MA 5220 – Set Theory – Homework file

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Homework 2.1. Georg Cantor made the observation that there exist different types of infinite sets and that there are more real numbers than natural numbers. Provide a proof by diagonalisation that given a countable list of real numbers, there is one differing from all of them, by completing the following proof. Given a sequence (b_k) of positive rational numbers with $b_i > \sum_{j>i} b_j$ for all $i \in \mathbb{N}$. Furthermore, assume that $a_0, a_1, \ldots$ is a countable list of real numbers. Now let $c_0 = 0$ and choose

$$c_{k+1} = \begin{cases} c_k + b_k & \text{if } a_k \leq c_k; \\ c_k - b_k & \text{if } a_k > c_k. \end{cases}$$

Provide an explicit formula for all b_i satisfying the above requirement and also prove that $c_\infty = lim_{k \to \infty} c_k$ exists by showing that the corresponding sequence converges in the real numbers and that c_∞ differs from all a_j.

Homework 2.2. Another way to prove this more direct is the following: Assume that (a_0, b_0) is an open interval of real numbers and $x_0, x_1, \ldots$ is an infinite sequence of real numbers inside this interval. Now do the following iteration for $n = 0, 1, \ldots$: Choose an open interval (a_{n+1}, b_{n+1}) properly inside (a_n, b_n) which does not contain x_n and which satisfies that a_{n+1}, b_{n+1} are both properly away from each of a_n, b_n, x_n by at least $(b_n - a_n)/10$. Provide an explicit algorithm doing this (with sufficient case distrinctions) and prove that the intersection of all these intervals contains exactly one real number d which is different from all x_n. (For notation, (a, b) is usually a pair of sets; if it is an open interval of real numbers, this is explicitly mentioned as in this homework.)

Homework 2.3. Bernard Bolzano wrote in 1848 a book “Paradoxien des Unendlichen” (Paradoxes of the infinite) which was published in 1851 three years after his death. In this book he showed that finite and infinite sets behave differently. For example, there are infinite sets A, B such that $A \subset B$ (proper subset) but A, B have a bijection f between them. Cantor defined in 1874 that sets have the same cardinality if and only if there is a bijection between them. Prove that for any two open intervals (a, b), (c, d) in the real numbers (which might have end points $-\infty$ or $+\infty$ and which always satisfy $a < b$ and $c < d$), there is always a strictly increasing bijections from (a, b) to (c, d). Provide explicit formulas and make the necessary case distinctions, the formulas should be as easy as possible.

Homework 2.4. Assume that $A \subset B$ and $A = \{x_1, x_2, \ldots, x_n\}$ for some natural number n. Prove that B has at least $n+1$ elements and that there is no injective mapping from B to A. Do this by induction over n. Start the base-case with $n = 1$; that a nonempty set cannot be mapped to an empty one is trivial.

Homework 2.6. Construct a bijection from the nonzero reals to the open interval $(0, 1)$ which is noncontinuous only at the natural numbers but not at any other point.

Homework 2.7. Kunen's book states the axiom of pairs as follows in ZFC:

$$\forall x, y \in V\, \exists z \in V\, [x \in z \wedge y \in z].$$

Prove using ZF the following more intuitive version of the axiom:

$$\forall x, y \in V\, \exists z \in V\, [z = \{x, y\}].$$

Homework 2.8. Kunen's book states the axiom of union as follows in ZFC:

$$\forall x \in V \exists y \in V \forall z \in x \forall u \in z\, [u \in y].$$

Prove using ZF that the following more intuitive versions of these axioms hold:

$$\forall x \in V \exists y \in V \forall u \in V\, [u \in y \leftrightarrow \exists z \in x\, [u \in z]\,].$$

Homework 2.9. An at most countable set is a set A such that there is a surjective function f from $\mathbb{N}$ to A. Prove that the set of all finite binary strings is at most countable.

Homwork 2.10. Let A be the set of all real numbers strictly above 1 which do not have the digits $0, 1, 8, 9$ in their decimal representation. Here the number $2.333\ldots$ should not be written as $02.333\ldots$, so leading zeroes do not exclude a number from A. Answer the following questions and prove them:

(a) Is the set nowhere dense? That is, does the set satisfy that for each open interval (a, b) there is a subinterval (c, d) which is disjoint to A?

(b) Is the set A finite or countable or uncountable? Prove the answer.

Homework 2.11. Consider $B = \bigcup A$ for some set A. How many elements must A have at least so that it is guaranteed that B has at least $n+1$ elements? Provide a formula in dependence of n and explain how it is computed.

