On Martin's Pointed Tree Theorem

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Abstract. We investigate the reverse mathematics strength of Martin's pointed tree theorem (MPT) and one of its variants, weak Martin's pointed tree theorem (wMPT).

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1. Introduction

A set A of reals is cofinal if for every real x there is some $y \in A$ so that $y \geq_T x$. A *pointed* tree T is a perfect tree so that for every infinite path $x \in [T]$, $T \leq_T x$. In 1968, Martin [9] proved the following theorem.

Theorem 1.1 (Martin [9]). *Assume that every game is determined. Then given any set $A \subseteq \omega^\omega$, if A is cofinal, then there is a pointed tree $[T] \subseteq A$.*

Theorem 1.1 builds a significant connection between descriptive set theory and recursion theory. It has been a central goal in descriptive set theory to prove lower bounds on the consistency strength of some descriptive set theory theorems (see, for example, Harrington [4] or Koellner and Woodin [7]). Despite the seemingly simple form of Theorem 1.1, the proof of its consistency relative to ZF requires the existence of infinitely many Woodin cardinals; which is far beyond the strength of ZF. One of the reasons for the importance of Theorem 1.1 is that it was used by Slaman and Steel [18] as a critical tool in their study of Martin's conjecture, one of the central open problems in recursion theory. We study a natural version of Martin's theorem and a variant, wMPT, before a recursion theory background.

Definition 1.2. *Martin's pointed tree theorem*, MPT, states that given a tree $T \subseteq \omega^{<\omega}$, if $[T]$ is cofinal, then T has a pointed subtree.

Definition 1.3. *Weak Martin's pointed tree theorem*, wMPT, states that given a tree $T \subseteq \omega^{<\omega}$, if $[T]$ is cofinal, then T has a perfect subtree.

In this paper, we are mainly interested in the reverse mathematical strengths of these statements.

Reverse mathematics is used to gauge the complexity of mathematical theorems by determining precisely which axioms are needed to prove a given theorem. For example, Martin and Steel [10] proved the conclusion of Theorem 1.1 under the hypothesis that there are infinitely many Woodin cardinals. A typical question in reverse mathematics would be whether this hypothesis is necessary for the proof of a given statement; and in the case of Theorem 1.1 it indeed is, as shown by Koellner and Woodin [7].

When studying reverse mathematics before a recursion theory background one often focuses on second order arithmetical theories. In other words one focuses on the question of which theorems can be proven assuming only a certain subset of second order arithmetical axioms. In this context, there exists a group of sets of axioms, the so-called *big five* (see Simpson [17]). These sets of axioms are distinguished from others in that most of the mathematical theorems whose reverse mathematical strength has been classified so far have turned out to be equivalent to one of these five. This is why most researchers in the area consider the *big five* systems to be of central importance for the subject, and why they have received much attention. In this paper, we use the *big five* to measure the strength of MPT and wMPT.

Based on Martin's proof of Theorem 1.1, it seems that for a proof of MPT we need $\mathbf{\Delta}^0_4$-DET. However, using results from [5] and [12], we show that over ACA_0 we have that MPT is equivalent to ATR_0. As a consequence, the question arises whether we can prove the same equivalence over weaker axiom systems. We prove that, over RCA_0, MPT does not even imply WKL_0. So MPT can be viewed as a natural theorem incomparable with WKL_0 and ACA_0 but "joining" ACA_0 to ATR_0. However the question of whether the equivalence can be proven over WKL_0 remains open.

wMPT is obviously implied by MPT. It is also related to other classical results in descriptive set theory. The perfect set theorem (PST) says that every uncountable set has a perfect subtree. In the reverse mathematics setting PST corresponds to RPST, the statement that *every uncountable closed set has a perfect subset*. A natural analogue of wMPT in descriptive set theory is the statement that *every cofinal set has a perfect subset* (CPST). It turns out (see Solovay [19] and Chong and Yu [2]) that PST and CPST have the same consistency strength. Obviously RPST also implies wMPT. Simpson [17] shows that, over ACA_0, RPST is equivalent to ATR_0. However, we prove that wMPT is strictly weaker than ATR_0 (and therefore than RPST) and incomparable with WKL_0 and ACA_0. Hence we have two mathematical statements which have the same consistency strength but different reverse mathematics strength. Another interesting conclusion is that wMPT can be viewed as a natural example of a theorem that is not equivalent to any of the *big five*.

We also would like to point out the interesting technique used to prove Theorem 4.2, which states that, over WKL_0, wMPT does not imply ACA_0. It demonstrates a natural application of algorithmic randomness theory to reverse mathematics. In fact, in this article, in all proofs where genericity was used it could be replaced with randomness; we keep the genericity because it simplifies the proofs. However, the use of randomness in the proof of Theorem 4.2 seems necessary since no generic real can be hyperimmune-free.

The article is organized as follows: In section 2 we review some background knowledge; in section 3 we investigate MPT and wMPT over RCA_0; in section 4 we investigate them over WKL_0; and in section 5 over ACA_0.

2. Preliminaries

2.1. General notations

For every real $x \in \omega^\omega$, let $\boldsymbol{x}$ be the Turing degree of x.

