MA 4207 - Mathematical Logic
Course-Webpage `http://www.comp.nus.edu.sg/~fstephan/mathlogicug.html`
This file contains all homeworks for MA4207.

Assignments for Week 2
Homework 2.1
Cantor's function $x, y \mapsto (x+y)\cdot(x+y+1)/2+y$ is a bijection from $\mathbb{N}\times\mathbb{N}$ onto $\mathbb{N}$. Construct a bijection from $\mathbb{Z}\times\mathbb{Z}$ onto $\mathbb{Z}$.

Homework 2.2
Prove that there is no set X such that its powerset $\{Y : Y \subseteq X\}$ has five elements.

Homework 2.3
Show that a power set has always more elements than the given set, that is, fill out the missing details at the following proof-sketch. Recall that $\text{Card}(A) \leq \text{Card}(B)$ iff there is a one-one function from A to B and show that $\text{Card}(\mathbb{P}(A)) \not\leq \text{Card}(A)$.

Proof-Sketch: The $\emptyset$ has 0 and $\mathbb{P}(\emptyset)$ has one element, namely $\emptyset$, hence one cannot have a one-one mapping from $\mathbb{P}(\emptyset)$ to $\emptyset$. Now assume that A is not empty and $f : A \to \mathbb{P}(A)$ is a function. Show that there is a set $B \subseteq A$ which is not in the range of f. Then consider any function $g : \mathbb{P}(A) \to A$ and prove that this function cannot be one-one, as otherwise a surjective f from A to $\mathbb{P}(A)$ would exist. Hence $\text{Card}(\mathbb{P}(A)) \not\leq \text{Card}(A)$.

Homework 2.4
Use Homework 2.3 to prove that there is no set X such that $\text{Card}(\mathbb{P}(X)) = \aleph_0$. The fact that every set X is either finite or satisfies $\aleph_0 \leq \text{Card}(X)$ can be used in the proof.

Homework 2.5
Determine the cardinality of the set $\{X \subseteq Y : \text{Card}(X) = 3\}$ for each of the following sets Y: (a) $Y = \{0,1\}$; (b) $Y = \{0,1,2,3,4\}$; (c) $Y = \{0,1,2,3,4,5,6,7,8,9,10\}$; (d) $Y = \mathbb{N}$.

Homework 2.6
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is a power of $2\}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.

Homework 2.7
A set X is well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element, that is, an element $z \in Y$ with $z < u$ for all $u \in Y - \{z\}$. A set X is strongly well-ordered iff every non-empty subset $Y \subseteq X$ contains a minimal element and a maximal element. Determine for which cardinals κ there is a strongly well-ordered set X with $\text{Card}(X) = \kappa$.

Homework 2.8
Consider the following sets X and Y and show that they satisfy $\text{Card}(X) = \text{Card}(Y)$ by constructing explicitly a bijection g from X to Y. Here

$$X = \{f : \text{dom}(f) = \mathbb{N} \text{ and } \text{ran}(f) \subseteq \mathbb{N}\} \text{ and}$$

$$Y = \{f : \text{dom}(f) = \mathbb{N} \text{ and } \text{ran}(f) \subseteq \mathbb{N} \text{ and } \forall n\,[f(n) < f(n+1)]\}$$

and one has to define which function $g(f)$ is for each $f \in X$.

Homework 2.9
Use the Theorem of Schröder and Bernstein to show that the following two sets V and W have the same cardinality:

$$V = \{f : \text{dom}(f) = \mathbb{N} \text{ and } \text{ran}(f) = \mathbb{N}\} \text{ and}$$
$$W = \mathbb{P}(\mathbb{N}).$$

Homework 2.10
Which of the following sets is a function is a valid pair-function on the natural numbers: Note that a valid pair-function must satisfy for all $a, b, c, d \in \mathbb{N}$ that $\langle a, b\rangle = \langle c, d\rangle$ iff $a = c$ and $b = d$. Furthermore, say which of these functions is a bijection from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$. The candidate functions are the following ones:

1. $a, b \mapsto 2^a \cdot 3^b$;
2. $a, b \mapsto (a + b)^3 + b^3$;
3. $a, b \mapsto (a + b)^2 - (a - b)^2$;
4. $a, b \mapsto (0 + 1 + 2 + \ldots + (a + b)) + a$;
5. $a, b \mapsto 2^{a \cdot b} + b$.
6. $a, b \mapsto 2^{(a+1)\cdot(b+1)} + b$.

Homework 2.11
Consider the set $\mathbb{D} = \{q : q$ is a rational number with $0 \leq q < 1$ such that its denominator is of the form $2^i \cdot 3^j$ for some $i, j \in \mathbb{N}\}$. Construct a bijection from $\mathbb{N}$ to $\mathbb{D}$ explicitly.

Homework 2.12
Let $f_0, f_1, \ldots$ be a sequence of functions from $\mathbb{N}$ to $\mathbb{N}$. Construct a function g which dominates all these functions, that is, which satisfies for each f_k that $\exists x \in \mathbb{N}\, \forall y \in \mathbb{N}$ $[f_k(x + y) \leq g(x + y)]$.

Homework 2.13
Can a corresponding result also be proven for the set $\mathbb{M}$ of all finite and countable ordinals? Here note that any countable set of members of $\mathbb{M}$ has an upper bound in $\mathbb{M}$. So the question is now whether for a sequence f_k of functions from $\mathbb{M}$ to $\mathbb{M}$ with k running over all members of $\mathbb{M}$, whether there is a function g from $\mathbb{M}$ to $\mathbb{M}$ such that $g(x) \geq f_k(x)$ for all $k \leq x$. Such a function would dominate all f_k.

Homework 2.14
Recall that two ordinals x, y with $x < y$ satisfy that $x \in y$ when y is viewed as a set. Furthermore, note that for all sets x, y, if $x \in y$ then $y \notin x$. Use these to prove that

ordinals are linearly ordered and that the ordering $<$ on the ordinals coincides with the element-relation $\in$.

Homework 2.15
An ordered set $(X, <)$ is called well-ordered iff every nonempty subset of it has a mimimum. Use Zorn's Lemma to prove that every set X of ordinals is well-ordered. For this, consider the mapping $f : x \mapsto \{y \in X : y \geq x\}$ and show that for each nonempty subset Z of X the range Y of f on Z has a maximal element $f(u)$ for some $u \in Z$. Then show that this u is a mimimum of Z.

Homework 2.16
Let $\mathbb{M}$ be the set of at most countable ordinals from Homework 2.13. Consider all functions of the form $f : \mathbb{M} \to \mathbb{M}$ with $x < y \to f(y) \leq f(x)$ for all $x, y \in \mathbb{M}$. What is the cardinal of this function set. For this result, one can use that in well-ordered sets, every strictly descending sequence is finite and that all sets of ordinals are well-ordered by the natural ordering on the ordinals.

Homework 2.17
Let $\mathbb{H}$ be all ordinals whose cardinal is strictly below $\aleph_\omega$. Prove that there is a countable set of ordinals in $\mathbb{H}$ such that this set has no common upper bound in $\mathbb{H}$. For proving this, use that the cardinal $\aleph_\omega$ is a limit of the cardinals $\aleph_k$ with $k \in \mathbb{N}$.

Homework 2.18
There is an operation called ordinal addition such that for all ordinals x, y, z with $x < y < z$ it holds that $z + x < z + y$. Here it is important that z comes first, as for ordinal addition, the order matters and $0 + \omega = 1 + \omega$. Write these two statements in first-order logic with variables quantifying over ordinals and $+$ referring to the ordinal addition and $<, =$ meaning the less than and equal relation on ordinals.

Homework 2.19
Use this fact to prove the following statements: There is a one-one mapping from the set of finite ordinals into the set of countable ordinals where ω is the least countable ordinal; furthermore, there is also a one-one mapping from the set of countable ordinals into the set of all ordinals with cardinal $\aleph_1$.

Assignments for Week 3

Homework 3.1

Let $f(n)$ be the maximum number of negation symbols in a well-formed formula which does not contain any subformula of the form $(\neg(\neg\alpha))$ and which contains at most n atoms. Here $(\neg(A_1 \vee (\neg(A_2 \vee (\neg A_1)))))$ has 3 atoms and n is 3, as repeated atoms are counted again. Determine the value $f(n)$ in dependence of n.

Homework 3.2

Prove by induction that a well-formed formula of length n contains less than $n/3$ connectives and at most $(n+3)/4$ atoms.

Homework 3.3

Use the truth-table method to prove that the following formulas are equivalent:

- $((\neg A_1) \vee (\neg A_2))$;
- $(\neg(A_1 \wedge A_2))$;
- $((A_1 \vee A_2) \leftrightarrow (A_1 \oplus A_2))$.

Homework 3.4

Use the truth-table method to check whether the following statement is correct:

$$\{(A_1 \vee A_2), (A_2 \vee A_3), (A_1 \vee A_3)\} \models ((A_1 \wedge A_2) \vee A_3).$$

Homework 3.5

Use the truth-table method to check whether the following statement is correct:

$$\{(A_1 \to A_2), (A_2 \to A_3), (A_3 \to A_4)\} \models ((A_1 \to A_3) \wedge (A_2 \to A_4))$$

Homework 3.6

Use the truth-table method to check whether the following statement is correct:

$$\{(A_1 \to A_2), (A_2 \to A_3), (A_3 \to A_4)\} \models (A_4 \to A_1)$$

Homework 3.7

List out the truth-table for the formula $(((A_1 \oplus A_2) \wedge (\neg A_3)) \oplus (A_1 \vee A_3))$.

Homework 3.8

List out the truth-table for the formula $((A_1 \oplus A_3) \vee ((A_1 \oplus A_2) \wedge (A_2 \oplus A_3)))$.

Homework 3.9

Consider the following formulas:

$$\begin{aligned}
\phi_1 &= (((A_1 \vee A_2) \vee A_3) \wedge ((A_4 \vee A_5) \vee A_6)); \\
\phi_2 &= (((A_1 \vee A_2) \wedge (A_3 \vee A_4)) \wedge (A_5 \vee A_6)); \\
\phi_3 &= (((((A_1 \oplus A_2) \oplus A_3) \oplus A_4) \oplus A_5) \oplus A_6).
\end{aligned}$$

There are $2^6 = 64$ ways to assign the truth-values to the sentence symbols (or atoms) $A_1, \ldots, A_6$. Determine for each of the formulas ϕ_1, ϕ_2, ϕ_3, how many of these assignments make the formula true and how many of these assignments make the formula false.

