MA 3205 – Set Theory – Homework due Week 6

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Homework. The homework follows the lecture notes. You have to hand in one starred homework in Weeks 3–6, in Weeks 7–9 and in Weeks 10–13. Further homework can be checked on request. Homework to be marked should be handed in after the lecture on Tuesday of the week when the homework is due.

Note that the first midterm test is on Tuesday 15 September 2009 in the first half of the lecture. Please revise the first 8 chapters of the lecture notes for this midterm test.

Exercise 7.4. Let X be finite. Prove that the set of all functions from X to X is finite.

Exercise 7.10. Prove that V_ω satisfies the following property: if $x \in V_\omega$ and $y \subseteq x$ or $y \in x$, then $y \in V_\omega$. Show that $\mathbb{N}$ does not satisfy this property, but that some proper infinite subset of V_ω does.

Exercise 7.11. Determine all $x_0 \in V$ which satisfy that there are no $x_1, x_2, x_3, x_4 \in V$ with $x_1 \in x_0, x_2 \in x_1, x_3 \in x_2, x_4 \in x_3$. The set $\{\{\emptyset\}\}$ is such an x_0, although $x_1 = \{\emptyset\}$ and $x_2 = \emptyset$ exist, x_3 and x_4 do not exist. The set $\{\{\emptyset, \{\{\emptyset\}\}\}\}$ does not qualify.

Exercise 8.9*. Let $D = \{f : \mathbb{N} \to \mathbb{N} \mid \forall n\,(f(S(n)) \leq f(n))\}$ be the set of all decreasing functions. Show that D is countable.

Exercise 8.12. Let $\mathbb{A}$ be the set of algebraic real numbers, that is, the set of all $r \in \mathbb{R}$ for which there are $n \in \mathbb{N}$ and $z_0, z_1, \ldots, z_n \in \mathbb{Z}$ such that $z_n \neq 0$ and $z_0 + z_1 r + z_2 r^2 + \ldots + z_n r^n = 0$. Note that such a polynomial of degree n can have up to n places r which are mapped to 0. Show that $\mathbb{A}$ is countable by giving a one-to-one mapping from $\mathbb{A}$ into $\mathbb{N}$.

Exercise 8.14. Call a set A *hereditarily at most countable* iff for every $B \in \mathcal{TC}(A)$ it holds that B is at most countable. For example, $\mathbb{N}$ and V_ω are hereditarily at most countable. Now assume that X, Y are hereditarily at most countable. Show that the following sets are hereditarily at most countable as well: $X \cup Y$, $X \cap Y$, $X - Y$ and $\{X\}$. Furthermore, show that whenever $\mathcal{TC}(Z)$ is at most countable then Z is hereditarily at most countable.