If Φ is a Turing functional, then we use $\Phi^{x\upharpoonright n}[m]$ to denote the finite string computed from oracle x at stage m with use n.

If $x \leq_{\mathrm{T}} y$ via a total Turing functional, then we write $x \leq_{\mathrm{tt}} y$ and say that x is truth-table reducible to y.

We refer the reader to Lerman [8] and Odifreddi [15] for more recursion theoretical background.

Given a tree T, we use $[T]$ to denote the collection of the infinite paths through T. For a finite string σ, let $[\sigma]$ denote the collection of reals extending σ.

A set $D \subseteq 2^{<\omega}$ is *dense* if for every $\sigma \in 2^{<\omega}$ there is some $\tau \in D$ so that $\tau \succ \sigma$, i.e., τ is an extension of σ.

A real $g \in 2^\omega$ is *arithmetically generic*, if for every arithmetical dense set $D \subseteq 2^{<\omega}$ there is some n so that either

- $g \upharpoonright n \in D$, or
- $\forall \sigma\, (\sigma \succ g \upharpoonright n \to \sigma \notin D)$.

A real x is *hyperimmune-free* if every x-recursive function is dominated by a recursive function. It is obvious that if x is hyperimmune-free and $y \leq_{\mathrm{T}} x$, then $y \leq_{\mathrm{tt}} x$. By Jockusch and Soare's Hyperimmune-Free Basis Theorem [6], every nonempty Π^0_1 subset of 2^ω contains a hyperimmune-free real.

We say that $x \gg y$ if there is a real $z \leq_{\mathrm{T}} x$ in 2^ω so that $\forall e (z(e) \neq \Phi^y_e(e))$. There is a nonempty Π^0_1 subset of 2^ω in which every real x has the property $x \gg \emptyset$. So by the Hyperimmune-Free Basis Theorem there is a hyperimmune-free real $x \gg \emptyset$.

A partial function $p\colon \omega \to \omega$ is recursively bounded if there is a recursive function $f\colon \omega \to \omega$ so that for every n, $p(n)\downarrow \to p(n) < f(n)$. The following result should be well known.

Proposition 2.1 (Folklore). *If $x \gg \emptyset$, then for every partial recursive function $\Psi\colon \omega \to \omega$ which is recursively bounded, there is a total x-recursive function g extending Ψ.*

Proof. Let a partial recursive function $\Psi\colon \omega \to \omega$ with a recursive bound f be given. Define a partial recursive function $\Xi\colon \omega^2 \to 2$ so that

$$\Xi(e, \langle n, m\rangle) = \begin{cases} 1 & \Psi(n)\downarrow = m, \\ 0 & \exists k < f(n)(k \neq m \wedge \Psi(n)\downarrow = k), \\ \uparrow & \text{otherwise.} \end{cases}$$

By the *s-m-n*-Theorem, there is a recursive function h so that

$$\Xi(e, \langle n, m\rangle) = \Phi_{h(\langle n,m\rangle)}(e).$$

Let $z \leq_{\mathrm{T}} x$ be in 2^ω so that $\forall e (z(e) \neq \Phi_e(e))$. Define $g(n) = m$ if m is the least number $< f(n)$ so that $z(h(\langle n, m\rangle)) = 0$ if any; otherwise let $g(n) = 0$. Obviously g is a total x-recursive function extending Ψ. $\square$

Note that for every recursive tree $T \subseteq 2^{<\omega}$ and real $x \gg \emptyset$, if $[T]$ is not empty, then there must be some $y \leq_{\mathrm{T}} x$ such that $y \in [T]$.

We need the following technical lemma to build a perfect tree by projection. For $\sigma, \tau \in \omega^n$ we define the notation $\sigma \oplus \tau$ as $\sigma(0)\tau(0)\sigma(1)\tau(1)\ldots\sigma(n)\tau(n)$. The motivation behind this lemma and notation is that we will later work with game-theoretic trees, for games with two players making moves alternately.

Lemma 2.2. *Suppose that T is a perfect tree so that for every $f_0 \oplus g_0, f_1 \oplus g_1 \in [T]$, if $g_0 = g_1$, then $f_0 = f_1$. Then there is a perfect tree $S_1 \leq_{\mathrm{T}} T$ so that for every infinite path $g \in [S_1]$, there is some f so that $f \oplus g \in [T]$.*

Proof. Fix a tree T as in the assumption. We T-recursively build a helper tree S_0 and the desired tree S_1 stage by stage.

At stage 0, let $\varepsilon \oplus \varepsilon = \varepsilon \in S_0$ and $\varepsilon \in S_1$.

We assume that at stage n, for every $\sigma \oplus \tau \in S_0$, there are $\tau_0 | \tau_1$ extending τ so that there are σ_0, σ_1 extending σ so that $\sigma_0 \oplus \tau_0, \sigma_1 \oplus \tau_1 \in T$.