Homework 3.10
For the formulas from Homework 3.9, is the statement

$$\{\phi_1, \phi_2, \phi_3\} \models (((((A_1 \wedge A_2) \wedge A_3) \wedge A_4) \wedge A_5) \wedge A_6)$$

true or false? Prove your answer.

Homework 3.11
For the formulas from Homework 3.9, is the statement

$$\{\phi_1, \phi_2, \phi_3\} \models (((((A_1 \vee A_2) \vee A_3) \vee A_4) \vee A_5) \vee A_6)$$

true or false? Prove your answer.

Homework 3.12
Using the connectives $\vee, \wedge, \rightarrow, \leftrightarrow, \oplus, \neg$, construct a formula using atoms A_1, A_2, A_3, A_4 which says that at least two and at most three of these atoms are true.

Homework 3.13
Using the connectives $\vee, \wedge, \rightarrow$, construct a formula using atoms $A_1, A_2, A_3, A_4, A_5, A_6$ which says that either all six atoms are false or all six atoms are true.

Homework 3.14
Use the truth-table method to prove the associativity of $\leftrightarrow$, that is, prove that $(A_1 \leftrightarrow (A_2 \leftrightarrow A_3))$ and $((A_1 \leftrightarrow A_2) \leftrightarrow A_3)$ are the same. Furthermore, check whether there is a truth-assignment ν with $\overline{\nu}((A_1 \leftrightarrow (A_2 \leftrightarrow A_3))) = 1$ and $\nu(A_1) \neq \nu(A_2)$.

Homework 3.15
Use the truth-table method to check whether $(A_1 \oplus (A_2 \oplus A_3))$ and $(A_1 \leftrightarrow (A_2 \leftrightarrow A_3))$ are equivalent.

Homework 3.16
Make the truth-tables of $\wedge$, $\oplus$ and $\neg$ for $\{0, u, 1\}$-valued logic where the value u stands for an unknown value of 0 and 1 and where the output u is taken iff one cannot derive from the inputs what the output is. Note that two inputs u need not to represent the same of 0 and 1.

Homework 3.17
Make the truth-tables of $\rightarrow$, $\leftrightarrow$ and $\vee$ for the $\{0, u, 1\}$-valued logic from 3.16.

Homework 3.18
Assume that $B = \{\emptyset\}$ and $D = \{A : A \subseteq \mathbb{N}\}$ and consider infinitely many constructor functions $f_k(A) = \{k\} \cup \{x + k + 1 : x \in A\}$ with $k \in \mathbb{N}$. What is C^* in this set-up? Are the constructor functions one-one and disjoint in range?

Assignments for Week 4

Homework 4.1

Let $atom(\phi)$ be the set of atoms used in ϕ, so $atom(((A_1 \vee A_2) \wedge A_2)) = \{A_1, A_2\}$ and $atom((0 \vee 1)) = \emptyset$. Let WFF be the set of well-formed formulas. Let

$$C_1 = \{\phi \in WFF : \forall \nu \, [\text{if } \nu(A) = 1 \text{ for some } A \in atom(\phi) \text{ then } \overline{\nu}(\phi) = 1]\}.$$

For which of the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ is C_1 closed under the connective? Here one says that C is closed under the connective $\oplus$ if all formulas $\phi, \psi \in C$ satisfy that $(\phi \oplus \psi) \in C$. Similarly for other connectives.

Homework 4.2

Let $atom(\phi)$ and WFF be defined as in Homework 4.1. Let

$$C_2 = \{\phi \in WFF : \forall \nu \, [\text{if } \nu(A) = 0 \text{ for at most one } A \in atom(\phi) \text{ then } \overline{\nu}(\phi) = 1]\}.$$

For which of the connectives $\neg, \wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ is C_2 closed under the connective?

Homework 4.3

Let $C_3 = \{\phi \in WFF : \text{every } A \in atom(\phi) \text{ occurs in } \phi \text{ exactly once and } 0, 1 \text{ do not occur in } \phi\}$. Prove by induction that C_3 does not contain any tautology and also not contain any antitautology. Here a tautology is a formula which is always true (independent of the choice of the truth-values of the atoms) and an antitautology is a formula which is always false.

Homework 4.4

Define on WFF by recursion the functions $atom$ ($\phi \mapsto atom(\phi)$) and $maxatom$ ($\phi \mapsto \max\{k : A_k \in atom(\phi)\}$), where the atoms $A_1, A_2, \ldots$ can be used and $\max \emptyset = 0$. So $maxatom((A_1 \vee (A_4 \wedge A_5))) = 5$ and $maxatom((0 \vee 1)) = 0$.

Homework 4.5

Define on WFF by recursion the function $numcon(\phi)$ as the number of connectives $\wedge, \vee, \rightarrow, \leftrightarrow, \oplus$ occurring in ϕ. Furthermore define the function $numneg(\phi)$ as the number of negations occurring in ϕ. Determine the best-possible constants c, m, n such that

$$|\phi| \leq c \cdot numcon(\phi) + n \cdot numneg(\phi) + m$$

for all WFF ϕ.

Homework 4.6

Let $C_6 = \{\phi \in WFF : \text{every } A \in atom(\phi) \text{ occurs in } \phi \text{ exactly once}\}$; note that formulas in C_6 might have occurrences of the constants 0 and 1. Define by recursion a function F from C_6 into the rational numbers between 0 and 1 which returns for each formula $\phi \in C_6$ the truth-probability $n/2^m$ where n is the number of rows in the truth-table of ϕ evaluated to 1 and m is the number of atoms used in the formula so that 2^m is the overall number of rows in the truth-table of ϕ. For example, $F(1) = 1$, $F((A_1 \oplus (A_2 \vee A_3))) = 1/2$ and $F(((A_2 \vee A_5) \wedge (A_3 \vee 0))) = 3/8$.

Homework 4.7
Let $C_7 = \{\phi \in WFF : \phi$ can use the constants $0, 1$ and the only connectives in ϕ are $\wedge$ and $\vee\}$. Prove by induction that a formula $\phi \in C_7$ is a tautology iff $\overline{\nu}(\phi) = 1$ for the truth-assignment ν with $\nu(A_k) = 0$ for all k.

Homework 4.8
Let $C_8 = \{\phi \in WFF : \phi$ can use the constants $0, 1$ and the only connectives in ϕ are $\wedge$ and $\vee\}$. Prove by induction that a formula $\phi \in C_8$ is an antitautology iff $\overline{\nu}(\phi) = 0$ for the truth-assignment v with $\nu(A_k) = 1$ for all k.

Homework 4.9
Let U be a finite set of atoms and $C_9 = \{\phi \in WFF : atom(\phi) \subseteq U\}$. Prove that there is a finite set F of formulas such that for every formula $\phi \in C_9$ there is a $\psi \in F$ with $(\psi \leftrightarrow \phi)$ being a tautology.

Homework 4.10
The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Insert back the needed brackets for getting a member of WFF.

1. $A_1 \wedge \neg A_2 \vee A_3 \rightarrow A_4 \wedge \neg A_5$;

2. $A_1 \vee \neg A_2 \wedge \neg A_3 \leftrightarrow A_4 \rightarrow A_5$;

3. $\neg\neg A_1 \vee \neg A_2$.

Homework 4.11
Is there a formula using the connectives $\oplus$ and $\neg$ but no other connectives where the value of the formula depends on the placement of brackets?

Homework 4.12
Let $\nu(A_1) = 1$, $\nu(A_2) = 1$, $\nu(A_3) = 0$. The below formulas are given in Polish notation. Write them as WFF and evaluate them according to ν:

1. $\neg \leftrightarrow \oplus A_1 A_2 A_3$;

2. $\wedge \vee \neg \vee A_1 A_2 A_3 A_1$;

3. $\oplus \wedge A_1 A_2 \wedge A_2 A_3$.

Homework 4.13
Write the following formula in Polish notation: $\neg((A_1 \vee \neg A_2) \wedge (A_2 \vee \neg A_3) \wedge (A_3 \vee \neg A_1))$.

Homework 4.14
Consider the formula ϕ in 9 atoms $A_1, \ldots, A_9$ and consider the set S of all $(A_i \oplus A_j \oplus A_k)$ with $A_i \in \{A_1, A_2, A_3\}$, $A_j \in \{A_4, A_5, A_6\}$ and $A_k \in \{A_7, A_8, A_9\}$. Now let ϕ_1 be the conjunction of all formulas in S; that is, if $S = \{\psi_1, \psi_2, \ldots, \psi_n\}$ then $\phi_1 = \psi_1 \wedge \psi_2 \wedge \ldots \wedge \psi_n$ when the brackets are omitted. Determine how many assignments of these atoms make the formula true and how many make it false.

Homework 4.15
Let ϕ_2 to be taken the disjunction of the formulas in set S from Homework 4.14. Determine how many assignments of these atoms make the formula true and how many make it false.

Homework 4.16
The following formulas have brackets omitted according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Insert back the needed brackets for getting a member of WFF and evaluate the formulas for the truth-assignment ν which sets all atoms to 1.

1. $A_1 \vee \neg A_2 \wedge A_3 \leftrightarrow A_4 \rightarrow \neg A_5 \vee A_3 \wedge A_6 \oplus A_7$;
2. $A_1 \rightarrow A_2 \rightarrow A_3 \rightarrow A_4 \rightarrow \neg A_5$;
3. $\neg\neg\neg A_1 \oplus A_2 \oplus A_3$.

Homework 4.17. The following formulas have some brackets omitted. Insert them back according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Evaluate them with $A_1 = 1$, $A_2 = 0$, $A_3 = 1$, $A_4 = 0$.

1. $A_1 \wedge A_2 \oplus A_3 \vee A_4$;
2. $\neg(\neg A_1 \vee \neg A_2) \wedge A_3 \vee A_4$;
3. $A_1 \vee A_2 \oplus A_1 \vee A_3 \oplus A_2 \vee A_3 \oplus A_3 \vee A_4$.

Homework 4.18. The following formulas have some brackets omitted. Insert them back according to the rule that the binding strengths of the connectives is ordered as $\neg, \wedge, \vee, \oplus, \rightarrow, \leftrightarrow$. Reinsert the operators and either find an satisfying assignment or prove that such does not exist.

1. $A_1 \wedge \neg A_1 \vee A_2 \wedge \neg A_3 \vee A_3 \wedge \neg A_2$;
2. $A_1 \wedge \neg A_2 \oplus A_2 \wedge \neg A_3 \oplus A_3 \wedge \neg A_1$.

Homework 4.19. The following formulas have some brackets omitted. Find a way to reinsert brackets such that they become tautologies and explain why this works.

