MA 3205 – Set Theory – Homework for Week 6

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 12.09.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 7.4. Let X be finite. Prove that the set of all functions from X to X is finite.

Exercise 7.10. Prove that V_ω satisfies the following property: if $x \in V_\omega$ and $y \subseteq x$ or $y \in x$, then $y \in V_\omega$. Show that $\mathbb{N}$ does not satisfy this property, but that some proper infinite subclass of V_ω does.

Exercise 7.11. Determine all $x_0 \in V$ which satisfy that there are no $x_1, x_2, x_3, x_4 \in V$ with $x_1 \in x_0, x_2 \in x_1, x_3 \in x_2, x_4 \in x_3$. The set $\{\{\emptyset\}\}$ is such an x_0, although $x_1 = \{\emptyset\}$ and $x_2 = \emptyset$ exist, x_3 and x_4 do not exist. The set $\{\{\emptyset, \{\{\emptyset\}\}\}\}$ does not qualify.

Exercise 8.9. Let $D = \{f : \mathbb{N} \to \mathbb{N} \mid \forall n\,(f(S(n)) \leq f(n))\}$ be the set of all decreasing functions. Show that D is countable.

Exercise 8.12. Let the elements of A be ordered such that the symbol $\emptyset$ comes first and the comma comes last. Let $<_{ll}$ be the length-lexicographic ordering on the set A^* of all strings over A. Now let $f : V_\omega \to A^*$ map every set x in V_ω to first expression describing x. Then $\emptyset <_{ll} \{\}$, thus the symbol "$\{\}$" is never used to describe the empty set; this convention is also applied in this text. Check which of the following facts are true:

1. the length of $f(x)$ is odd for every $x \in V_\omega$;
2. if $f(x) = \{y\}$ then $f(S(x)) = \{y, \{y\}\}$;
3. $f(2) = \{\emptyset, \{\emptyset\}\}$.

Furthermore, find a formula giving the length of $f(n)$ for every $n \in \mathbb{N}$ and determine which of the following numbers is the length of $f(10)$: 42, 100, 1000, 1001, 1022, 1023, 1024, 2047, 4096, 256^2, $10^{10} - 1$, 2^{256}, 1023^{1023}.

If the length of $f(x)$ is n and $f(y)$ is m, what is the length of $f((x, y))$ for the ordered pair (x, y)?

Exercise 8.13. Let $\mathbb{A}$ be the set of algebraic real numbers, that is, the set of all $r \in \mathbb{R}$ for which there are $n \in \mathbb{N}$ and $z_0, z_1, \ldots, z_n \in \mathbb{Z}$ such that $z_n \neq 0$ and $z_0 + z_1 r + z_2 r^2 + \ldots + z_n r^n = 0$. Note that such a polynomial of degree n can have up to n places r which are mapped to 0. Show that $\mathbb{A}$ is countable by giving a one-to-one mapping from $\mathbb{A}$ into $\mathbb{N}$.