Homework 2.12. Assume that $C = \bigcup A$. Furthermore, assume that the symmetric difference of two elements $D, E \in A$ has cardinality 4 or less whenever $D \neq E$. Assume that $n = |\bigcap A|$ and $m = |A|$. Provide good lower and upper bounds for $|\bigcup A|$ using the parameters m, n; it is possible to use the $O(...)$ notation and try to get something like $f(m, n) + O(g(m))$ for some functions f, g. Prove the correctness of the bounds found, for the lower bound one calls it $f(m, n) + \Omega(g(m))$, as one wants to say that $|\bigcup A|$ is at least as big as $f(m, n)$ plus some positive constant factor times $g(m)$.

Homework 3.1. Let $S(x)$ denote the successor function $x \mapsto x \cup \{x\}$. Let $\varphi(n)$ be the following predicate: $\forall y\,[(\emptyset \in y \wedge \forall z \in y\,[S(z) \in y]) \rightarrow n \in y]$. This predicate defines that n is a natural number coded in set theory, as natural numbers are defined as $0 = \emptyset$ and $S(x) = x \cup \{x\}$ defines the successor number of x. Show the following: If a set x satisfies the axiom of Infinity, that is, contains the emptyset and is closed under S then $\omega = \{n \in x : \varphi(n)\}$ is defined as well and represents the set of natural numbers.

Homework 3.2. Consider two sets x, y which both satisfy the following predicate ψ: $\psi(x)$ denotes $\emptyset \in x \wedge \forall n \in x \exists m \in x\,[n = S(m)] \wedge \forall n \in x\,[S(n) \in x]$. Use the axioms of Foundation and Extensionality and the closure of sets under finite unions, intersections and set difference to show that $x = y$.

Homework 3.3. Consider the partial order $\in$ on sets and all chain notions in this homework relate to $\in$. A set x is called a chain if $\forall m, n \in x\,[m = n \vee m \in n \vee n \in m]$. For example, $\{x, \{x\}\}$ is a two-element chain. An ascending chain is a set x such that for all $n \in x$ there is an $m \in x$ with $n \in m$. A descending chain is a set x such that for all $n \in x$ there is an $m \in x$ with $m \in n$. Prove that it follows from ZF that there are sets which are ascending chains but no sets which are descending chains. Which axioms are needed to prove that?

Homework 3.4. Let x be a set of three elements. List out all possibilities how a directed graph of these three elements of x with respect to the relation $\in$ can look like.

Homework 3.5. Let $trcl(x)$ be the set of x, all elements of x, all elements of elements of x and so on. Now define the predicate φ as

$$\varphi(x) \leftrightarrow \forall y \in trcl(x)\,[y \text{ has at most one element}].$$

Is there a set x satisfying that $\varphi(x)$ holds and that $trcl(x)$ is infinite?

Homework 3.6. Let φ as in Homework 3.5 and $W = \{x : \varphi(x)\}$? Is W a set? If W is a set, is it empty, finite, countable or uncountable?

Homework 3.7. The rank of a set is given by $rank(\emptyset) = 0$ and $rank(A) = sup\{rank(B)+1 : B \in A\}$. Consider the alternative definition $rank'(A) = sup\{rank(B) : B \in A\} + 1$. Provide an example where $rank(A) \neq rank'(A)$. Note that the rank can be an ordinal which is not an integer.

Homework 3.8. Recall the rank from Homework 3.7. What are the possible ranks of a set which exactly one infinite set in its transitive closure?

Homework 3.9. Recall the rank from Homework 3.7. What are the possible ranks of a set which exactly three infinite sets in its transitive closure?

Homework 3.10. Kuratowski's definition of an ordered pair (x, y) is $\{\{x\}, \{x, y\}\}$. Analogously one can define an ordered triple as follows: $(x, y, z) = \{\{x\}, \{x, y\}, \{x, y, z\}\}$. Is now $(x, x, y) \neq (x, y, y)$ whenever $x \neq y$?

Homework 3.11. Check whether the tuple $x = (x_1, x_2, \ldots, x_n)$ is well-represented by the set $T_x = \{(1, x_1), (2, x_2), \ldots, (n, x_n)\}$, that is, is the mapping from the set of all n-tuples x with $n \geq 2$ to the corresponding T_x is one–one? State YES or NO and prove the answer.

Homework 3.12. Assume that $f : A \to B$ is a function and C a proper subset of A. Let $g : C \to B$ be the restriction of f to the domain C, that is, for all $x \in C$, $g(x) = f(x)$ and for all $x \notin C$, $g(x)$ is undefined. Which of the properties "injective", "surjective" and "bijective" are inherited from f to g. Explain the anwer.