1. $A_1 \oplus \neg A_1 \wedge A_2 \oplus \neg A_2$;
2. $A_1 \vee A_2 \vee A_3 \oplus \neg A_1 \vee A_2 \vee A_3$;
3. $\neg A_1 \leftrightarrow A_2 \oplus A_2 \leftrightarrow A_1$.

Assignments for Week 5

Homework 5.1

Consider six-valued logic with values $\{0.0, 0.2, 0.4, 0.6, 0.8, 1.0\}$ ordered in the same way as rational numbers and define that $x \wedge y$ is the minimum and $x \vee y$ is the maximum of the inputs; furthermore, $\neg x$ is $1.0 - x$ (with the usual numerical operation). Now, for any formula α, let $\min(\alpha) = \min\{\overline{\nu}(\alpha) : \nu$ is a six-valued truth-assignment of the atoms$\}$. Similarly one defines $\max(\alpha)$. Which are the possible values which $\min(\alpha)$ and $\max(\alpha)$ can take where α ranges over all formulas obtained by connecting atoms using $\wedge, \vee, \neg$?

Homework 5.2

If one uses the connectives $\wedge, \vee, \neg, \oplus$ for the six-valued logic from Homework 5.1 with $x \oplus y$ being defined as $\min\{x + y, 2 - x - y\}$, what are the possible values of $\min(\alpha)$ and $\max(\alpha)$, which are defined as in Homework 5.1.

Homework 5.3

For the six-valued logic from Homework 5.1 and 5.2, say that two formulas α, β are equivalent iff they for all six-valued truth-assignments ν to the atoms satisfy $\overline{\nu}(\alpha) = \overline{\nu}(\beta)$. Check whether the following equivalences of formulas hold in the six-valued logic:

1. $(A_1 \wedge A_2) \equiv (A_2 \wedge A_1)$;
2. $(A_1 \wedge (A_2 \vee A_3)) \equiv ((A_1 \wedge A_2) \vee (A_1 \wedge A_3))$;
3. $(A_1 \wedge (A_2 \oplus A_3)) \equiv ((A_1 \wedge A_2) \oplus (A_1 \wedge A_3))$.

Homework 5.4

For the six-valued logic from Homework 5.1 and 5.2, let $x \leftrightarrow y$ be calculated by $1 - \max\{x - y, y - x\}$. Using the equivalence definition from Homework 5.3, check whether the following formulas are equivalent:

1. $(A_1 \leftrightarrow A_2) \equiv \neg(A_1 \oplus A_2)$;
2. $(A_1 \leftrightarrow A_2) \equiv (A_1 \oplus \neg A_2)$;
3. $\neg(A_1 \wedge A_2) \leftrightarrow (\neg A_1 \vee \neg A_2) \equiv 1.0$;
4. $((A_1 \leftrightarrow A_2) \leftrightarrow A_3) \equiv (A_1 \leftrightarrow (A_2 \leftrightarrow A_3))$.

Homework 5.5

Construct a circuit for $A_1 \oplus A_2 \oplus A_3$ with the gates $\wedge, \vee, \neg$; these gates can have multiple inputs.

Homework 5.6

Construct a circuit for $A_1 \oplus A_2 \oplus A_3$ using *nand* and *nor* and *not* gates, which can have multiple inputs.

Homework 5.7
Assume that a company uses only chips which output 0 when all inputs are 0 – as all-0-inputs and outputs are considered as an "error-information" and every useful information is coded by input-vectors which are not everywhere 0. Now a vendor offers to produce the chips as specified using only "exclusive-or-gates" ($\oplus$) and "inclusive-or-gates" ($\vee$) at a very competitive price. The company boss finds it suspicious and asks the company's technician: Can this work? Provide the correct answer and prove why it can work or why it cannot work.

Homework 5.8
Recall that $maj(x, y, z)$ is 1 iff at least two of the inputs x, y, z are 1. Let C_8 consist of all formulas which are atoms or which are formed from other formulas $\alpha, \beta, \gamma \in C_8$ by taking $maj(\alpha, \beta, \gamma)$ or $\neg\alpha$. Prove by induction that each Boolean function B^n_α formed from an $\alpha \in C_8$ satisfies $B^n_\alpha(x_1, \ldots, x_n) = \neg B^n_\alpha(\neg x_1, \ldots, \neg x_n)$ for $x_1, \ldots, x_n \in \{0, 1\}$.

Homework 5.9
How many Boolean functions can be formed by using input variables $x_1, \ldots, x_n$ and the constants and connectives from $\{0, 1, \wedge\}$.

Homework 5.10
How many Boolean functions can be formed using input variables $x_1, \ldots, x_n$ and the constants and connectives from $\{0, 1, \neg, \oplus\}$.

Homework 5.11
Assume that α can use some of the atoms $A_1, \ldots, A_5$, the truth-values 0 and 1 and up to two connectives $\oplus$. How many functions of the form B^5_α can be formed using such α?

Homework 5.12
Assume that α can use some of the atoms $A_1, \ldots, A_4$, the truth-values 0 and 1 and at most one connective $\wedge$ and at most one connective $\vee$. How many functions of the form B^4_α can be formed using such α?

Homework 5.13
Use as few of "and" ($\wedge$) and "inclusive or" ($\vee$) as possible in order to make a formula α with four atoms A_1, A_2, A_3, A_4 such that the following conditions hold:

- If at least three of the atoms A_1, A_2, A_3, A_4 are true then α is true;
- If at most one of the atoms A_1, A_2, A_3, A_4 are true then α is false;
- If exactly two of the atoms A_1, A_2, A_3, A_4 are true then there is no constraint on which value α takes.

Use the last condition in order to optimise the number of connectives in the formula.

Homework 5.14
Recall that $maj(x, y, z)$ is 1 iff at least two of the inputs x, y, z are 1. Is there an $n \in \{1, 2, 3, 4\}$ for which the set $\{maj, B^n_{A_1 \oplus A_2 \oplus \ldots \oplus A_n}\}$ complete? For those n where

it is incomplete, can it be made complete by adding the logical constants $0, 1$ to the set of connectives? If so, which of these are needed?

Assignments for Week 6
Homework 6.1
Which of the following statements are true? Prove your answers.
(a) $\{\alpha, \beta\} \models c \vee d \Leftrightarrow \{\alpha, \beta\} \models c$ or $\{\alpha, \beta\} \models d$.
(b) $\{\alpha, \beta\} \models c \wedge d \Leftrightarrow \{\alpha, \beta\} \models c$ and $\{\alpha, \beta\} \models d$.
(c) $\{\alpha, \beta\} \models \alpha \oplus \beta \Leftrightarrow \alpha \wedge \beta$ is not satisfiable.
Here a formula α is satisfiable iff there is a choice of truth-values of the atoms such that α becomes true.

Homework 6.2
Which of the following statements are true? Prove your answers.
(a) $S \models \alpha \Leftrightarrow S \cup \{\alpha\}$ is satisfiable.
(b) $S \models \alpha \Leftrightarrow S \cup \{\neg\alpha\}$ is not satisfiable.
(c) $S \models \alpha \rightarrow \beta \Leftrightarrow S \cup \{\neg\alpha\} \models \neg\beta$.
Here a set S of formulas is satisfiable iff there is a choice of truth-values of the atoms such that all formulas in S are true.

Homework 6.3
Make an infinite set S of formulas such that every subset of two formulas is satisfiable but no subset of three or more formulas is.

Homework 6.4
Is the set $\{\leftrightarrow, \neg, \oplus, 0, 1\}$ of connectives and constants complete? Do the subsets $\{\leftrightarrow, 1\}$ and $\{\leftrightarrow, 0\}$ have the same expressive power or less expressive power than $\{\leftrightarrow, \neg, \oplus, 0, 1\}$?

Homework 6.5
Make a formula in A_1, A_2, A_3, A_4 with as few of the connectives $\wedge$ and $\vee$ as possible, but which might use as many $\neg$ as needed such that the following constraints are satisfied: If none or all four of the atoms are 1 then the output is 0 and if one or three of the atoms are 1 then the output is 1; there is no requirement on what happens if exactly two atoms are 1. Use Enderton's Square Method.

Homework 6.6
Make a formula in A_1, A_2, A_3, A_4 with as few of the connectives $\wedge$ and $\vee$ as possible, but which might use as many $\neg$ as needed such that the following constraints are satisfied: If none or three of the atoms are 1 then the output is 0 and if one or all four of the atoms are 1 then the output is 1; there is no requirment on what happens if exactly two atoms are 1. Use Enderton's Square Method.

Homework 6.7
For switching circuits based on relays and with the possibility to use both normal and negated inputs, construct a circuit which uses as few input-invocations as possible in order to compute the majority-function in three variables.

Homework 6.8
Consider the three-valued fuzzy logic with truth-values from $Q = \{0, 1/2, 1\}$. Can the

set of $\{\wedge, \vee, \neg, \oplus, \leftrightarrow, \rightarrow\}$ plus the three truth-values be used to generate all functions from $Q^2 \to Q$?

Homework 6.9
Consider fuzzy logic with truth-values from some finite Q satisfying the constraints from Chapter 1.5. Find a set containing only three connectives which is, together with the truth-values, as powerful as the set $\{\wedge, \vee, \neg, \oplus, \leftrightarrow, \rightarrow\}$ with respect to the ability to generate functions from $Q^2 \to Q$.

Homework 6.10
Consider fuzzy logic with $Q = \{r \in \mathbb{R} : 0 \leq r \leq 1\}$. Provide some examples of functions from Q to Q which are not equal to B^1_α for some α generated by rational truth-values and the connectives of fuzzy logic in Chapter 1.5.

Homework 6.11-6.13
Corollary 17A says that if $S \models \alpha$ then there is a finite subset S' of S such that $S' \models \alpha$. This proof does not directly translate to fuzzy logic and indeed, if one defines $S \models \alpha$ in fuzzy logic in the wrong way, then it is false. For the following homeworks, consider $Q = \{r \in \mathbb{R} : 0 \leq r \leq 1\}$ and $S = \{q \to A_1 : q \in \mathbb{Q} \text{ and } 0 \leq q < 1\}$ and $\alpha = A_1$.

Homework 6.11
Assume that one defines $S \models \alpha$ as "All Q-valued truth-assignments ν satisfy that if $\overline{\nu}(\beta) = 1$ for all $\beta \in S$ then $\overline{\nu}(\alpha) = 1$" and show that then $S \models \alpha$ but no finite subset S' of S satisfies $S' \models \alpha$.

