.8, IV [13]),

$$\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi \Leftrightarrow \exists P(x \in P \wedge P \Vdash \varphi).$$

Lemma 4.13. *If x is a generic real, then*
(1) *$\mathfrak{A}(\omega_1^{\mathrm{CK}}, x)$ satisfies Δ_1^1-comprehension. So $\omega_1^x = \omega_1^{\mathrm{CK}}$.*
(2) *x is Δ_1^1-dominated and Δ_1^1-semi-traceable.*
(3) *x is not Δ_1^1-traceable.*

Proof. (1). The proof of (1) is exactly same as the proof of Theorem 5.4 IV, [13].

(2). By Proposition 4.9, it suffices to show that for every function $g \leq_h x$, there are an increasing Δ_1^1 function f and a Δ_1^1-function $F : \omega \to \omega^{<\omega}$ with $|F(n)| \leq n$ so that for every n, there exists some $m \in [f(n), f(n+1))$ so that $g(n) \in F(n)$. Since $g \leq_h x$ and $\omega_1^x = \omega_1^{\mathrm{CK}}$, there is a ranked formula φ so that for every n, $g(n) = m$ if and only if $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(n,m)$. So there is a condition $S \Vdash \forall n \exists! m \varphi(n,m)$. Fix a condition $T \subseteq S$. As in the proof of Lemma 4.12, we can build a hyperarithmetic sequence of conditions $\{P_\sigma\}_{\sigma \in 2^{<\omega}}$ as in the proof so that

$$P_{\sigma^\frown i} \Vdash \varphi(|\sigma|, m_{\sigma^\frown i}) \text{ for } i \leq 1$$

if $\log|\sigma| - 1$ is a fat number of P_σ and $\sigma \in P_\sigma$. Let Q as defined in the proof of Lemma lemma fusion lemma. Define a Δ^1_1 function $f(0) = 0$, $f(n+1)$ be the least number $k > f(n)$ so that m_σ is defines for some σ with $f(n) < |\sigma| < k$. Let $F(n) = \{0\} \cup \{m_\sigma \mid |\sigma| = n\}$ be a Δ^1_1 function. Then

$$Q \Vdash \forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i)).$$

So

$$Q \Vdash \exists F \exists f (\forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i))).$$

Since T is an arbitrary condition stronger than S, this means

$$S \Vdash \exists F \exists f (\forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i))).$$

Since $x \in S$,

$$\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \exists F \exists f (\forall n |F(n)| \leq n \wedge \forall n \exists m \in [f(n), f(n+1)) \exists i \in F(m)(\varphi(m,i))).$$

So x is Δ^1_1-dominated and Δ^1_1-semi-traceable.

(3). Suppose $f : \omega \to \omega$ is a Δ^1_1 function so that for every n, there is a number $m \in [2^n, 2^{n+1})$ with $f(m) = x(m)$. Then there is a ranked formula φ so that $f(n) = m \Leftrightarrow \mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \varphi(n,m)$. Moreover, $\mathfrak{A}(\omega_1^{\mathrm{CK}}, x) \models \forall n \exists m \in [2^n, 2^{n+1})(\varphi(m, x(m)))$. So there is a condition $T \Vdash \forall n \exists m \in [2^n, 2^{n+1})(\varphi(m, \dot{x}(m)))$ and $x \in T$. Let n be a number so that T is fat at n and $\sigma \in 2^{2^n - 1}$ be a finite string in T. Let μ be a finite string so that $\mu(m) = 1 - f(m + 2^n - 1)$. Define $S = \{\sigma^\frown \mu^\frown \tau \mid \sigma^\frown \mu^\frown \tau \in T\} \subseteq T$. Then $S \Vdash \forall m \in [2^n, 2^{n+1})(\neg \varphi(m, x(m)))$. But S is stronger than T, a contradiction. By Corollary 4.8, x is not Δ^1_1-traceable. □

We may now separate Δ^1_1-traceability from the conjunction of Δ^1_1-semi-traceability and Δ^1_1-dominability.

Theorem 4.14. *There are $2^{\aleph_0}$ many Δ^1_1-dominated and Δ^1_1-semi-traceable reals which are not Δ^1_1-traceable.*

Proof. This is immediate from Lemma 4.13. Note that there are $2^{\aleph_0}$ many generic reals. □

5. Lowness for Kurtz randomness

Definition 5.1. *Given a relativizable class of reals $\mathcal{C}$ (for instance, $\mathcal{C}$ is the class of random reals), a real x is low for $\mathcal{C}$ if $\mathcal{C} = \mathcal{C}^x$.*

We shall prove that lowness for Δ^1_1-randomness is different from lowness for Δ^1_1-Kurtz randomness.