We claim that the assumption holds at stage 0. Since T is perfect, there must exist two distinct paths $f_0 \oplus g_0$, $f_1 \oplus g_1 \in [T]$. If we always have $g_0 = g_1$ then, by the assumption on T, f_0 must be equal to f_1, which is a contradiction to the fact that T is perfect. So fix g_0, g_1 and some n such that $g_0(n) \neq g_1(n)$. Write $\tau_0 = g_0 \upharpoonright n+1$, $\tau_1 = g_1 \upharpoonright n+1$, $\sigma_0 = f_0 \upharpoonright n+1$, and $\sigma_1 = f_1 \upharpoonright n+1$.

At stage $n+1$, for every leaf $\sigma \oplus \tau \in S_0$, select $\sigma_0 \oplus \tau_0, \sigma_1 \oplus \tau_1 \in T$ as defined above and put them into S_0. Also put τ_0, τ_1 into S_1. With the same argument as above for stage 0 the assumption remains true at stage $n+1$.

Now it is clear that S_1 is a perfect tree. Suppose that $g \in [S_1]$, then there must be $\sigma_0 \oplus \tau_0 \prec \sigma_1 \oplus \tau_1 \prec \ldots$ constructed at stages $0, 1, \ldots$, respectively, so that $\tau_0 \prec \tau_1 \prec \cdots \prec g$. Let $f = \bigcup_{i \in \omega} \sigma_i$. Then $f \oplus g \in [T]$. Thus S_1 is as required. $\square$

2.2. Reverse mathematics

We refer to Simpson [17] for the background on reverse mathematics. We recall that RCA_0 is the most basic axiom system for the second order arithmetical theory. The axioms in the stronger system WKL_0 state that every infinite binary tree has an infinite path. The even stronger system ACA_0 includes all arithmetical comprehension axioms, and ATR_0 ensures that arithmetical transfinite recursion is allowed. Together with $\mathbf{\Pi}^1_1$-CA_0, these systems form the famous *big five* hierarchy.

Given a theory T and a proposition φ, to show that $T \nvdash \varphi$, we will use model-theoretical arguments. A model $\mathscr{M}$ for the second order arithmetical language has the form $(N, M, 0, 1, +, \times, <, \in)$, which is a two-sorted model. The first sort N contains the natural numbers, and the second sort M is the family of subsets of N that exist in the model at hand. In this paper, we will always have $N = \omega$ and $M \subseteq 2^\omega$, that is, we only focus on so-called ω-models.

2.3. Game theory

We recall the basic game theoretical notions used in this article, and refer to Moschovakis [13] for more details. Given a set $A \subseteq \omega^\omega$, we define an infinite game G_A with perfect information as follows: The game has two players labelled **I** and **II**. The game is played by letting the players choose natural numbers alternately for ω-many steps. Each game generates a real $x = (n_0, m_0, \dots, n_i, m_i, \dots) \in \omega^\omega$ where n_i and m_i are the numbers played by **I** and **II**, respectively, at their i-th move. If $x \in A$, then **I** wins the game. Otherwise, **II** wins.

A strategy is a function $h\colon \omega^{<\omega} \to \omega$. For a set $A \subseteq \omega^\omega$ and the corresponding game G_A, if h is **I**'s strategy and **II** plays g, then as usual $h * g$ denotes the outcome generated by h and g. If h is **II**'s strategy and **I** plays f, then $f * h$ denotes the outcome generated by h and f.

I has a *winning* strategy h for the game G_A if for every $g \in \omega^\omega$, the real $h * g \in A$. **II** has a *winning* strategy h for the game G_A if for every $f \in \omega^\omega$, the real $f * h \notin A$.

A game G_A is *determined* if either **I** or **II** has a winning strategy. Given a class $\Gamma \subseteq \mathscr{P}(\omega^\omega)$, Γ-DET says that G_A is determined for every $A \in \Gamma$.

The following remarkable connection between game theory and recursion theory was established by Martin. Call a set A of reals *Turing-invariant* if for every $x \in A$, $y \equiv_\mathrm{T} x$ implies $y \in A$.

Theorem 2.3 (Martin [9]). *Assume every set is determined. Then every set A of reals that is Turing-invariant and cofinal contains an upper cone with respect to Turing reducibility.*

Γ-TD says that if an $A \in \Gamma$ which is Turing-invariant is also cofinal, then it contains an upper cone of Turing degrees.

The connection between game theory and reverse mathematics was initiated by Blass and Steel.

Theorem 2.4 (Blass [1] and Steel [20]). *Over* RCA_0, ATR_0 *implies* $\mathbf{\Pi}^0_1$-DET.

2.4. Algorithmic randomness

In this article, the theory of algorithmic randomness will only serve as a tool. We refer the reader to Nies [14] and Downey and Hirschfeldt [3] for details on the topic.

Given a Turing machine M, define its Kolmogorov complexity function as

$$C_M(\sigma) = \min\{|\tau| \mid M(\tau) = \sigma\}.$$

The universal Turing machine U induces an optimal Kolmogorov complexity function up to a constant. That is, for every Turing machine M, there is a constant c_M so that $\forall \sigma (C_U(\sigma) \leq C_M(\sigma) + c_M)$. Usually, U is fixed and the subscript omitted.