Homework 6.12
Assume that one defines $S \models \alpha$ as "All Q-valued truth-assignments ν and all $\varepsilon > 0$ satisfy that there is $\beta \in S \cup \{1\}$ with $\overline{\nu}(\beta) \leq \overline{\nu}(\alpha) + \varepsilon$" and show that then $S \models \alpha$ but no finite subset S' of S satisfies $S' \models \alpha$.

Homework 6.13
Assume that one defines $S \models \alpha$ as "All Q-valued truth-assignments ν and all $q \in Q$ satisfy that if $\overline{\nu}(\beta) \geq q$ for all $\beta \in S$ then $\overline{\nu}(\alpha) \geq q$" and show that then $S \models \alpha$ but no finite subset S' of S satisfies $S' \models \alpha$.

Assignments for Week 7

Homework 7.1

Recall that a set is recursively enumerable iff it is empty or is the range of a function computed by an effective procedure (also called recursive function). Consider now three X, Y, Z be effectively enumerable sets which all contain 0 and which are the ranges of functions F_X, F_Y, F_Z which are given by effective procedures. Make recursive functions G, H such that the range of G is $X \cup Y \cup Z$ and the range of H is $(X \cap Y) \cup (X \cap Z) \cup (Y \cap Z) = \{u : u \text{ is in at least two of the sets } X, Y, Z\}$.

Homework 7.2

Prove that the following set S is decidable: S is the set of all n such that there are infinitely many natural numbers m for which both m and $m + n$ are powers of 2.

Homework 7.3

Prove that the following set S is decidable: S contains all numbers x for which there are infinitely many pairs y, z of prime numbers satisfying that $y < z \leq y + x$.

Homework 7.4

A binary tree T is a set of binary strings such that whenever $\sigma\tau \in T$ then $\sigma \in T$ (where $\sigma\tau$ is the concatenation of σ and τ). König's Lemma says that every infinite binary tree contains an infinite branch. Now let $A_1, A_2, \ldots$ be the atoms and let $S = \{\alpha_1, \alpha_2, \ldots\}$ be a set of formulas. Now let T be a binary tree which on level n contains all those $\sigma \in \{0, 1\}^n$ which satisfy for all formulas $\beta \in \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, if no atom A_k with $k > n$ occurs in β then every ν with $\nu(A_k) = \sigma(k)$ makes β true. Prove the following: If T is infinite then T has an infinite branch and each infinite branch defines a ν with $\nu \models S$; if T is finite then there is a first level n on which T has no nodes and $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is not satisfiable.

Homework 7.5

Let $S = \{\alpha : \alpha \text{ is a well-formed formula and for all } \nu,\ \overline{\nu}(\alpha) = 1 \text{ iff more than a half of the atoms } A_k \in \mathit{atom}(\alpha) \text{ satisfy } \nu(A_k) = 1\}$. Prove that S is decidable.

Homework 7.6

Let ν be computed by an effective procedure mapping each $k \in \mathbb{N}$ to the truth-value assigned to atom A_k. Let $S = \{\alpha \in WFF : \overline{\nu}(\alpha) = 1\}$. Which of the following options is correct?

(a) S is decidable; (b) S is recursively enumerable but not decidable; (c) S is not recursively enumerable.

Homework 7.7

Assume that you know that addition, subtraction and multiplication are effectively computable. Use now recursion in one variable to show that (a) the integer division $n, m \mapsto \max\{k : k \cdot m \leq n\}$ and (b) $n \mapsto \binom{2n}{n}$ are effectively computable functions. Note that the recursion can use case-distinctions; for example, the inductive definition of the remainder $f(a, b)$ of a by b is $f(0, b) = 0$ and if $f(a, b) + 1 < b$ then $f(a+1, b) = f(a, b) + 1$ else $f(a + 1, b) = 0$.

Homework 7.8
Prove that if S is a satisfiable set of formulas then $WFF - S$ is not a satisfiable set of formulas.

Homework 7.9
Assume that S_1, S_2, S_3 are satisfiable sets of formulas. What about the set $T = (S_1 \cup S_2) \cap (S_1 \cup S_3) \cap (S_2 \cup S_3)$? Prove that T is satisfiable or give an example of S_1, S_2, S_3 where the resulting T is not satisfiable.

Homework 7.10
Let $S = \{\alpha \in WFF : \overline{\nu}(\alpha) = 1\}$ for some ν and $T = \{\alpha : (\alpha \vee A_1), (\alpha \vee (\neg A_1)) \in S\}$. Is T satisfiable? Is $S = T$?

Homework 7.11
Call a set S of formulas almost-zero-satisfiable (azs) iff there is a ν with $\nu(A_k) = 0$ for almost all atoms and $\overline{\nu}(\alpha) = 1$ for all $\alpha \in S$. Does the notion "azs" satisfy the compactness theorem? That is, for any infinite set $S \subseteq WFF$, if every finite subset is almost-zero-satisfiable, is then S itself also almost-zero-satisfiable?

Homework 7.12
Are there infinite sets S, T of wff such that every finite subset T' of T there is a finite subset S' of S such that $S' \models \alpha$ for all $\alpha \in T'$ but it does not hold that $S \models T$, that is, there is some ν which is true on all members of S but not all members of T.

Homework 7.13
Assume that there are infinitely many logical atoms. Is there a set S of formulas such that for all ν mapping atoms to $\{0, 1\}$, $\nu \models S$ iff there are exactly three atoms A, B, C with $\nu(A) = 1, \nu(B) = 1, \nu(C) = 1$?

Assignments for Week 8

Homework 8.1

Let $Ap(x)$ say "x is an apple", $Ba(x)$ say "x is a banana", $Cb(x)$ say "x is a cranberry" and $Cu(x)$ say "x is a currant". Furthermore, let $Ye(x)$ say "x is yellow", $Re(x)$ say "x is red" and $Bl(x)$ say "x is black". Now translate the following English sentences into logic:

1. There are yellow apples and red apples.
2. All bananas are yellow.
3. Cranberries are always red.
4. There are red currants and black currants and every currant has one of these two colours.

Homework 8.2

Given the notation from Homework 8.1, translate the following formulas into normal English language sentences:

$$\begin{aligned} \forall x \quad & [Ap(x) \rightarrow \neg Ba(x)]; \\ \exists x \quad & [Bl(x) \wedge \neg Ap(x) \wedge \neg Ba(x)]; \\ \forall x\, \forall y \quad & [Re(x) \wedge Bl(y) \rightarrow x \neq y]; \\ \forall x\, \exists y \quad & [(Cu(x) \wedge Re(x)) \rightarrow (Cu(y) \wedge Bl(y))]. \end{aligned}$$

Homework 8.3

Assume that there is a set X of five fruits satisfying the following formulas.

$$\begin{aligned} \forall x \quad & [Ap(x) \vee Ba(x) \vee Cu(x)]; \\ \forall x \quad & [(\neg Ap(x) \wedge \neg Ba(x)) \vee (\neg Ap(x) \wedge \neg Cu(x)) \vee (\neg Ba(x) \wedge \neg Cu(x))]; \\ \forall x \quad & [Bl(x) \vee Re(x) \vee Ye(x)]; \\ \forall x \quad & [(\neg Bl(x) \wedge \neg Re(x)) \vee (\neg Bl(x) \wedge \neg Ye(x)) \vee (\neg Re(x) \wedge \neg Ye(x))]; \\ \forall x \quad & [Ap(x) \rightarrow \neg Bl(x)]; \\ \forall x \quad & [Ba(x) \rightarrow Ye(x)]; \\ \forall x \quad & [Cu(x) \rightarrow \neg Ye(x)]; \\ \exists u\, \exists v\, \exists w\, \exists x\, \exists y \quad & [Re(u) \wedge v \neq w \wedge Ye(v) \wedge Ye(w) \wedge x \neq y \wedge Bl(x) \wedge Bl(y)]. \end{aligned}$$

Calculate the number of models (up to isomorphism) which satisfy these formulas with five elements.

Homework 8.4

Use the formulas from Homework 8.3, but assume that X has 6 elements. Calculate the number of models (up to isomorphism) which satisfy these formulas with six elements.

Homework 8.5
Use the formulas from Homework 8.3, but assume that X has at most 4 elements. Calculate the number of models (up to isomorphism) which satisfy these formulas with up to four elements.

Homework 8.6
Assume that equality is in the logical language, but no predicate or function. Make a set S of formulas which says that the number of elements of a structure satisfying S is either a prime number or infinite. This set S is infinite.

Homework 8.7
Assume that a structure X with one function symbol f satisfies

$$\forall x\, [f(x) \neq x \wedge f(f(x)) = x].$$

What can be said about the number of elements in the base set X?

Homework 8.8
Make a formula using the language of natural numbers with addition and order which says that there are infinitely many numbers which are not multiples of any of 2, 3 and 5. This formula should not use multiplication.

Homework 8.9
Consider the structure $(\mathbb{N}, +, -, \cdot, <, =, 0, 1, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every natural number. Make formulas which express the following:

1. Each number is either 0 or 1 or the multiple of a prime number;

2. There are infinitely many prime numbers of the form $5n + 1$.

Homework 8.10
Consider the structure $(\mathbb{N}, +, -, \cdot, <, =, 0, 1, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every natural number. Make formulas which express the following:

1. Every even number other than 0 and 2 is the sum of two prime numbers;

2. There are infinitely many numbers x such that $x - 1$ and $x + 1$ are both prime numbers.

Homework 8.11
Consider the structure $(\mathbb{Z}, +, -, \cdot, <, =, 0, -1, 1, -2, 2, \ldots)$ and the corresponding first-order logical language of arithmetic with constants for every integer. Make formulas which express the following:

1. The number 23 is not the sum of three squares;

2. A number is the sum of four squares if it is greater or equal 0.

Homework 8.12
For first-order logic, assume that the logical language has only equality and variables and quantifiers and the logical connectives. The formula $\exists x, y, z\, [x \neq y \wedge x \neq z \wedge y \neq z]$ can only be satisfied by a structure with at least three elements. Is there, in this logical language, a formula α which can only be satisfied by structures with infinitely many elements? Is there a set S of formulas such that S is only satisfied by structures with infinitely many elements?

Homework 8.13
Let $(F, +, -, \cdot, f, =, 0, 1, 2)$ be the finite field with the three elements $0, 1, 2$ and let $f : F \to F$ be any function. Which of the following statements are true for this structure (independently of how f is chosen)?

