# MA 3205 – Set Theory – Homework due Week 13

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**Homework.** The homework follows the lecture notes. You have to hand in at least three starred homeworks throughout the semester. Further homework can be checked on request. Homework to be marked should be handed in after the lecture on Tuesday of the week when the homework is due.

**Exercise 17.12.** Consider the following partial ordering given on the set $\mathbb{N}^{\mathbb{N}}$ of all functions from $\mathbb{N}$ to $\mathbb{N}$:

$$f \sqsubset g \Leftrightarrow \exists n \, \forall m > n \, (f(m) < g(m)).$$

This partial ordering only shares some but not all of the properties of the ordering $<_{lin}$ considered in the lecture. In order to see this, show the following two properties:

- For countably many functions $f_0, f_1, \ldots$ there is a function $g$ such that $\forall n \in \mathbb{N} \, (f_n \sqsubset g)$;
- There are uncountably many $f$ below the exponential function $n \mapsto 2^n$. Namely for every $A \subseteq \mathbb{N}$ the function $c_A : n \mapsto \sum_{m \in n} 2^{n-m-1} \cdot A(m)$ is below the exponential function.

Note that $c_A \sqsubset c_B \Leftrightarrow A <_{lex} B$. Thus there is an uncountable linearly ordered set of functions below the exponential function.

**Exercise 17.13*.** Use the Axiom of Choice to prove the following: If $|A| = \aleph_1$ and every $B \in A$ satisfies $|B| \leq \aleph_1$ then $|\bigcup A| \leq \aleph_1$.

**Exercise 18.9.** Verify that Hausdorff's axioms are true for the set $\mathbb{R}$. That is, verify that $(\mathbb{R}, \{A \subseteq \mathbb{R} \mid A \text{ is open}\})$ is a Hausdorff space.

**Exercise 18.10.** Let $\alpha$ be any ordinal. Define a topology on $\alpha$ by saying that $\beta$ is open iff $\beta$ is an ordinal and $\beta \subseteq \alpha$. Verify that the first three axioms of Hausdorff are satisfied, but not the last fourth one.

**Exercise 18.11.** Find a topology on the set of ordinals up to a given ordinal $\alpha$ which satisfies the Axioms of Hausdorff and in which an ordinal $\beta \in \alpha$ is isolated iff it is either a successor ordinal or 0.