For the purposes of this article we use the following definition of randomness which, by a result of Miller and Yu [11], is equivalent to the classical definition of Martin-Löf randomness (see, e.g., Downey and Hirschfeldt [3]): A real $x \in 2^\omega$ is *random* if for every recursive function $f\colon \omega \to \omega$ with

$$\sum_{n \in \omega} 2^{-f(n)} < \infty,$$

there is a constant c so that for all n, $C(x\upharpoonright n) \geq n - f(n) - c$. In particular, if x is random, then there exists a c such that for all n, $C(x\upharpoonright n) \geq \frac{n}{2} - c$. There is a nonempty Π^0_1 set that only contains random reals. So if $x \gg \emptyset$, then x computes a random real.

3. Over RCA_0

Lemma 3.1. *Suppose that $T \subseteq \omega^{<\omega}$ is a recursive tree so that there is a nonrecursive infinite path $x \in [T]$ that is Turing-below some arithmetically generic real $g \in 2^\omega$. Then T has a recursive perfect subtree.*

Proof. Suppose that T is a recursive tree with a nonrecursive infinite path $x \in [T]$ so that $x = \Phi^g$ for some arithmetically generic real g.

Let $D_0 = \{\sigma \mid \Phi^\sigma \notin T\}$. Clearly, D_0 is arithmetical. Since g is arithmetically generic and $\Phi^g \in [T]$, there must be some n_0 so that for every $\sigma \succ g\upharpoonright n_0$, $\sigma \notin D_0$ and so $\Phi^\sigma \in T$. Let

$$D_1 = \left\{ \sigma \;\middle|\; \begin{array}{c} \sigma \succ g\upharpoonright n_0 \wedge \exists n \forall m \geq n \forall \tau_0 \succ \sigma \forall \tau_1 \succ \sigma : \\ (\Phi^{\tau_0}(m)\downarrow \wedge \Phi^{\tau_1}(m)\downarrow \rightarrow \Phi^{\tau_0}(m) = \Phi^{\tau_1}(m)) \end{array} \right\}.$$

Since g is arithmetically generic and Φ^g is not recursive, there must be some $n_1 \geq n_0$ so that for every $\sigma \succ g\upharpoonright n_1$, $\sigma \notin D_1$. So for every $\sigma \succ g\upharpoonright n_1$ and n, there are $\tau_0 \succ \sigma$, $\tau_1 \succ \sigma$ and $m > n$ so that $\Phi^{\tau_0}(m) \neq \Phi^{\tau_1}(m)$.

Now, by using this property of D_1, it is routine to construct a recursive perfect tree $S \subseteq [g\upharpoonright n_1]$ so that the the set $T_1 = \{\Phi^\sigma \mid \sigma \in S\}$ is also a recursive perfect tree. By the property of n_0, $T_1 \subseteq T$. □

To construct a model satisfying MPT, we have to relativize Lemma 3.1 accordingly.

Given a tree $T \subseteq \omega^{<\omega}$, a real z and an index i, let $\mathscr{HK}_i(z,T)$ be a $z \oplus T$-recursive tree so that

$$[\mathscr{HK}_i(z,T)] = \left\{ f \oplus g \;\middle|\; \begin{array}{c} g \in [T] \wedge \forall n (f(n) \text{ is the least } m \\ \text{with } \Phi_i^{g\upharpoonright m}(n)[m]\downarrow \wedge \Phi_i^g(n) = z(n)) \end{array} \right\}.$$

The idea of $\mathscr{HK}_i(z,T)$ originates from Harrington and Kechris [5].

Obviously $\mathscr{HK}_i(z,T) \leq_{\mathrm{T}} T \oplus z$. Note that if Φ_i^g is undefined or different from z then no path of the form $f \oplus g$ can be contained in $\mathscr{HK}_i(z,T)$. As a consequence, if $z \geq_{\mathrm{T}} T$, then it holds for every $f \oplus g \in [\mathscr{HK}_i(z,T)]$ that $g \in [T]$ and $g \geq_{\mathrm{T}} z \geq_{\mathrm{T}} z \oplus T \geq_{\mathrm{T}} \mathscr{HK}_i(z,T)$.

Theorem 3.2. *Over* RCA_0, MPT *does not imply* WKL_0.

Proof. Choose a sequence $g_0, g_1, g_2, \dots$ of elements of 2^ω such that g_0 is arithmetically generic and such that for all n, g_{n+1} is arithmetically generic relative to $g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_n$. Let $\mathscr{M} = (\omega, M, \dots)$, where $M = \{x \mid \exists n (x \leq_{\mathrm{T}} \oplus_{i \leq n} g_i)\}$. Obviously $\mathscr{M} \not\models \mathrm{WKL}_0$, as WKL_0 would guarantee the existence of PA-complete sets, but no such sets can exist in $\mathscr{M}$ as generics do not compute PA-complete sets (see, e.g., Downey and Hirschfeldt [3, Corollary 2.24.6]).

