MA 5219 - Logic and Foundations of Mathematics 1
Homework due in Week 10, Tuesday.

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Hand in each starred homework; 1 mark per homework (if it is correct), up to 10 marks in total for homework.

12.1* First-Order Logic. Which of the following sets of formulas in first-order logic are recursive (= decidable), recursively enumerable but not recursive or even not recursively enumerable? The underlying logical language is that of first-order logic with two operations $+, \cdot$ permitted on the members of the model and three constants 0, 1, 2. Furthermore, the logical language uses equality.
(a) $A = \{\phi : \phi$ is a tautology, that is, ϕ is true in all models$\}$;
(b) $B = \{\phi : \phi$ is not true in any model$\}$;
(c) $C = \{\phi : \phi$ is true in some but not all models$\}$.
Note that formally, one has to say that for a set D of formulas, D is recursive iff $\{gn(\phi) : \phi \in D\}$ is recursive where $gn(\phi)$ is the Gödel number of ϕ in some numbering system (where the numbers need not to coincide with the members of models of ϕ but are just members of $\mathbb{N}$). Similarly one defines when a set D of formulas is recursively enumerable.

12.2 Isomorphy and equivalence of models. Assume that a logic language only consists of formulas using equality, constants, variables, Boolean combinations of equalities and quantified open formulas of such Boolean combinations. Furthermore, let X be the set of all $c_i \neq c_j$, $c_i \neq x_j$ and $x_i \neq x_j$ for all pairwise distinct i, j. Which of the following statements are true:
(a) Any two models of X are isomorphic;
(b) Any two models of X are elementary equivalent but they might not be isomorphic;
(c) There is a formula α which is true in some but not all models of X.

12.3 Decidability. Given the set X of formulas from 12.2, can one decide whether $X \models \alpha$ for any formula α? If one cannot decide that set, can one enumerate the set of all formulas α which satisfy $X \models \alpha$?

12.4 Primitive Recursive Functions. Consider the function $f(x) = 2^{2^{2^x}}$ and g given by $g(x, y) = 1$ if $2^{f(x)} > y$ and $g(x, y) = 0$ if $2^{f(x)} \leq y$. Are the functions f and g primitive-recursive?