Theorem 5.2. *If x is Δ^1_1-dominated and -semi-traceable, then x is low for Δ^1_1-Kurtz random. Actually x is low for Δ^1_1-Kurtz test. In other words, every $\Delta^1_1(x)$ open set with measure 1 has a Δ^1_1 open subset of measure 1. Hence lowness for Δ^1_1-randomness is different from lowness for Δ^1_1-Kurtz randomness.*

Proof. Suppose x is Δ^1_1-dominated and -semi-traceable and U is a $\Delta^1_1(x)$ open set with measure 1. Then there is a real $y \leq_h x$ so that U is $\Sigma^0_1(y)$. So there is a recursive oracle set U^z so that $U^y = U$.

Define a $\Delta^1_1(x)$ function $\hat{f}(n) =$ the shortest string $\sigma \prec y$ so that $\mu(U^\sigma[\sigma]) > 1-2^{-n}$. By the assumption, there are an increasing Δ^1_1 function g and Δ^1_1 function f so that for every n, there is an $m \in [g(n), g(n+1))$ so that $f(m) = \hat{f}(m)$. Without loss of generality, we can assume that $\mu(U^{f(m)}[m]) > 1 - 2^{-m}$ for every m.

Define a Δ^1_1 open set V so that $\sigma \in V$ if and only if there exists some n so that $[\sigma] \subseteq \bigcap_{g(n)\leq m<g(n+1)} U^{f(m)}[m]$. By the property of f and g, $V \subseteq U^y = U$. But for every n,

$$\mu(\bigcap_{g(n)\leq m<g(n+1)} U^{f(m)}[m]) > 1 - \sum_{g(n)\leq m<g(n+1)} 2^{-m} \geq 1 - 2^{-g(n)+1}.$$

So

$$\mu(V) \geq \lim_n \mu(\bigcap_{g(n)\leq m<g(n+1)} U^{f(m)}[m]) = 1.$$

Hence x is low for Δ^1_1-Kurtz random.

In [1], it is proved that lowness for Δ^1_1-randomness is the same as Δ^1_1-traceability. By Theorem 4.14, lowness for Δ^1_1-randomness is different from lowness for Δ^1_1-Kurtz randomness. $\square$

Corollary 5.3. *There is a non-zero hyperdegree below $\mathcal{O}$ which is not a base for a cone of Δ^1_1-Kurtz randomness.*

Proof. Clearly there is a real $x <_h \mathcal{O}$ which is Δ^1_1-dominated and -semi-traceable. Then the hyperdegree of x is not a base for Δ^1_1-Kurtz random. $\square$

Actually the converse of Theorem 5.2 is also true.

Lemma 5.4. *If x is low for Δ^1_1-Kurtz random, then x is Δ^1_1-dominated.*

Proof. Firstly we show that if x low for Δ^1_1-Kurtz test, then x is Δ^1_1-dominated.

Suppose $f \leq_h x$ is an increasing function. Let $S_f = \{z \mid \forall n(z(f(n)) = 0)\}$. Obviously S_f is a $\Delta^1_1(x)$ closed null set. So there is a Δ^1_1 closed null set $[T] \supseteq S_f$ where $T \subseteq 2^{<\omega}$ is a Δ^1_1-tree. Define

$$g(n) = \min\{m \mid \frac{|\{\sigma \in 2^m \mid \sigma \in T\}|}{2^m} < 2^{-n}\} + 1.$$

Since $\mu([T]) = 0$, g is a well defined Δ^1_1-function. We claim that g dominates f.

For every n, $S_{f(n)} = \{\sigma \in 2^{f(n)} \mid \forall i \leq n(\sigma(f(i)) = 0)\}$ has cardinality $2^{f(n)-n}$. But if $g(n) \leq f(n)$, then since $S \subseteq [T]$, we have

$$|S_{f(n)}| \leq 2^{f(n)-g(n)} \cdot |\{\sigma \in 2^{g(n)} \mid \sigma \in T\}| < 2^{f(n)-g(n)} \cdot 2^{g(n)-n} = 2^{f(n)-n}.$$

This is a contradiction. So x is Δ^1_1-dominated.

Now suppose x is not Δ^1_1 dominated witnessed by some $f \leq_h x$. Then S_f is not contained in any Δ^1_1-closed null set. Actually, it is not difficult to see that for any σ with $[\sigma] \cap S_f \neq \emptyset$, $[\sigma] \cap S_f$ is not contained in any Δ^1_1-closed null set (otherwise, as proved above, one can show that f is dominated by some Δ^1_1 function). Then, by an induction, we can construct a Δ^1_1-Kurtz random real $z \in S_f$ as follows:

Fix an enumeration $P_0, P_1, ...$ of Δ^1_1-closed null set.

At stage $n+1$, we have constructed some $z \upharpoonright l_n$ so that $[z] \upharpoonright l_n \cap S_f \neq \emptyset$. Then there is a $\tau \succ z \upharpoonright l_n$ so that $[\tau] \cap S_f \neq \emptyset$ but $[\tau] \cap S_f \cap P_n = \emptyset$. Fix such a τ, let $l_{n+1} = |\tau|$ and $z \upharpoonright l_{n+1} = \tau$.