1. $\forall x, y\, [(x + y) \cdot (x + y) = (x \cdot x) + (y \cdot y) - (x \cdot y)]$;
2. $\forall x, y\, [(x + y) \cdot (x + y) \cdot (x + y) = (x \cdot x \cdot x) + (y \cdot y \cdot y)]$;
3. $\forall x, y\, [(x + y) \cdot (x + y) \cdot (x + y) \cdot (x + y) = (x \cdot x \cdot x \cdot x) + (y \cdot y \cdot y \cdot y)]$;
4. $\exists a, b, c\, \forall x\, [f(x) = a \cdot x \cdot (x - 1) + b \cdot x \cdot (x - 2) + c \cdot (x - 1) \cdot (x - 2)]$;
5. $\forall x\, [x \cdot x \cdot x \neq 2]$.

Assignments for Week 9

Homework 9.1

Let $(A, +, \cdot)$ and $(B, +, \cdot)$ be the remainder rings modulo a and b, respectively, $a, b \in \{2, 3, 4, 5, 6\}$. For which a, b is there a homomorphism f from $(A, +, \cdot)$ to $(B, +, \cdot)$ such that any two terms t_1, t_2 satisfy $(A, +, \cdot), s \models t_1 = t_2$ iff $(B, +, \cdot), s' \models f(t_1) = f(t_2)$, where $s'(v_k) = f(s(v_k))$ for all variables v_k.

Homework 9.2

Choose values for a, b from Homework 9.1 and a formula ϕ and a function f such that f is a homomorphism and the formula ϕ is true in $(A, +, \cdot)$ but not in $(B, +, \cdot)$.

Homework 9.3

Consider the model $(\{0, 1, \ldots, 9\}, +, \cdot)$ with addition and multiplication modulo 10, so $5 + 7 = 2$ and $5 \cdot 7 = 5$. Which are the sets defined by the following formulas:

1. $x \in A \Leftrightarrow \exists y\, [x = y \cdot y]$;
2. $x \in B \Leftrightarrow \forall y\, [x \cdot y = 0 \vee x \cdot y = 3 \vee x \cdot y = 5]$;
3. $x \in C \Leftrightarrow \forall y\, [x \neq y \cdot y \cdot y \cdot y]$.

Homework 9.4

Is the set $\{2, 4, 6, 8\}$ definable in the model of arithmetic modulo 10? Here the formula can use the operations $+, \cdot$ and the constants $0, 1$ and equality $=$ and connectives and quantifiers.

Homework 9.5

Is every function in the model $\{0, 1, \ldots, 9\}$ with addition and multiplication and all constants explicitly definable by a term? If so, give a proof; if not, explain why.

Homework 9.6

Let $\mathbb{Z} \cdot \{i\} + \mathbb{Z}$ be the set of all complex integer numbers. Show that this set together with $+$ and $\cdot$ is a ring. Prove that the basis element i is not definable by using an isomorphism which maps i to some other element.

Homework 9.7

Recall that a structure $(A, \circ, e)$ is a group iff it satisfies $\forall x, y, z\, [x \circ (y \circ z) = (x \circ y) \circ z]$, $\forall x\, [x \circ e = x \wedge e \circ x = x]$, $\forall x \exists y\, [x \circ y = e \wedge y \circ x = e]$.
Write down formally the axioms for an Abelian group, a ring with 1 and a commutative ring with 1, respectively.

Homework 9.8

Assume that $(A, +, \cdot, 0, 1)$ is a finite ring with $0 \neq 1$. Consider the formulas

$$\begin{aligned} x = 0 &\quad\Leftrightarrow\quad \forall y\, [x + y = y] \text{ and} \\ x = 1 &\quad\Leftrightarrow\quad \forall y\, [x \cdot y = y \wedge y \cdot x = y]. \end{aligned}$$

Are then all members of A definable with formulas like this? If yes then prove how this is done else provide a finite ring where some elements are not definable.

Homework 9.9
Let $(\mathbb{R}, +, \cdot, <, 0, 1)$ be the ordered field of the real numbers with the constants 0 and 1. Prove that all rational numbers and all real roots of polynomials are definable. Provide then examples of formulas $\phi_1, \phi_2, \phi_3, \phi_4$ such that x_k is the unique element satisfying ϕ_k where the formulas ϕ_k say the following:

1. $x_1 = 2/3$;
2. x_2 is the positive square-root of 3;
3. x_3 is the largest number satisfying $x_3^{10} - 4x_3^5 + 2 = 0$;
4. x_4 is the smallest number satisfying $3x_4^6 - 6x_4^4 + 3x_4^2 = 0$.

Homework 9.10
Consider a structure $(A, f, 0, 1, =)$ with $0, 1 \in A$ being constants and f a function from A to A. Make three formulas in the language of this structure which express the following conditions:

1. The first formula says that f has the range $\{0, 1\}$;
2. The second formula says that f is the inverse of itself;
3. The third formula says that every value in the range of f is the image of exactly two values.

Homework 9.11
Let $(A, P^A), (B, P^B)$ be two structures with $A = \{0\}$ and $B = \{1, 2\}$. choose the predicate P^B such that there is no strong homomorphism from B to A (independently of what P^A is) while there is for each possible choice of P^A a strong homomorphism from (A, P^A) to (B, P^B).

Homework 9.12
Let $(\mathbb{Z}, Succ, Even)$ be a structure with $Even(x)$ being true iff x is even and $Succ$ being the successor function. Let f be a function from the structure to itself. Prove that if f is a homomorphism then f is a strong homomorphism.

Homework 9.13
Consider the structure $(\mathbb{Z}, Neigh, Even)$ where $Even(x)$ is true iff x is even and $Neigh(x, y)$ is true iff $x = y + 1$ or $x = y - 1$. Construct a function g from $\mathbb{Z}$ to itself which is a homomorphism but not a strong homomorphism.

Homework 9.14
Assume that $(A, +, a, b, c, d)$ is an n-dimensional vector space for some n over the field $(\{0, 1, 2\}, +, \cdot)$ with three elements; here for the skalar multiplication, $x \cdot 0 = x + x + x$, $x \cdot 1 = x$ and $x \cdot 2 = x + x$, so that the multiplication with each fixed skalar is definable. Find the largest dimension n so that all elements in the vector space $(A, +, a, b, c, d)$ are definable when one chooses the right values for a, b, c, d and explain how the

formulas to define the elements look like; note that an isomorphism of the structure itself has to map a to a, b to b, c to c and d to d.

Homework 9.15
Assume that a structure $(X, +)$ satisfies the below axioms:

1. $\forall x\, \forall y\, \forall z\, [x + (y + z) = (x + y) + z]$;

2. $\forall x\, \forall y\, [x + y = y + x]$;

3. $\forall x\, \forall y\, [x + x = y + y]$;

4. $\forall x\, [x + x + x = x]$.

Assume furthermore, that the structure has four elements $0, a, b, c$ and that 0 is the element with $0 = x + x$ for all $x \in X$. Prove (informally, not in the deductive calculus) that the structure satisfies $a + b = c$ and show that a, b, c are not definable by constructing an isomorphism h with $h(a) \neq a$, $h(b) \neq b$ and $h(c) \neq c$.

Assignments for Week 10

Homework 10.1

This homework considers undirected graphs without self-loops. Consider the following two graphs:

```
(a) 0 - 1 - 2          (b) 0  1 - 2
        | /                   | /
        3                     3
```

(a) Prove that in graph (a) nodes 0 and 1 are definable and nodes 2 and 3 are not.
(b) Prove that in graph (b) the node 0 is definable and nodes 1, 2 and 3 are not.
(c) For which n is there a graph of 6 nodes such that exactly n out of these 6 nodes are definable?

Homework 10.2

Let the logical language contain a function symbol f for a function with one input. Show that Λ proves the formulas

$$\neg(f(x) = f(y)) \to \neg(f(y) = f(x)),\ f(x) = f(y) \to (f(y) = f(z) \to f(x) = f(z))$$

which is similar to some proofs in the lecture notes.

Homework 10.3

For the following formulas α and terms t, either write what α^z_t is or write that a substitution is not permitted. The formulas are $\exists x\,[\neg(x = z{+}1)]$, $\forall z\,[x = z]$, $f(x{\cdot}z) = f(0)$ and the terms are x, 0, $z + z$. Do not forget to make brackets where needed.

Homework 10.4

For the following formulas α and terms t, either write what α^z_t is or write that a substitution is not permitted. The formulas are $\exists x \forall y\,[x = y \cdot z]$, $\forall x\,\exists y\,[z = x + y]$, $\forall u\,[z \cdot z + 1 \neq u \cdot u + 2]$ and the terms are $x + y$, 0, $v \cdot w$. Do not forget to make brackets where needed.

Homework 10.5

Use the Deduction Theorem to show the following:
If $\Gamma \vdash \alpha \to \beta \to \gamma \to \delta$ then $\Gamma \vdash\ \gamma \to \alpha \to \beta \to \delta$.
Which other interchanges of $\alpha, \beta, \gamma, \delta$ are permitted and which not?

Homework 10.6

Prove the statement from Homework 10.5 using only tautologies and modus ponens.

Homework 10.7

Let the logical language have a predicate P and constant c. Prove formally that

$$\{\forall x\,\forall y\,[P(x) \to P(y)]\} \vdash P(c) \to \forall y\,[P(y)]$$

using the axioms of Λ, the Deduction Theorem and the Generalisation Theorem.

Homework 10.8
Let $(A, +, 0)$ be a structure with constant 0 and binary operation $+$. Make a formal proof for
$$\{\forall x\,[x + x = 0]\} \vdash \forall x\,[(x + x) + (x + x) = 0]$$
using axioms from Λ and the Generalisation Theorem.

Homework 10.9
Let $(A, +, 0)$ be a structure with constant 0 and binary operation $+$. Make a formal proof for
$$\{\forall x\,\forall y\,[x + y = y + x]\} \vdash \forall u\,[u + (u + u) = (u + u) + u]$$
using the axioms of Λ and the Generalisation Theorem.

Homework 10.10
For $(A, +, 0)$ as in Homework 10.9, make a formal proof for
$$\{\forall x\,\forall y\,\forall z\,[(x + y) + z = x + (y + z)]\} \vdash \forall u\,[u + (u + u) = (u + u) + u]$$
using the axioms of Λ and the Generalisation Theorem.

Homework 10.11
Is the statement
$$\{\forall x\,\forall y\,[x + y = y + x], \forall x\,\forall y\,\forall z\,[(x + y) + z = x + (y + z)]\} \models \forall x\,\forall y\,\exists z\,[x + z = y]$$
true? If the statement is true then make a formal proof else provide a model satisfying the left but not the right side of $\models$.