Now assume we are given a tree T that is cofinal in M. Then there must be some $x \in [T]$ and some n so that $T \leq_{\mathrm{T}} g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_n <_{\mathrm{T}} x$. Let j be an index of the second reduction, that is, $\Phi_j^x = g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_n$. On the other hand there must exist some $m > 0$ so that $x \leq_{\mathrm{T}} \oplus_{i \leq m+n} g_i$. Note that $\oplus_{n < i \leq m+n} g_i$ is $\oplus_{i \leq n} g_i$-arithmetically generic. Then $\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)$ is an $\oplus_{i \leq n} g_i$-recursive tree such that

1. for every $f \oplus g \in [\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)]$, $g \in [T]$ and $g \geq_{\mathrm{T}} \oplus_{i \leq n} g_i \oplus T \geq_{\mathrm{T}} \mathscr{HK}_j(\oplus_{i \leq n} g_i, T)$; and
2. there is an $f \oplus g \in [\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)]$, for example $f \oplus x$ for some f, so that

$$\oplus_{i \leq m+n} g_i \geq_{\mathrm{T}} f \oplus g >_{\mathrm{T}} \oplus_{i \leq n} g_i \geq_{\mathrm{T}} \mathscr{HK}_j(\oplus_{i \leq n} g_i, T).$$

Note that $\oplus_{n < i \leq m+n} g_i$ is $[\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)]$-arithmetically generic. Relativizing Lemma 3.1 to $\oplus_{i \leq n} g_i$, we apply it to $[\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)]$ to obtain a $\oplus_{i \leq n} g_i$-recursive perfect tree $S \subseteq \mathscr{HK}_j(\oplus_{i \leq n} g_i, T)$. By the property (1) of $\mathscr{HK}(\oplus_{i \leq n} g_i, T)$, S is pointed.

Note that it follows directly from the definition of $\mathscr{HK}_{(.)}(.,.)$ that if $f_0 \oplus g \in [\mathscr{HK}_j(\oplus_{i\leq n} g_i, T)]$ and $f_1 \oplus g \in [\mathscr{HK}_j(\oplus_{i\leq n} g_i, T)]$ then $f_0 = f_1$. In particular this is true for $f_0 \oplus g$ and $f_1 \oplus g \in S \subseteq \mathscr{HK}_j(\oplus_{i\leq n} g_i, T)$. Since S is a perfect tree, by Lemma 2.2, there is an S-recursive perfect tree T_1 so that for every $g \in [T_1]$, there is some f so that $f \oplus g \in [S]$. Note that $T_1 \subseteq T$. Moreover, by property (1), for every $g \in [T_1]$, $g \geq_{\mathrm{T}} S \geq_{\mathrm{T}} T_1$. So T_1 is a pointed subtree of T. □

We prove that ACA_0 does not imply wMPT.

Lemma 3.3. *There is a recursive tree T so that $[T]$ is countable and cofinal for the arithmetical reals, that is, for every arithmetical real x there exists some $y \in [T]$ with $y \geq_{\mathrm{T}} x$.*

Proof. It is well known (see, for example, Sacks [16]) that there is a sequence of uniformly recursive trees $\{T_n\}_{n\in\omega}$ so that for every n, $[T_n]$ contains a unique real x_n of Turing degree $\mathbf{0}^{(n)}$. Let $T = \bigcup_{n\in\omega} n^\frown T_n$. □

Proposition 3.4. ACA_0 *does not imply* wMPT.

Proof. Let $\mathscr{M} = (\omega, M, \dots)$, where $M = \{x \mid \exists n\,(x \leq_{\mathrm{T}} \emptyset^{(n)})\}$. Obviously, $\mathscr{M} \models \mathrm{ACA}_0$. Let T be as in Lemma 3.3. Then T is cofinal in M. However, T has no perfect subtree. Hence $\mathscr{M} \not\models$ wMPT. □

By Corollary 5.7, over RCA_0, wMPT does not imply MPT.

4. Over WKL_0

Lemma 4.1. *Suppose that $T \subseteq 2^{<\omega}$ is a recursive tree and that $x \in [T]$ is a real so that there is some random real $y \leq_{\mathrm{tt}} x$. Then for every real $z \gg \emptyset$, there must be some perfect tree $S \subseteq T$ so that $S \leq_{\mathrm{T}} z$.*

Proof. Suppose that Φ_e is a *tt*-reduction so that $\Phi_e^x = y$. Let c_0 be such that for all n we have $C(y\restriction n) \geq \frac{n}{2} + c_0$ and let f be a recursive, increasing function such that for all n it holds that $\Phi_e^{x\restriction f(n)}\restriction n\,[f(n)] = y\restriction n$. Without loss of generality we assume that $f(n) > 2^n$ for all n.

Let $T_1 \subseteq T$ be a recursive tree so that

$$[T_1] = \{z \in 2^\omega \mid z \in [T] \wedge \forall n\,(C(\Phi_e^{z\restriction f(n)}\restriction n\,[f(n)]) \geq \frac{n}{2} - c_0)\}.$$

Note that the condition on the right-hand side is Π^0_1, so that T_1 indeed exists. Note furthermore that $x \in [T_1]$.

Let g be a recursive function so that $g(0) = 0$ and for every n, $g(n+1) = f(f(g(n)))$.

Claim. There is an n_0 so that for all $n > n_0$ and every $\sigma \in T_1 \cap 2^{g(n)}$ with $[\sigma] \cap [T_1] \neq \emptyset$, there are two different $\sigma_0, \sigma_1 \in 2^{g(n+1)} \cap T_1$ extending σ so that $[\sigma_0] \cap [T_1] \neq \emptyset$ and $[\sigma_1] \cap [T_1] \neq \emptyset$.

