MA 3205 – Set Theory – Homework for Week 13

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Homework. The homework follows the lecture notes. Below the list of the homeworks for the tutorials from 14.11.2006 onwards. It is not mandatory to hand in homework; but it is recommended to solve the questions by yourself before going to the tutorials. Homework can be corrected on request.

Exercise 18.3. Show that the standard representation can be defined in set-theory: First define a representation for the set $A = \mathbb{Z} \cup \{sign\}$. Then look at the class of all functions $r : A \to \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, +, -\}$, call it B; the decimal point could be placed between $r(0)$ and $r(-1)$ and need not to be represented explicitly.

Define which elements of B represent real numbers and get $\mathbb{R}$ by comprehension, state the property explicitly. For this and further definitions, integer constants, integer addition and $<$ on the integers can be used to in order to deal with positions of digits. The selection should be made such that r represents $\sum_{z \in \mathbb{Z}} r(z) \cdot 10^z$ in the case that $r(sign)$ is $+$ and $-\sum_{z \in \mathbb{Z}} r(z) \cdot 10^z$ in the case that $r(sign)$ is $-$. Make sure that every real occurs in the representation exactly once. For example, fix the sign of 0 to either $+$ or $-$.

This representation has the disadvantage that $\mathbb{N} \not\subseteq \mathbb{R}$. So one distinguishes as in many programming languages like FORTRAN between the natural number 2 and the real number 2.0. Nevertheless, there is a one-to-one mapping $f : \mathbb{N} \to \mathbb{R}$ which maps every natural number to its representative in $\mathbb{R}$. f can be defined inductively using a $g : \mathbb{R} \to \mathbb{R}$ such that $f(S(n)) = g(f(n))$ for all natural numbers n. Give two properties $nat, succ$ such that $nat(r)$ is true iff r is in the range of f and $succ(r, q)$ is true if $nat(r) \wedge nat(q) \wedge q = g(r)$.

Exercise 18.4. In the early days of computing, integers were represented by bytes, more precisely, they were limited to the numbers -128 up to 127. The negative numbers started with a 1 and the positive (including 0) with a 0. So one had that $-128 = 10000000$, $-127 = 10000001$, $\ldots$, $-2 = 11111110$, $-1 = 11111111$, $0 = 00000000$, $1 = 00000001$, $2 = 00000010$, $\ldots$, $127 = 01111111$. This exercise shows how to transfer this idea to the representation of the reals. Consider the following set WS representing the reals Without Sign:

$$\begin{aligned} WS = \{r : \mathbb{Z} \to \{0, 1\} \quad | & \quad \forall n \in \mathbb{Z}\, \exists m < n\, (r(m) = 0) \\ \wedge & \quad \exists n \in \mathbb{Z}\, \forall m > n\, (r(m) = r(m-1))\} \end{aligned}$$

One can define on WS an addition $+$. Let $(r + q)(n) = 1$ if one of the following three conditions holds:

1. $r(n) = q(n)$ and $r(k) \neq q(k)$ for all $k < n$;

2. $r(n) = q(n)$ and there is $m < n$ such that $r(m) = 1$ and $q(m) = 1$ and $r(k) \neq q(k)$ for all k with $m < k < n$;

3. $r(n) \neq q(n)$ and there is an $m < n$ such that $r(n) = 0$ and $q(n) = 0$ and $r(k) \neq q(k)$ for all k with $m < k < n$.

Let $(r + q)(n) = 0$ otherwise. From $+$ one can define an ordering $<$ on WS by

$$r < q \Leftrightarrow \exists s \in WS\,(q = r + s \wedge \exists n \in \mathbb{Z}\,\forall m > n\,(s(m) = 0)).$$

Define an order-isomorphism F satisfying $\forall r, q \in WS\,(F(r + q) = F(r) + F(q))$ from $(WS, +, <)$ to $(\mathbb{R}, +, <)$.

Exercise 18.9. Verify that Hausdorff's axioms are true for the set $\mathbb{R}$. That is, verify that $(\mathbb{R}, \{A \subseteq \mathbb{R} \mid A \text{ is open}\})$ is a Hausdorff space.

Exercise 18.10. Let α be any ordinal. Define a topology on α by saying that β is open iff β is an ordinal and $\beta \subseteq \alpha$. Verify that the first three axioms of Hausdorff are satisfied, but not the last fourth one.

Exercise 18.11. Find a topology on the set of ordinals up to a given ordinal α which satisfies the Axioms of Hausdorff and in which an ordinal $\beta \in \alpha$ is isolated iff it is either a successor ordinal or 0.