Homework 10.12
Is the statement
$$\{\forall x\,\forall y\,\exists z\,[x + z = y], \forall x\,\forall y\,\forall z\,[(x + y) + z = x + (y + z)]\} \models \forall x\,\forall y\,\exists z\,[z + x = y]$$
true? If the statement is true then make a formal proof else provide a model satisfying the left but not the right side of $\models$.

Assignments for Week 11
Homework 11.1
Assume that α, β, γ are well-formed formulas. Give a formal proof of the statement

$$\{\beta, \gamma\} \models \alpha \to \beta$$

which only uses the formulas from Λ and the Modus Ponens.

Homework 11.2
Assume that $\{\alpha, \beta\}$ tautologically implies γ. The below derivation is incorrect. Say what the fault is and replace it by a corrected one:

1. $\{\alpha, \beta\} \vdash \alpha \to \beta \to \gamma$ (Axiom Group 1)
2. $\{\alpha, \beta\} \vdash \beta$ (Copy)
3. $\{\alpha, \beta\} \vdash \beta \to \alpha \to \beta$ (Axiom Group 1)
4. $\{\alpha, \beta\} \vdash \alpha \to \beta$ (Modus Ponens)
5. $\{\alpha, \beta\} \vdash \gamma$ (Modus Ponens)

For the following exercises, P, Q are predicates and a, b, c are constants.

Homework 11.3
Make a formal proof for

$$\{\forall x\,[P(x) \to Q(c)], \forall x\,[\neg P(x) \to Q(c)]\} \vdash Q(c).$$

Homework 11.4
Make a formal proof for $\{\forall x\,[P(x)], \exists y\,[\neg P(y)]\} \vdash Q(z)$.

Homework 11.5
Make a formal proof for $\emptyset \vdash \forall x\,\forall y\,[P(x) \to Q(y)] \to P(a) \to Q(b)$.

Homework 11.6
Is the statement $\emptyset \vdash P(x) \to \forall y\,[P(y)]$ correct? Explain your answer.

Homework 11.7
Is the statement $\emptyset \vdash P(x) \to \forall y\,[P(x)]$ correct? Explain your answer.

Homework 11.8
Is the statement $\emptyset \vdash P(x) \to \exists y\,[P(y)]$ correct? Explain your answer.

Homework 11.9
Let $(G, \circ, f, e)$ be a structure and Γ contain the following axioms:

- $\forall x, y, z\,[(x \circ y) \circ z = x \circ (y \circ z)]$;
- $\forall x, y\,[x \circ y = y \circ x]$;
- $\forall x\,[x \circ e = x]$;

- $\forall x\,[x \circ f(x) = e]$;
- $\forall x, y, z\,[x \circ y = x \circ z \rightarrow y = z]$;

So $(G, \circ)$ is an Abelian group with neutral element e and inversion f. Prove informally the following results:

- $\forall v, w\,[f(v) = f(w) \rightarrow v = w]$;
- $\forall v, w\,[v \circ w = e \rightarrow f(v) = w]$;
- $\forall v, w\,[f(v \circ w) = f(w) \circ f(v)]$.

Homework 11.10
Consider all structures $(A, \circ)$ where A has two elements and satisfies the axioms

$$\forall x\,[x \circ x = x] \text{ and } \forall x\,\forall y\,[x \circ y = y \circ x].$$

Show that all these structures are isomorphic.

Homework 11.11
Assume that $(\mathbb{N}, +, <, 0, 1, P)$ is a structure where $\mathbb{N}$ is the set of natural numbers and $+, <, 0, 1$ have the usual meaning on $\mathbb{N}$. Let the powers of 2 be the set $\{1, 2, 4, 8, 16, \ldots\}$ and make a formula α such that $(\mathbb{N}, +, <, 0, 1, P) \models \alpha$ iff $\forall x\,[Px \leftrightarrow \text{x is a power of 2}]$.

Note that such a formula only implicitly defines the powers of 2 and not explicitly; therefore this formula α does *not say* that the powers are definable from addition and order in $\mathbb{N}$.

Homework 11.12
Make a formula α which says that $f : A \rightarrow A$ is a one-to-one function but not an onto function. Provide a model $(A, f, =)$ which satisfies α. Can A be finite?

Assignments for Week 12

Homework 12.1

Let α, β, γ be any well-formed formulas. Which of the below formulas are valid, independent of what formulas for α, β, γ are chosen? If yes, give a formal proof, if not, find a counter example by chosing the right values for α, β, γ.

1. $\forall x\,[\alpha \to \beta] \to \forall x\,[\neg\beta] \to \forall x\,[\neg\alpha]$;
2. $\forall x\,\forall y\,[\alpha \to \beta] \to \forall x\,\forall y\,[\beta \to \gamma] \to \forall x\,\forall y\,[\gamma \to \alpha]$;
3. $\forall x\,[\alpha \to \beta] \to \forall x\,[\alpha \to \neg\beta] \to \forall x\,[\neg\alpha]$.

Homework 12.2

Assume that x, y are different variables. Which of the below statements are valid for all choices of α:

- $\forall x\,[\alpha^x_y \to \alpha]$;
- $\forall x\,[\alpha \to \alpha^x_y]$.

Either provide a proof that the formula is valid or a counter example (one choice of α and the corresponding structure and default) where the formula is false.

Homework 12.3

Assume that the logical language contains the predicate symbols P and Q. Make formal proofs for the following facts. You can use the Deduction and the Generalisation Theorems and use axioms of the first group in order to deal with connectives other than $\neg$ and $\to$.

1. $\{P(y)\} \vdash \forall x\,[x = y \to P(x)]$;
2. $\{\forall x\,[x = y \to Q(x)], \forall z\,[Q(z) \to P(z)]\} \vdash P(y)$;
3. $\{\forall x\,[P(x)], \forall x\,[Q(x)]\} \vdash \forall x\,[P(x) \wedge Q(x)]$.

Homework 12.4

Prove the following statement, perhaps by first proving that $\{\forall y\,[\neg(y = f(x))]\}$ is inconsistent and then using that therefore $\neg\forall y\,[\neg(y = f(x))]$ can be proven from $\emptyset$:

$$\emptyset \vdash \forall x\,\exists y\,[y = f(x)].$$

Homework 12.5

If the following sentence is valid then prove it else provide a structure where it is false:

$$\forall x\,\exists y\,[f(f(x)) = y \wedge f(f(y)) = x].$$

Homework 12.6

If the following sentence is valid then prove it else provide a structure where it is false:

$$\exists y\,\forall x\,[y = f(x)] \to \exists y\,\forall x\,[y \neq f(x)].$$

Homework 12.7
Let the logical language contain an unary function f and constants a, b and equality. If the following sentence is valid then prove it else provide a structure where it is false:

$$f(a) \neq f(b) \rightarrow \forall x\, \exists y\, [f(x) \neq f(y)].$$

Homework 12.8
Let S, T be any sets of sentences and let the logical language contain $=$ and one unary function symbol f, but no constants. Furthermore, assume that the theories generated by S, T are both 6-categorical, that is, each of these two theories has up to isomorphism one model of size 6. Is the following true: $S \cup T$ is also 6-categorical and the model is the same as those of S and T alone, respectively. Prove your answer.

Homework 12.9
Let S, T be any sets of sentences. Is the following true: If all infinite structures are models of S and all infinite structures are models of T then all infinite structures are models of $S \cup T$?

Homework 12.10
Is the set

$$\{\forall x\, \forall y\, \exists z\, [x \circ z = y], \forall x\, \forall y\, \exists z'\, [z' \circ x = y],$$
$$\exists x\, \exists y\, \exists z\, [x \circ z = y \wedge z \circ x \neq y], \forall x\, \forall y\, \forall z\, [x \circ (y \circ z) = (x \circ y) \circ z]\}$$

a consistent set of formulas? In other words, is there a structure $(A, \circ)$ such that $\circ$ is associative and for each x, y one can find from each side elements z, z' such that $x \circ z = y$ and $z' \circ x = y$; however, it might be that for some x, y, z with $x \circ z = y$, one has to take a different z' for achieving $z' \circ x = y$.

Homework 12.11
Consider the finite structure with domain $\{0, 1, \ldots, p-1\}$ and multiplication and addition modulo p, besides the usage of $+, \cdot$ the logical language also permits $-$ and the constants $0, 1, \ldots, p-1$. Make in a programming language of your choice a computer program which evaluates to 0 (false) or 1 (true) depending on whether the statement

$$\forall x\, \exists y\, \exists z\, [x = y \cdot y + z \cdot z]$$

is true in the structure and use the program to determine for which of $p = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ the formula is true, that is, for which of these modulo rings is every number the sum of two squares.

Assignments for Week 13

Homework 13.1

Are the following sets of sentences effectively enumerable:

1. $\{\alpha \in T$: every group satisfies $\alpha\}$; 2. $\{\alpha \in T$: every Abelian group satisfies $\alpha\}$?

Here T is the set of all sentences in the logical language with one operation $\circ$ and one constant e and one function f to denote the group operation, neutral element and inversion, respectively.

Homework 13.2

Let the logical language contain exactly one predicate P and no function symbols; the predicate P is unary (one input only). Recall that a sentence is a formula with no free variables. Make a sentence α such that, for each n, there are, up to isomorphism, exactly $n-1$ models of α with n elements.

Homework 13.3

Let the logical language contain the predicates $P_0, P_1, \ldots$ and let Γ for all n, m with $m < n$ contain the following formulas:

$$\exists x\, \forall y\, [P_n(x) \wedge (P_n(y) \rightarrow y = x)],\ \forall x\, [\neg P_n(x) \vee \neg P_m(x)].$$

How many models of finite cardinality, of cardinality $\aleph_0$ and or cardinality $\aleph_1$ does Γ have? Here isomorphic models should not be double counted.

Homework 13.4

Two structures are elementarily equivalent iff they satisfy the same sentences. Is there a structure which is elementarily equivalent to the real numbers with addition and multiplication, but not isomorphic to it? Explain your answer.

Homework 13.5

Assume that two sets of sentences Γ and Δ do not have any structure in common, that is, any structure of Γ fails to satisfy all formulas in Δ and every structure of Δ fails to satisfy all formulas of Γ, but both sets Γ and Δ are consistent. Is there a single sentence α such that all structures of Γ satisfy α and none of Δ does?

Homework 13.6

Let a structure $\mathcal{Z} = (\mathbb{Z}, \ldots, -2, P_{-2}, -1, P_{-1}, 0, P_0, 1, P_1, 2, P_2, \ldots)$ contain all integers and constants for all integers so that if c_n is the constant for n and P_n the predicate for n then $P_n(x)$ is true in the model iff $x \leq c_n$. Note that $\leq$ itself is not part of the logical language. Up to isomorphism, how many countable models are there which are elementarily equivalent to $\mathcal{Z}$? 0 or 1 or … or countably infinite or uncountably infinite models?