Subproof. If not, then for every m there is an $n \geq m$ and some $\sigma_0 \in 2^{g(n)} \cap T_1$ with $[\sigma_0] \cap [T_1] \neq \emptyset$, and a unique string σ in $g(n+1)$ extending σ_0 such that $[\sigma] \cap [T_1] \neq \emptyset$. Then we build a Turing machine M as follows: A pair (ν_1, τ) is enumerated into M if and only if

1. $\nu_1 \in 2^{g(n)} \cap T_1$ and $|\tau| = |f(g(n))|$ for some n; and
2. there is a unique string $\nu_2 \in 2^{g(n+1)} \cap T_1$ extending ν_1 such that for every $\nu_3 \in 2^{g(n+1)} \cap T_1$ extending ν_1 with $\nu_3 \neq \nu_2$ we have $[\nu_3] \cap [T_1] = \emptyset$; and
3. for the ν_2 above we have $\tau = \Phi_e^{\nu_2\restriction g(n+1)}[g(n+1)]$.

Recall that $f(n) > 2^n$ for every n. So if $M(\sigma) = \tau$, then $C_M(\tau) \leq \log|\tau|$. Hence $C(\tau) \leq \log|\tau| + c_M$ for some constant c_M.

Then for every m, there is an $n > m$ and some $z \in [T_1]$ such that

$$C(\Phi_e^{z\restriction g(n+1)}\restriction f(g(n))) \leq \log f(g(n)) + c_M.$$

This contradicts the choice of T_1. $\diamond$

Now define a partial recursive function $\varphi\colon 2^{<\omega} \times \omega \to 2^{<\omega}$ as follows: For every n, $\sigma \in 2^{g(n)}$ and $m \leq 2^{g(n+1)}$ let $\varphi(\sigma, m)$ be the m-th finite string τ in $2^{g(n+1)}$ for which we detect that $[\tau] \cap [T_1] = \emptyset$. By Proposition 2.1, there is a total z-recursive function f extending φ. For every $\sigma \in 2^{g(n)}$, let

$$S_\sigma = \{\tau \succ \sigma \mid \tau \in 2^{g(n+1)}\} \setminus \{f(\sigma, m) \mid m \leq 2^{g(n+1)-g(n)} - 2\}.$$

Then the sequence $\{S_\sigma\}_{\sigma \in 2^{<\omega}}$ is z-recursive. By the Claim, for every $n > n_0$ and $\sigma \in T_1 \cap 2^{g(n)}$ with $[\sigma] \cap [T_1] \neq \emptyset$,

1. $|S_\sigma| \geq 2$; and
2. if $\tau \in S_\sigma$, then $[\tau] \cap [T_1] \neq \emptyset$; and
3. any two different strings in S_σ are incompatible.

Now, using these facts, it is easy to z-recursively construct a perfect tree $S \subseteq T_1 \subseteq T$. $\square$

Theorem 4.2. *Over* WKL_0, wMPT *does not imply* ACA_0.

Proof. Let $x_0 = \emptyset \ll x_1 \ll x_2 \ll \ldots$ be a sequence of hyperimmune-free reals. We can see inductively that such a sequence exists, because we can at step n build an x_{n-1}-recursive tree P_n such that all of its paths y have $y \gg x_{n-1}$; we can then apply the Hyperimmune-Free Basis Theorem relative to x_{n-1} to get x_n.

Let $\mathscr{M} = (\omega, M, \ldots)$ where $M = \{y \mid \exists n\,(y \leq_{\mathrm{T}} x_n)\}$.

Obviously, $\mathscr{M} \models \mathrm{WKL}_0$ but $\mathscr{M} \not\models \mathrm{ACA}_0$.

Let $T \subseteq \omega^{<\omega}$ be a tree that is cofinal in M. We code T into a tree $T_1 \subseteq 2^{<\omega}$ by letting $\sigma \in T_1$ if and only if there are a $\tau \in T$ and a $k < \omega$ such that $\sigma = 0^{\tau(0)}10^{\tau(1)}\ldots 0^{\tau(|\tau|-1)}10^k$. Note that $\{\boldsymbol{x} \mid x \in [T]\} \setminus \{\boldsymbol{x} \mid x \in [T_1]\} = \emptyset$, that is, the Turing degree spectrum of T_1 is at least as large as that of T. Furthermore note that $\{\boldsymbol{x} \mid x \in [T_1]\} \setminus \{\boldsymbol{x} \mid x \in [T]\} \subseteq \{\mathbf{0}\}$, that is, the coding leaves the Turing degree spectrum of the tree unchanged, modulo the computable degree. Therefore T_1 is also cofinal in M.

WKL_0 implies that a T_1-random real y exists; since T_1 is cofinal, there must be some $x \in [T_1]$ with $x \geq_{\mathrm{T}} y$ in M. Since x is hyperimmune-free, in fact $y \leq_{\mathrm{tt}} x$. Fix a number n so that $x_n \gg T_1$. Applying Lemma 4.1 by relativizing it to T_1, there must be some x_n-recursive perfect tree $S_1 \subseteq T_1$. Then it is easy to see that there must be some x_n-recursive perfect tree $S \subseteq T$. Thus $S \in M$. $\square$

Note that by Corollary 5.7, over WKL_0, wMPT does not imply MPT.

