The Power of Frequency Computation

(Extended Abstract)

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Abstract. The notion of *frequency computation* concerns approximative computations of n distinct parallel queries to a set A. A is called (m,n)-recursive if there is an algorithm which answers any n distinct parallel queries to A such that at least m answers are correct. This paper gives natural combinatorial characterizations of the fundamental inclusion problem, namely the question for which choices of the parameters m, n, m', n', every (m,n)-recursive set is (m',n')-recursive. We also characterize the inclusion problem restricted to recursively enumerable sets and the inclusion problem for the polynomial-time bounded version of frequency computation. Furthermore, using these characterizations we obtain many explicit inclusions and noninclusions.

1 Introduction

Frequency computation is a classical approximation notion introduced by Rose [19] in the early sixties and subsequently developed (in historical order) by Trakhtenbrot [20], Kinber [11], and Dëgtev [8] until the early eighties. Recently an upsurge of interest occurred in connection with the investigation of bounded query computations in complexity theory [1, 2, 3, 6] and computability theory [4, 5, 9]. Very recently frequency computations were also studied in inductive inference [7, 13, 17]. In the present paper we characterize the fundamental inclusion problem of frequency computation by a combinatorial property. This yields a short proof of McNicholl's new result [18] that the inclusion problem is decidable. Similar characterizations are provided for the restriction of the inclusion problem to r.e. sets and for a polynomial time version of frequency computation.

We now discuss the definitions and results in more detail. For natural numbers m, n $(1 \leq m \leq n)$ a set A is called (m,n)-recursive (in short $A \in \Omega(m,n)$) if any n distinct parallel queries to A can be effectively answered such that at least m of the answers are correct (i.e., at most $(n-m)$ errors are made). More formally, $A \in \Omega(m,n)$ if there is a recursive function $f : \omega^n \rightarrow \{0,1\}^n$, mapping n-tuples of numbers to n-tuples of bits, such that for all n pairwise distinct numbers $x_1, \ldots, x_n$:

$$f(x_1, \ldots, x_n) = (b_1, \ldots, b_n) \;\Rightarrow\; |\{i : A(x_i) = b_i\}| \geq m.$$

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Trivially, every recursive set is (m,n)-recursive for all m,n, and every (n,n)-recursive set is recursive. Trahktenbrot [20] proved that there exist nonrecursive $(1,2)$-recursive sets, and, answering a question of Myhill, that if $\frac{m}{n} > \frac{1}{2}$ then every (m,n)-recursive set is already recursive – see [10, 14] for a recent survey and related results. Intuitively, no nonrecursive set can be approximated with frequency greater than $\frac{1}{2}$, and this bound is tight. This result is nontrivial as Kinber [11] showed that for $n > 1$ there can be no uniform method to extract a decision procedure from an $(n-1,n)$-algorithm. In particular, Trahktenbrot's inclusions $\Omega(m,n) \subseteq \Omega(1,1)$ for $\frac{1}{2} < \frac{m}{n} < 1$ cannot be shown by local modifications of the (m,n)-algorithm.

In general, the inclusion problem concerns the power of (m,n)-computations with different parameters m,n, namely the question whether, for any given m,n,h,k, every (m,n)-recursive set is (h,k)-recursive. As we have seen above, Trakhtenbrot's result answers this for $\frac{h}{k} > \frac{1}{2}$. For frequency less than or equal to $\frac{1}{2}$, Dëgtev [8] characterized the special case $\{(1,n,h,k) : \Omega(1,n) \subseteq \Omega(h,k)\}$ and stated first results on explicit inclusions and noninclusions. Additional results were obtained in [15], but the question whether the full inclusion problem $\{(m,n,h,k) : \Omega(m,n) \subseteq \Omega(h,k)\}$ is decidable was left open. Very recently, McNicholl [18] answered it positively. In his proof he also uses a finite combinatorial property, but this is rather implicit and the proof is very lenghty (however, due to its greater generality, it could also be useful for other related types of inclusion problems which appear in the literature).

In this paper, the inclusion problem is characterized by natural combinatorial conditions. This yields a new and much shorter proof of McNicholl's result. We first sketch Dëgtev's approach of local (m,n)-computations and apply it to obtain a characterization of the polynomial-time version of frequency computation. Then we weaken Dëgtev's condition to obtain our characterization of the inclusion problem. A new proof, which has no counterpart in Dëgtev's approach is required to show that our condition is sufficient. In section 4 it is shown that the recursively enumerable (r.e.) case "When is every *r.e.* (m,n)-recursive set (h,k)-recursive?" is also decidable. Finally, using our characterizations, we obtain several explicit inclusions and noninclusions.