Homework 13.7

Let a structure $\mathcal{Q}$ contain the domain $\mathbb{Q}$ and for each rational number q a constant c_q and a predicate P_q such that $P_q(x)$ is true iff $x \leq q$. Note that $\leq$ itself is not part of the logical language. Up to isomorphism, how many countable models are there which are elementarily equivalent to $\mathcal{Q}$? 0 or 1 or ... or countably infinite or uncountably infinite models?

Homework 13.8
Recall that a theory is $\aleph_0$-categorical iff it has an infinite model and every two countable infinite models are isomorphic. Let the logical language have only one unary predicate P and equality $=$. Show that every complete theory of this logical language either has only a finite model or has an infinite model and is $\aleph_0$-categorical.

Homework 13.9
Let $Mod(S)$ denote the set of models of S. Show the following for sets S, T of sentences:

1. If $S \subseteq T$ then $Mod(T) \subseteq Mod(S)$;
2. $Mod(S \cup T) \subseteq Mod(S) \cap Mod(T)$;
3. If $Mod(S) = Mod(T)$ then $Mod(S) = Mod(S \cup T)$.

Homework 13.10
Is there a sentence α such that α has a model with κ members in the domain iff $\kappa = n^2$ for some $n \in \{1, 2, 3, \ldots\}$ or $\kappa \geq \aleph_0$, where the underlying logical language has one unary predicate P and one binary operation $\circ$ (α can use these).

Homework 13.11
Let $(G, \circ, e)$ be a group with 8 elements. Show that every group $(H, \bullet, d)$ which is elementarily equivalent to $(G, \circ, e)$ is also isomorphic to $(G, \circ, e)$.

Homework 13.12
Provide an example of an infinite group $(G, \circ, e)$ such that every group which is elementarily equivalent to $(G, \circ, e)$ and has the same number of elements as $(G, \circ, e)$ is also isomorphic to $(G, \circ, e)$. Hint: Use an Abelian group also satisfying some torsion axiom, say $\forall x\, [x \circ x \circ x = e]$. The number of repetitions of x in the torsion rule should be a prime number.

Homework 13.13
Let the logical language have one unary predicate P and equality. Furthermore, assume that a theory T has for each n an axiom which says that at least n elements x satisfy $P(x)$ and another n elements satisfy $\neg P(x)$. Show that this theory is not $\aleph_1$-categorical and determine the number of models of cardinality $\aleph_1$ it has – note that one can split a set of cardinality $\aleph_1$ into two sets of cardinality κ, λ iff $\max\{\kappa, \lambda\} = \aleph_1$. The cardinals up to $\aleph_1$ are $0, 1, 2, \ldots, \aleph_0, \aleph_1$.

Homework 13.14
Assume that a model $\mathfrak{A}$ has the domain of rational numbers $\mathbb{Q}$ and infinitely many constants $c_k = -2^{-k}$, $d = 0$ and $e_k = 2^{-k}$. The countable models elementarily equivalent to this structure are dense linear orders with the c_k all strictly below d and going strictly upwards and the e_k all being strictly above d and going strictly downwards. Up to isomorphism, how many countable models are there which are elementarily equivalent to $\mathfrak{A}$? Provide of each of these models one isomorphic copy

by saying what the values if c_k, d and e_k are.

Homework 13.15
Prove formally, using the Axioms of Λ, Modus Ponens and copying from the set of preconditions and the Generalisation Theorem the following:

$$\{\exists x\,[P(x)], \forall y[Q(y)]\} \vdash \forall z\,[P(z) \to Q(z)]$$

Selflearning Assignments with Solutions
Homework 14.1
Give an example for a set S of formulas in sentential logic such that

- for all $\alpha, \beta \in S$, the formulas $(\alpha \vee \beta), (\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ and $\neg\neg\alpha$ are also in S;
- for all α, either $\alpha \in S$ or $\neg\alpha$ in S but not both.

Solution. Let ν be such that $\nu(A) = 1$ for all atoms A. Now let

$$S = \{\alpha : \overline{\nu}(\alpha) = 1\}$$

and one can see that due to the definition of $\overline{\nu}$, it is always true that exactly one of $\alpha, \neg\alpha$ are in S. Furthermore, if α, β are in S then $\overline{\nu}$ makes both of them true and it follows that $\overline{\nu}$ also makes the formulas $(\alpha \vee \beta), (\alpha \wedge \beta)$, $(\alpha \rightarrow \beta)$, $(\alpha \leftrightarrow \beta)$ and $\neg\neg\alpha$ true, thus they are also in S.

Homework 14.2
How many Boolean functions of the form B^n_α can be built where α uses the atoms $A_1, \ldots, A_n$ and combines them either with $\wedge$ or with $\rightarrow$? Other connectives and logical constants are not allowed. List out the numbers of functions for $n = 1, 2, 3$.

Solution. The number of functions is 2^{2^n-1}. Note that $1 \rightarrow 1$ and $1 \wedge 1$ both evaluate to 1. Thus if $A_1, \ldots, A_n$ are all 1 then the output is 1. If at least one of them is 0, then $A_1 \wedge A_2 \wedge \ldots \wedge A_n$ has the value 0 and one can use this as a replacement for the constant 0; furthermore, $\neg\alpha$ is then realisable by the formulas $F(\alpha) = (\alpha \rightarrow (A_1 \wedge A_2 \wedge \ldots \wedge A_n))$. If at least one of the atoms is 0 then $F(\alpha)$ has the value $\neg\alpha$ else α has the value 1. It follows that one can translate a formula β using only $\wedge$ and $\neg$ – and every Boolean function in n variables can be represented by such a formula – and then one replaces in this formula all subformulas $\neg\alpha$ by $F(\alpha)$ until one gets a formula γ which only contains $\rightarrow$ and $\wedge$ and atoms. Now it holds that

$$B^n_\gamma = F^n_{\beta \vee (A_1 \wedge A_2 \wedge \ldots \wedge A_n)}$$

and therefore the given Boolean function is only changed to 1 in the case that all inputs are 1. For all other input-vectors, the original value is maintained. Thus one can choose $2^n - 1$ values freely and make the corresponding β and the number of such $\{0, 1\}$-valued functions is 2^{2^n-1}. For $n = 1, 2, 3, 4$ these values are $2, 8, 128, 32768$. For $n = 1$, the two functions are the identity-function $B^1_{A_1}$ and the constant-1-function $B^1_{A_1 \rightarrow A_1}$.

Homework 14.3
Prove by induction that for every formula using only $\oplus$, $\leftrightarrow$ and $\neg$ as connectives, which is built from the atoms $A_1, A_2, \ldots, A_n$, either all possible assignments of these n values or half of them or none of them evaluates the formula to true.

Solution. What one is proving by induction is the following statement: Given a formula α using the above indicated connectives, one defines $Depend(\alpha)$ to be the set of all atoms A such that there are ν, μ assigning values to the atoms different only on A_k with $\overline{\nu}(A_k) \neq \overline{\mu}(A_k)$. One shows now by induction the following statement for formulas α of the given type:

(*) If ν, μ differ exactly on A_k and $A_k \in Depend(\alpha)$ then $\overline{\nu}(\alpha) = \neg\overline{\mu}(\alpha)$.

To see (*), first note that for constants, the sets $Depends(0)$ and $Depends(1)$ are empty and thus the statement is true; furthermore, if $\alpha = A_k$ then $Depend(A_k) = A_k$ and it is obvious that if ν, μ differ on A_k then $\overline{\nu}(A_k) = \neg\overline{\mu}(A_k)$. Now for an induction, consider α, β which are satisfying (*):

1. Consider $\gamma = \neg\alpha$. One let $Depend(\gamma) = Depend(\alpha)$ and considers any ν, μ which differ only in one atom A_k: If $A_k \in Depend(\alpha)$ then $\overline{\nu}(\gamma) = \neg\overline{\nu}(\alpha) = \neg\neg\overline{\mu}(\alpha) = \neg\overline{\mu}(\gamma)$; If $A_k \notin Depend(\alpha)$ then $\overline{\nu}(\gamma) = \neg\overline{\nu}(\alpha) = \neg\overline{\mu}(\alpha) = \overline{\mu}(\gamma)$. So for ν, μ only differing in A_k, $\overline{\nu}(\gamma) = \neg\overline{\mu}(\gamma)$ iff $A_k \in Depend(\gamma)$.

2. Consider $\gamma = \alpha \oplus \beta$. One let $Depend(\gamma)$ be the symmetric difference of $Depend(\alpha)$ and $Depend(\beta)$, that is, contain exactly those atoms which are in exactly one of the sets $Depend(\alpha)$ and $Depend(\beta)$. Consider any ν, μ which differ only in one atom A_k: If $A_k \in Depend(\gamma)$ then A_k is exactly in one of the sets $Depend(\alpha)$ and $Depend(\beta)$, without loss of generality say in the first. Now $\overline{\nu}(\alpha)$ and $\overline{\mu}(\alpha)$ differ while $\overline{\nu}(\beta)$ and $\overline{\mu}(\beta)$ are the same. It follows that one of $\overline{\nu}(\alpha \oplus \beta)$, $\overline{\mu}(\alpha \oplus \beta)$ is $0 \oplus c$ while the other one is $1 \oplus c$, where $c \in \{0, 1\}$. Hence $\overline{\nu}(\gamma) = \neg\overline{\mu}(\gamma)$. If $A_k \notin Depend(\gamma)$ by $A_k \notin Depend(\alpha), A_k \notin Depend(\beta)$ then $\overline{\nu}(\alpha) = \overline{\mu}(\alpha)$, $\overline{\nu}(\beta) = \overline{\mu}(\beta)$ and $\overline{\nu}(\gamma) = \overline{\nu}(\alpha \oplus \beta) = \overline{\mu}(\alpha \oplus \beta) = \overline{\mu}(\gamma)$. If $A_k \notin Depend(\gamma)$ by $A_k \in Depend(\alpha), A_k \in Depend(\beta)$ then $\overline{\nu}(\alpha) = \neg\overline{\mu}(\alpha)$, $\overline{\nu}(\beta) = \neg\overline{\mu}(\beta)$ and $\overline{\nu}(\gamma) = \overline{\nu}(\alpha \oplus \beta) = \overline{\mu}(\neg\alpha \oplus \neg\beta) = \overline{\mu}(\alpha \oplus \beta) = \overline{\mu}(\gamma)$. So for ν, μ only differing in A_k, $\overline{\nu}(\gamma) = \neg\overline{\mu}(\gamma)$ iff $A_k \in Depend(\gamma)$.

