MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 8, Tuesday.

Frank Stephan. Departments of Computer Science and Mathematics, National University of Singapore, 10 Lower Kent Ridge Road, S17#07-04.

Email fstephan@comp.nus.edu.sg
Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Telephone office 65162759
Office hours Thursday 14.00-15.00h

There will be a starred homework next week.

8.1 Birkhoff rules. Adjust the axioms of the Birkhoff rules to incorporate binary associative operations like $\circ$ as follows (where the fourth rule is given for a function f of arity 3 but applies for functions of all arities):

$$\frac{\emptyset}{t = t} \qquad \frac{s = t}{t = s}$$

$$\frac{s = t, t = r}{s = r} \qquad \frac{s_1 = t_1, s_2 = t_2, s_3 = t_3}{f(s_1, s_2, s_3) = f(t_1, t_2, t_3)}$$

$$\frac{\emptyset}{(r \circ s) \circ t = r \circ s \circ t} \qquad \frac{\emptyset}{r \circ (s \circ t) = r \circ s \circ t}$$

$$\frac{s = t}{s \circ r = t \circ r} \qquad \frac{s = t}{r \circ s = r \circ t}$$

$$\frac{s = t}{q \circ s \circ r = q \circ t \circ r} \qquad \frac{s = t}{s^\sigma = t^\sigma} \text{ for all global substitutions } \sigma$$

In the following let x, y, z be variables and d, e be constants.
(a) Assume now that the additional axioms $x \circ x \circ x \circ x \circ x \circ x = e$, $e \circ x = x$, $x \circ e = x$ and $d \circ d \circ d \circ d \circ d = e$ are given. Prove that $d = e$.
(b) Assume now that the additional axioms $x \circ x \circ x \circ x \circ x \circ x = e$, $e \circ x = x$, $x \circ e = x$ and $d \circ d \circ d = e$ are given. Prove that one can neither derive $d = e$ nor derive $x \circ y = y \circ x$ by producing a model in which these two formulas are false.
(c) Assume now that the additional axioms $x \circ d = d$, $x \circ e = x$, $e \circ x = x$, $f(x \circ y) = f(y) \circ f(x)$, $f(f(x)) = x$ and $x \circ x \circ f(x) = x$ are given, where f is a unary function symbol. Derive $f(x) \circ x \circ x = x$. Furthermore, provide a model which shows that $f(x) = x$ cannot be derived.

8.2 Second Order Logic. Second order language permits to quantify over sets. Using that the subsets of the natural numbers are uncountable, give a set X of formulas such that every second order model of X is uncountable. X should of course be satisfiable, that is, have at least one model.