Then $z \in S_f$ is a Δ^1_1-Kurtz random.

So x is not low for Δ^1_1-Kurtz random. □

Lemma 5.5. *If x is low for Δ^1_1-Kurtz random, then x is Δ^1_1-semi-traceable.*

Proof. The proof is a shift of the main result in [7].

Firstly we show that if x low for Δ^1_1-Kurtz test, then x is Δ^1_1-semi-traceable.

Suppose that x is low for Δ^1_1-Kurtz test and $f \leq_h x$. Partition ω into finite intervals $D_{m,k}$ for $0 < k < m$ so that $|D_{m,k}| = 2^{m-k-1}$. Moreover, if $m < m'$, then $\max D_{m,k} < \min D_{m',k'}$ for any $k < m$ and $k' < m'$. Let $n_m = \max\{i \mid i \in D_{m,k} \wedge k < m\}$ for every $m \in \omega$. Note that $\{n_m\}_{m\in\omega}$ is a recursive increasing sequence.

For every function h, let

$$P^h = \{x \in 2^\omega \mid \forall m (x(h \upharpoonright n_m) = 0)\}$$

be a closed null set. Obviously P^f is a $\Delta^1_1(x)$ closed null set. Then there is a Δ^1_1 closed null set $Q \supseteq P^f$. We define a Δ^1_1 function g as follows.

For each $k \in \omega$, let d_k be the least number d so that $|\{\sigma \in 2^d \mid \exists x \in Q(x \succ \sigma)\}| \leq 2^{d-k-1}$. Note that $\{d_k\}_{k\in\omega}$ is a Δ^1_1-sequence. Define

$$Q_k = \{\sigma \mid \sigma \in 2^{d_k} \wedge \exists x \in Q(x \succ \sigma)\}.$$

Then $\{Q_k\}_{k\in\omega}$ is a Δ^1_1 sequence of clopen sets and $|Q_k| \leq 2^{d_k-k-1}$ for each $k < d_k$. Then Greenberg and Miller [7] constructed a finite tree $S \subseteq \omega^{<\omega}$ and a finite sequence $\{S_m\}_{k<m\leq l}$ for some l with the following properties:

(1) $[S] = \{h \in \omega^\omega \mid P^h \subseteq [Q_k]\}$;
(2) $S_m \subseteq S \cap \omega^{n_m}$;
(3) $|S_m| \leq 2^{m-k-1}$;
(4) every leaf of S extends some string in $\bigcup_{k<m\leq l} S_m$.

Moreover, both the finite tree S and sequence can be obtained uniformly from Q_k.

Now for each m with $k < m \leq l$ and $\sigma \in S_m$, we pick a distince $i \in D_{m,k}$ and define $g(i) = \sigma(i)$. For the other undefined $i \in D_{m,k}$, let $g(i) = 0$.

So g is a well-defined Δ^1_1 function.

For each k, $P^f \subseteq Q \subseteq [Q_k]$. So $f \in [S]$. Hence there must be some $i > n_k$ so that $f(i) = g(i)$.

Thus x is Δ^1_1-semi traceable.

Now suppose x is not Δ^1_1-semitraceable. Let $f \leq_h x$ witnesses the property of x. Then P^f is not contained in any Δ^1_1 closed null set. It is shown in [7] that for any σ, assuming that $[\sigma] \cap P^f \neq \emptyset$, $[\sigma] \cap P^f$ is not contained in any Δ^1_1 closed null set. Then by an easy induction, one can construct a Δ^1_1-Kurtz random real in P^f.

So x is not low for Δ^1_1-Kurtz random. □

So we have the following theorem.

Theorem 5.6. *For any real $x \in 2^\omega$, the following are equivalent:*

(1) *x is low for Δ^1_1-Kurtz test;*
(2) *x is low for Δ^1_1-Kurtz randomness;*
(3) *x is Δ^1_1-dominated and -semi-traceable.*

It is unknown whether there exists a nonhyperarithmetic real which is low for Π^1_1-Kurtz random. But we know that lowness for Π^1_1-Kurtz randomness is a stronger notion.

Proposition 5.7. *If x is low for Π^1_1-Kurtz random, then x is low for Δ^1_1-Kurtz random.*

Proof. Suppose that x is low for Π^1_1-Kurtz random, y is Δ^1_1-Kurtz-random and there is a $\Delta^1_1(x)$ closed null set A with $y \in A$. Then by Theorem 2.7, the set

$$B = \bigcup \{C \mid C \text{ is a } \Delta^1_1 \text{ closed null set}\}$$

is a Π^1_1 null set. So $A - B$ is a $\Sigma^1_1(x)$ nonempty set. Thus there must be some real $z \in A - B$ with $\omega_1^z = \omega_1^x = \omega_1^{\mathrm{CK}}$. Since $z \notin B$, z is Δ^1_1-Kurtz random. So by Proposition 3.3, z is Π^1_1-Kurtz random. This contradicts to that x is low for Π^1_1-Kurtz random. □

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