3. If $\gamma = \alpha \leftrightarrow \beta$ then again $Depend(\gamma)$ is the symmetric difference of $Depend(\alpha)$ and $Depend(\beta)$. The proof in this case is the same as in the case of $\oplus$; alternatively, one could also replace $\alpha \leftrightarrow \beta$ by $\neg(\alpha \oplus \beta)$ and do the two prior inductive steps.

Thus the induction gives that for each formula α there are two cases: Either $Depend(\alpha) = \emptyset$ and then the truth-table of α assigns in all rows the same value or $Depend(\alpha)$ contains at least one atom A_k and if one puts this atom A_k into the last column of the truth-table and on can group the rows in pairs of rows where the truth-entries differ only for A_k and thus one of these rows carries the value 0 while the other one carries the value 1; hence half of the rows has a 0 and half has a 1. Here an example for $A_h \oplus \neg(A_k \oplus A_h)$:

A_h	A_k	$A_h \oplus \neg(A_k \oplus A_h)$
0	0	1
0	1	0
1	0	1
1	1	0

Homework 14.4
Make a formal proof that

$$\forall x \, \forall y \, [\alpha \rightarrow \beta] \rightarrow \forall y \, \forall x \, [\neg\beta \rightarrow \neg\alpha]$$

is a valid formula.

Solution. Recall that tautologies in sentential logic can be made to axioms in first-order logic by replacing the atoms by logical symbols; furthermore, any formula of the form $\forall x \, [\gamma] \rightarrow \gamma$ is an axiom. Thus one can make the following proof.

1. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \forall x \, \forall y \, [\alpha \rightarrow \beta]$ (Copy);
2. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \forall x \, \forall y \, [\alpha \rightarrow \beta] \rightarrow \forall y \, [\alpha \rightarrow \beta]$ (Axiom Group 2);
3. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \forall y \, [\alpha \rightarrow \beta]$ (Modus Ponens);
4. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \forall y \, [\alpha \rightarrow \beta] \rightarrow \alpha \rightarrow \beta$ (Axiom Group 2);
5. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \alpha \rightarrow \beta$ (Modus Ponens);
6. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash (\alpha \rightarrow \beta) \rightarrow (\neg\beta \rightarrow \neg\alpha)$ (Axiom Group 1);
7. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \neg\beta \rightarrow \neg\alpha$ (Modus Ponens);
8. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \forall x \, [\neg\beta \rightarrow \neg\alpha]$ (Generalisation Theorem);
9. $\{\forall x \, \forall y \, [\alpha \rightarrow \beta]\} \vdash \forall y \, \forall x \, [\neg\beta \rightarrow \neg\alpha]$ (Generalisation Theorem);
10. $\emptyset \vdash \forall x \, \forall y \, [\alpha \rightarrow \beta] \rightarrow \forall y \, \forall x \, [\neg\beta \rightarrow \neg\alpha]$ (Deduction Theorem).

Homework 14.5
Is the statement

$$\{Px \rightarrow Py\} \vdash \forall z \, [Px \rightarrow Pz]$$

correct? If so, make a formal proof, if not, make a model with default values of the variables for which it is false.

Solution. This statement is not correct. Assume that a model is given with variable defaults, that the model has at least the values $0, 1$, that $P(x)$ is equivalent to $x = 0$ and that x, y have the default value 0. Then $Px \rightarrow Py$ is true and $Px \rightarrow P1$ is false; in particular $\forall z \, [Px \rightarrow Pz]$ is false.

Homework 14.6
Which of the following statements can be proven? If so, then give the formal prove, else explain why one cannot do it.

(a) $\{x = 0\} \vdash \forall x \, [x = 0]$;

(b) $\{\forall y \, [y = 0]\} \vdash \forall x \, [x = 0]$;

(c) $\{\forall y\, \forall x\, [x = y]\} \vdash \forall x\, [x = 0]$.

Solution. Statement (a) does not hold. The reason is that one can consider the model $(\mathbb{N}, 0)$ and then one sees, that a variable assignment can make $x = 0$ true while the conclusion that all values in the model equal to 0 is false. Note that the Generalisation Theorem can only be applied if the variable x does not occur free in the precondition.

Statement (b) can be proven along the same lines as one can prove the Principle of Alphabetical Variants, indeed, it follows from this principle directly. A formal proof, using only Axioms and the Generalisation Theorem, is the following:

1. $\{\forall y\, [y = 0]\} \vdash \forall y\, [y = 0]$ (Copy);
2. $\{\forall y\, [y = 0]\} \vdash \forall y\, [y = 0] \rightarrow x = 0$ (Axiom 2);
3. $\{\forall y\, [y = 0]\} \vdash x = 0$ (Modus Ponens);
4. $\{\forall y\, [y = 0]\} \vdash \forall x\, [x = 0]$ (Generalisation Theorem).

Here the Generalisation Theorem can be applied, as the variable x does not occur, actually does not occur at all, in the preconditions.

Statement (c) can also be proven and the proof is even easier, as one only needs axioms from Λ:

1. $\{\forall y\, \forall x\, [x = y]\} \vdash \forall y\, \forall x\, [x = y]$ (Copy);
2. $\{\forall y\, \forall x\, [x = y]\} \vdash \forall y\, \forall x\, [x = y] \rightarrow \forall x\, [x = 0]$ (Axiom 2);
3. $\{\forall y\, \forall x\, [x = y]\} \vdash \forall x\, [x = 0]$ (Modus Ponens).

Note that in formal proofs, the axioms from Λ and the copying from the preconditions and the usage of Modus Ponens are always allowed.

Homework 14.7
Choose a logical language and a theory T in this language such that

- T is finite axiomatisable;
- T is $\aleph_0$-categorical and $\aleph_1$-categorical;
- T has a finite model of m elements iff $m = 3^n$ for some n.

Furthermore, is T complete? Explain your answer.

Solution. The idea is to use the language of Abelian groups where an element three times added to itself gives 0. These structures are equivalent to vector spaces over $\mathbb{F}_3$ and it is known from linear algebra that each two such vector spaces are isomorphic iff they are vector spaces over the same field and their bases have the same cardinality. Note that scalar multiplication over $\mathbb{F}_3$ with 0 gives the constant 0 function and with

1 gives the identity function and with 2 gives the sum of an element with itself. Thus one can define scalar multiplication by cases and does not need to incorporate it into the logical language. So the only symbols added into the language are 0 (neutral element) and + (addition modulo 3 in a vector space). The axioms postulated are now the following ones:

1. $\forall x\, \forall y\, \forall z\, [(x + y) + z = x + (y + z)]$;
2. $\forall x\, \forall y\, [x + y = y + x]$;
3. $\forall x\, [x + 0 = x]$;
4. $\forall x\, [x + (x + x) = 0]$.

Now, if κ is an infinite cardinal then, by results of linear algebra, a vector space over $\mathbb{F}_3$ has a basis of cardinality κ iff the vector space itself has cardinality κ; thus every such vector space and, therefore, also every structure satisfying the above axioms is κ-categorical; in particular these structures are $\aleph_0$-categorical and $\aleph_1$-categorical. Furthermore, the finite vector spaces of dimension n have all 3^n elements and every finite vector space has a finite basis (dimension). It is not needed for this homework to reprove the facts known from basic lectures like linear algebra.

Homework 14.8 Consider the logical language containing one unary function f and the set

$$S = \{\forall x\, \forall y\, [x \neq y \rightarrow f(x) \neq f(y)],\ \forall x\, [x \neq f(x)],\ \forall x\, \exists y\, [f(y) = x]\}$$

and let $Th(S)$ be the set of all sentences which can be proven from S. Check whether the $Th(S)$ is 5-categorical, that is, whether all models of cardinality 5 of $Th(S)$ are isomormphic. Provide all models for $\kappa = 5$ and check to which κ this generalises.

Solution. The answer is that $Th(S)$ is not 5-categorical. There are two models, (a) the model of a 5-cycle and (b) the model of a 2-cycle plus a 3-cycle. So if one calls the elements $0, 1, 2, 3, 4$ and makes tables of f in the two models, the tables are the following:

Inputs	0	1	2	3	4
f in Model (a)	1	2	3	4	0
f in Model (b)	1	0	3	4	2

For other small κ, note that the theory is not 1-categorical, as it has no model of size 1. The theory is 2-categorical and 3-categorical, as these sizes permit only one cycle and that cycle is of length κ. The theory is not κ-categorical for any $\kappa \geq 4$, as one can make, for finite κ, (a) one κ-cycle and (b) one cycle of length 2 and one of length $\kappa - 2$ and for $\kappa \geq 6$ one can also further models. For infinite κ, it is not κ-categorical as one can make, for any $n \geq 2$, a model consisting of κ n-cycles.

Homework 14.9

Assume that the logical language contains one unary function f and equality $=$.

Provide two sentences α and β such that the theories $Th(\{\alpha\})$ and $Th(\{\beta\})$ are κ-categorical for all $\kappa \geq 1$ and such that $Th(\{\alpha, \beta\})$ is complete and therefore either κ-categorical for exactly one finite κ or κ-categorical for exactly the infinite κ.

Solution.
The idea is to choose a finite κ such that the models of α and β coincide for exactly this κ and not for any other κ. Here one chooses $\kappa = 1$, as that is most easy to handle. Now the formulas are as follows: α is $\forall x \, \forall y \, [f(x) = f(y)]$; β is $\forall x \, [f(x) = x]$. So α requires that the function is constant and β requires that it is the identity. This can be combined if and only if the domain has exactly one element. Note that for $Th(\{\alpha\})$, two models of the same size are isomorphic, the isomorphism maps the elements in the two ranges to each other, as they are unique, and maps the other elements in a one-one way to each other. For two models of $Th(\{\beta\})$ of the same size, any bijection is an isomorphism, as the image of the identity-function is again the identity-function. $Th(\{\alpha, \beta\})$ has the unique model $(\{0\}, f, =)$ with $f(0) = 0$ (up to isomorphism) and therefore $Th(\{\alpha, \beta\})$ is complete. That all elements are equal in any model of $\{\alpha, \beta\}$ can be seen by the fact that given x, y, β implies that $x = f(x)$ and $y = f(y)$ and α implies that $f(x) = f(y)$ and thus $x = y$, so any two elements of the model are the same.

Please review also the old exams on the course homepage and other material available and read the course notes.