Question 4.3. Over WKL_0, does MPT imply ACA_0?

5. Over ACA_0

The following lemma is easy.

Lemma 5.1. *Over* RCA_0, *if T is a pointed tree, then for every real $x \geq_{\mathrm{T}} T$, there is a $y \in [T]$ so that $x \equiv_{\mathrm{T}} y$.*

Proof. As T is in particular perfect, we can choose a path y such that y encodes x by branching in T according to the bits of x. Then $x \geq_{\mathrm{T}} x \oplus T \geq_{\mathrm{T}} y \geq_{\mathrm{T}} y \oplus T \geq_{\mathrm{T}} x$. $\square$

The following important theorem can be used to transfer results about Σ^0_3 sets to recursive trees.

Theorem 5.2 (Harrington and Kechris [5]). *Over* RCA_0, *for every Σ^0_3 class A, there is a recursive tree T so that $\{\boldsymbol{x} \mid x \in [T]\} = \{\boldsymbol{x} \mid x \in A\}$. Moreover, the proof can be relativized.*

Recall the definition of Γ-TD from page 4. Montalbán and Shore proved the following theorem by combining Theorem 5.2 with a number of classical recursion theory results.

Theorem 5.3 (Montalbán and Shore [12]). *Over* ACA_0, $\boldsymbol{\Sigma}^0_3$-TD *implies* ATR_0.

Now we are ready to show that MPT implies ATR_0.

Proposition 5.4. *Over* ACA_0, MPT *implies* ATR_0.

Proof. By Theorem 5.3, it is sufficient to prove that MPT implies $\mathbf{\Sigma}^0_3$-TD.

We only prove the lightface version. Given any Σ^0_3, cofinal, and Turing-invariant set A, by Theorem 5.2, there is a recursive tree T so that $\{\boldsymbol{x} \mid x \in [T]\} = \{\boldsymbol{x} \mid x \in A\}$. So $[T]$ is also cofinal. By MPT, T has a pointed subtree T_1. By Lemma 5.1, for every real $x \geq_{\mathrm{T}} T_1$, there is a $y \in [T_1]$ so that $x \equiv_{\mathrm{T}} y$. Therefore A contains an upper cone of Turing degrees. □

Remark 5.1. Note that, over RCA_0, $\mathbf{\Sigma}^0_3$-TD does not imply wMPT. This is because RCA_0 has a model $\mathscr{M}$ that consists only of recursive reals. Then $\mathscr{M}$ satisfies $\mathbf{\Sigma}^0_3$-TD vacuously, but it contains a recursive tree T consisting exactly of one recursive path. Such a T is then vacuously cofinal, but not perfect. So $\mathscr{M}$ does not satisfy wMPT.

In the following theorem we will show that ATR_0 implies MPT. For this purpose, let $T \subseteq \omega^{<\omega}$ be any tree and define the T-recursive tree $\mathscr{HK}(T)$ as follows.

$$[\mathscr{HK}(T)] = \left\{ (i^\frown f \oplus g) \oplus h \;\middle|\; \begin{array}{c} g \in [T] \wedge h = (i^\frown f \oplus g) \oplus \Phi_i^g \wedge \\ \forall n\, (f(n) \text{ is the least } m \text{ with } \Phi_i^{g\upharpoonright m}(n)[m]\downarrow) \end{array} \right\}.$$

The idea for $\mathscr{HK}(T)$ is again taken from Harrington and Kechris [5].

Theorem 5.5. ATR_0 *implies* MPT.

Proof. Given any cofinal tree T, $\mathscr{HK}(T)$ is clearly also cofinal. By Theorem 2.4, $\mathscr{HK}(T)$ is determined. It is well known (see Martin [9] or Montalbán and Shore [12]) that if $\mathscr{HK}(T)$ is cofinal, then **II** cannot have a winning strategy. So **I** has a winning strategy, say w. Let $x \in 2^\omega$ be a real with $x \equiv_{\mathrm{T}} w$.

Let S be a w-recursive tree so that $[S] = \{w * (z \oplus x) \mid z \in 2^\omega\}$. Then

1. if $(i^\frown f_0 \oplus g) \oplus (z_0 \oplus x) \in [S]$ and $(i^\frown f_1 \oplus g) \oplus (z_1 \oplus x) \in [S]$, then $f_0 = f_1$ and $z_0 = z_1$. Together with Lemma 2.2 this implies that there is a w-recursive perfect tree $T_1 \subseteq T$.
2. for every $(i^\frown f \oplus g) \oplus (z \oplus x) \in [S]$, we have that $g \in [T]$ and $g \geq_{\mathrm{T}} z \oplus x \geq_{\mathrm{T}} w \geq_{\mathrm{T}} S$. This implies that T_1 is pointed.

Thus ATR_0 implies MPT. □

However, even over ACA_0, wMPT is strictly weaker than ATR_0.

