MA 5219 - Logic and Foundations of Mathematics 1
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogic.html`
Homework for Week 9, not needed to hand in.

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Hand in each homework which you want to be checked; 1 mark per each correct starred homework; up to 10 marks in total for homework - there will be more than 10 starred homeworks, so you have several chances to try.

9.1 Axiomatisable Theories. Let the logical language L contain variable symbols, constants $0, 1, 2$, the addition operation $+$ and the multiplication operation $\cdot$ as well as equality and existential and universal quantifiers. Which of the following theories are axiomatisable? In the following, $\mathbb{F}_3$ and $\mathbb{F}_9$ are the fields with 3 and 9 elements, respectively, where 0 is the neutral additive element, 1 the neutral multiplicative element and $2 = 1 + 1$.

1. $\{\alpha \in L : \mathbb{F}_3 \models \alpha\}$;
2. $\{\alpha \in L : \mathbb{F}_9 \models \alpha\}$;
3. $\{\alpha \in L : \mathbb{F}_3 \models \alpha \wedge \mathbb{F}_9 \models \alpha\}$;
4. $\{\alpha \in L :$ there is an $n \in \mathbb{N}$ such that all fields with at least n elements make α true$\}$.

Give a short reason why the corresponding theories are or are not axiomatisable.

9.2 Axiomatisable Henkin sets. Assume that X is a recursively enumerable Henkin set and that also the set of constants C is recursively enumerable. What can be said about $T = \{\alpha : X \vdash \alpha\}$? Is T (a) decidable or (b) recursively enumerable and undecidable or (c) not recursively enumerable? Explain your answer.

9.3 Number of models. (a) Make an axiomatisable theory T such that T has exactly 5 models (up to isomorphism). It is sufficient to give the axiomatisation X which generates T. For this, the logical language should be defined accordingly and it should be said which symbols are used (beside the logical ones).
(b) Make an axiomatisable theory T which is generated by a finite set X and which has infinitely many countable models. The theory should use only one function symbol f which can be defined accordingly; the equality $=$ can be used as well (and $=$ has the usual meaning). Here a model is countable iff it has as many elements as $\mathbb{N}$.