MA 3205 – Set Theory – Homework due Week 8

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Homework. The homework follows the lecture notes. You have to hand in one starred homework in Weeks 3–6, in Weeks 7–9 and in Weeks 10–13. Further homework can be checked on request. Homework to be marked should be handed in before the tutorial (as there is self-study in place of lectures in Week 8).

Exercise 10.6*. Let $(A, <)$ be a linearly ordered set and $B = A^{\mathbb{N}}$. Define

$$f <_{lex} g \Leftrightarrow \exists k \in \mathbb{N}\, (f \restriction k = g \restriction k \wedge f(k) < g(k)).$$

Here $f \restriction k$ is the restriction of f to k: $f \restriction k = \{(x, f(x)) \mid x \in k\}$.

Furthermore, let $C = A^*$. The lexicographic ordering on A^* is defined such that either the smaller word is shorter than the longer one or that the first word has a member of A strictly before the second one at the first position where they differ. That is, if m is the domain of f and n the domain of g, then

$$f <_{lex} g \Leftrightarrow \exists k \in S(m) \cap n\, ((f \restriction k = g \restriction k) \wedge (k = m \vee (k < m \wedge f(k) < g(k)))).$$

Show that $(B, <_{lex})$ and $(C, <_{lex})$ are linearly ordered sets. Assuming that $A = \{0, 1, 2, \ldots, 9\}$ with the usual ordering, put the following elements of C into lexicographic order: 120, 88, 512, 500, 5, 121, 900, 0, 76543210, 15, 7, 007, 00.

Exercise 10.10. Determine which of the following subsets of the real numbers $\mathbb{R}$ have a lower and upper bound. If so, determine the infimum and supremum and check whether these are even the least and greatest element of these sets.

1. $A = \{a \in \mathbb{R} \mid \exists b \in \mathbb{R}\, (a^2 + b^2 = 1)\}$;
2. $B = \{b \in \mathbb{R} \mid b^3 - 4 \cdot b < 0\}$;
3. $C = \{c \in \mathbb{R} \mid \sin(c) > 0\}$;
4. $D = \{d \in \mathbb{R} \mid d^2 < \pi^3\}$;
5. $E = \{e \in \mathbb{R} \mid \sin(\frac{\pi}{2} \cdot e) = \frac{e}{101}\}$.

Exercise 10.13. Consider the ordering $\sqsubset$ given by

$$\begin{aligned} (m, n) \sqsubset (i, j) \quad &\Leftrightarrow \quad (m < i) \\ &\vee \quad (m = i \wedge m \text{ is even} \wedge n < j) \\ &\vee \quad (m = i \wedge m \text{ is odd} \wedge n > j) \end{aligned}$$

on $A = \{0,1,2,3,4,5\} \times \mathbb{N}$. Construct an order-preserving mapping from $(\mathbb{Z},<)$ into $(A,\sqsubset)$ where $<$ is the natural ordering of $\mathbb{Z}$.

The set $(\mathbb{Z},<)$ there are nontrivial isomorphisms onto itself, that is, isomorphism different from the identity. For example, $z \mapsto z+8$. Does $(A,\sqsubset)$ also have nontrivial isomorphisms onto itself? If so, is there any element which is always mapped to itself?

Exercise 10.22. Show that in a complete ordered set $(A,<)$ every nonempty subset which is bounded from below has an infimum in A.

Tasks for Week 8. In Week 8 there are no physical lectures but only tutorials, seminars and so on. This is to practice E-Learning and learning from books, manuscripts on the internet and so on. You find the manuscript of the lecture for set theory on the page

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http://www.comp.nus.edu.sg/~fstephan/settheory.html
http://www.comp.nus.edu.sg/~fstephan/settheory.ps
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and the first of these two pages also has links to homework and other information.

In Week 8, please revise Sections 9 to 13 and study by yourself Sections 14 and 15 (pages 62 to 69 in the lecture notes). Section 14 is about the rank of sets; the rank of a set x is an ordinal to measure how complicated the structure of the element-relation on the transitive closure of x is. It is a quite important notion in set theory. Section 15 deals with ordinals and the way how to add them and to do other operations with them. Operations on ordinals and cardinals are quite important in set theory and therefore it is good to learn it.

The lecture on Friday 02.10.2009 will be taught by Feng Qi and the tutorials on Wednesday 07.10.2009 will be taught given by Yang Yue, as Frank Stephan is on academic leave to attend the conference ALT 2009 in Porto.