Theorem 5.6. *Over* ACA_0, wMPT *does not imply* ATR_0.

Proof. Let $g_0 \in 2^\omega$ be arithmetically generic and for every n let $g_{n+1} \in 2^\omega$ be arithmetically generic relative to $g_0 \oplus g_1 \oplus g_2 \oplus \ldots \oplus g_n$. Let $\mathscr{M} = (\omega, M, \ldots)$, where $M = \{x \mid \exists n (x \leq_{\mathrm{T}} (\oplus_{i \leq n} g_i)^{(n)})\}$. Obviously $\mathscr{M} \models \mathrm{ACA}_0$, but $\mathscr{M} \not\models \mathrm{ATR}_0$ as $\emptyset^{(\omega)}$ is not in M.

Now let some tree $T \subseteq \omega^{<\omega}$, which is cofinal in M, be given. Then there must be some real $x \in [T]$ and some n so that $T \leq_{\mathrm{T}} \emptyset^{(n)} \oplus g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_n <_{\mathrm{T}} x$ and x is not arithmetical in $g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_n$. Fix j such that $\Phi_j^x = \emptyset^{(n)} \oplus g_0 \oplus g_1 \oplus g_2 \cdots \oplus g_n$. On the other hand, there is some $m > 0$ so that $x \leq_{\mathrm{T}} \emptyset^{(m+n)} \oplus (\oplus_{i \leq m+n} g_i)$. Note that $\oplus_{n<i\leq m+n} g_i$ is $\emptyset^{(m+n)} \oplus (\oplus_{i \leq n} g_i)$-arithmetically generic. Then $\mathscr{HK}(\oplus_{i\leq n} g_i, T)$ is a $\emptyset^{(n)} \oplus (\oplus_{i\leq n} g_i)$-recursive tree so that

1. for every $f \oplus g \in [\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)]$, $g \in [T]$ and $g \geq_{\mathrm{T}} \emptyset^{(n)} \oplus (\oplus_{i \leq n} g_i) \oplus T \geq_{\mathrm{T}} \mathscr{HK}_j(\oplus_{i\leq n} g_i, T)$; and
2. there is an $f \oplus g \in [\mathscr{HK}_j(\oplus_{i \leq n} g_i, T)]$, for example $f \oplus x$ for some f, so that

$$\emptyset^{(m+n)} \oplus (\oplus_{i \leq m+n} g_i) \geq_{\mathrm{T}} f \oplus g >_{\mathrm{T}} \emptyset^{(n)} \oplus (\oplus_{i \leq n} g_i) \geq_{\mathrm{T}} \mathscr{HK}_j(\oplus_{i \leq n} g_i, T).$$

$$\begin{tikzcd}
 & \mathrm{ATR}_0 \equiv \mathrm{ACA}_0 + \mathrm{MPT} \arrow[dl] \arrow[dd] \arrow[dr] & \\
\mathrm{ACA}_0 \arrow[dd] & & \mathrm{MPT} \arrow[dd] \\
 & \mathrm{WKL}_0 + \mathrm{wMPT} \arrow[dl] \arrow[dr] & \\
\mathrm{WKL}_0 \arrow[dr] & & \mathrm{wMPT} \arrow[dl] \\
 & \mathrm{RCA}_0 &
\end{tikzcd}$$

Figure 1. All arrows represent strict implications.

Note that $\oplus_{n<i\leq m+n} g_i$ is $[\mathscr{HK}_j(\emptyset^{(n)} \oplus (\oplus_{i\leq n} g_i), T)]$-arithmetically generic. Relativizing Lemma 3.1 to $\emptyset^{(n)} \oplus (\oplus_{i\leq n} g_i)$, and noting the fact that x is not arithmetical in $\oplus_{i\leq n} g_i$, we apply it to $[\mathscr{HK}_j(\emptyset^{(n)} \oplus (\oplus_{i\leq n} g_i), T)]$ and obtain that there exists a $\emptyset^{(m+n)} \oplus (\oplus_{i\leq n} g_i)$-recursive perfect tree $S \subseteq \mathscr{HK}_j(\oplus_{i\leq n} g_i, T)$.

Note that $f_0 \oplus g \in S \subseteq [\mathscr{HK}(\oplus_{i\leq n} g_i, T)]$ and $f_1 \oplus g \in S \subseteq [\mathscr{HK}(\oplus_{i\leq n} g_i, T)]$ implies $f_0 = f_1$. Using the fact that S is a perfect tree, by Lemma 2.2, there is a $\emptyset^{(m+n)} \oplus (\oplus_{i\leq n} g_i)$-recursive perfect tree T_1 so that for every $g \in [T_1]$, there is some f so that $f \oplus g \in [S]$. Note that $T_1 \subseteq T$. So T has a perfect subtree in M. Thus $\mathscr{M} \models$ wMPT. $\square$

Corollary 5.7. *Over* ACA_0, wMPT *does not imply* MPT.

We remark that by a method similar as in the proof of Theorem 5.6 and by iterating the Turing jump through the recursive ordinals it can be shown that, even over $\mathbf{\Delta}^1_1$-CA_0, wMPT does not imply ATR_0.

Figure 1 gives an overview of the obtained results.

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