Definitions and Notation. We are using standard notation. $\omega = \{0,1,2,\ldots\}$. We identify a set A with its characteristic function; so we write $A(x) = 1$ for $x \in A$ and $A(x) = 0$ for $x \notin A$.

The concatenation of strings v,w is denoted by $v \cdot w$ or simply by vw; $V_1 \cdot V_2 = \{u \cdot v : u \in V_1, v \in V_2\}$. For $v = (b_1,\ldots,b_n) \in \{0,1\}^n$ and $i, i_1, \ldots, i_j$ such that $1 \le i \le n$ and $1 \le i_1 \le \ldots \le i_j \le n$, $v[i]$ denotes the i-th component b_i, and $v[i_1,\ldots,i_j]$ denotes the projection $(b_{i_1},\ldots,b_{i_j})$. $V[i_1,\ldots,i_j] = \{v[i_1,\ldots,i_j] : v \in V\}$, $\#v = |\{i : v[i] = 1\}|$. If $X = \{i_1,\ldots,i_j\}$ then $V[X]$ abbreviates $V[i_1,\ldots,i_j]$. We write $\sigma \preceq \tau$ if σ is an initial segment of τ. A *tree* is a set of strings closed under initial segments.

φ_i is the i-th function in a standard enumeration of all partial recursive functions of one argument. $\varphi_{i,s}(x)$ denotes the result, if any, of performing s steps in the computation of $\varphi_i(x)$. A set is called k-r.e. for $k = 1,2,\ldots$ if there is a recursive function $g(x,s)$ such that $A(x) = \lim_{s\to\infty} g(x,s)$, $|\{s : g(x,s) \neq g(x,s+1)\}| \leq k$ and $g(x,0) = 0$ for all x. Intuitively, the characteristic function of a k-r.e. set can be computed with at most k mindchanges. Clearly a set is r.e. iff it is 1-r.e.

2 A Local Version of (m,n)-Computation

Dëgtev [8] (and implicitly already Kinber [11]) introduced the following local version of (m,n)-computability in order to study the combinatorial aspect of frequency computation.

Definition 1. [8] Let $s \geq n$. A set $V \subseteq \{0,1\}^s$ of binary vectors is (m,n)-admissible if for all $x_1 < \ldots < x_n \in \{1,\ldots,s\}$ there is $(b_1,\ldots,b_n) \in \{0,1\}^n$ such that

$$|\{j : b_j = v[x_j]\}| \geq m \text{ for all } v \in V.$$

We extend the definition to the case $s < n$ by appending $(n-s)$ zeros to each of the vectors. (In other words, if $s < n$, i.e., $t := n - s > 0$, then V is (m,n)-admissible iff either $t \geq m$ or V is $(m-t, n-t)$-admissible in the original sense.)

Example: (1) $\{0,1\}^s$ is (m,n)-admissible iff $n - m \geq s$.
(2) Let $2m \leq n$. $\{0,1\}^{n-2m} \times \{0^{2m}, 1^{2m}\}$ is (m,n)-admissible and not $(m+1,n)$-admissible.

This notion leads to a combinatorial condition that is sufficient for inclusions.

Fact 1. [8] *If every (m,n)-admissible set $V \subseteq \{0,1\}^k$ is (h,k)-admissible, then $\Omega(m,n) \subseteq \Omega(h,k)$.*

Trakhtenbrot's Theorem implies that the condition is not necessary when $m > \frac{n}{2}$. It is open whether it is necessary when $m \leq \frac{n}{2}$. However, Dëgtev settled the case $m = 1$.

Theorem 2. [8] (1) *If there is a $(1,n)$-admissible set $V \subseteq \{0,1\}^k$ which is not (h,k)-admissible, then there exists a 2-r.e. set $A \in \Omega(1,n) - \Omega(h,k)$.*
(2) *If in addition $\{0^k, 1^k\} \subseteq V$ then A can be chosen to be an r.e. set.*

For the polynomial-time version of frequency computation, the admissibility condition is also a necessary condition for inclusions, as we now show. Let $\Omega_p(m,n)$ denote the class of all recursive sets which are (m,n)-recursive via an algorithm which runs in polynomial time w.r.t. to the length of the input (i.e., the sum of the lengths of the binary representations of the x_i). Kinber [11] (see also Amir, Gasarch [1]) proved that there are arbitrarily complex sets in $\Omega(n-1,n)$. Thus, the analog of Trahktenbrot's Theorem (which would be that every set in $\Omega(m,n)$ with $\frac{m}{n} > \frac{1}{2}$ is in $\mathbf{P}$) fails badly. On the other hand, the lack of nonuniform inclusions enables the following characterization

Theorem 3. *Let $1 \leq m \leq n, 1 \leq h \leq k$. $\Omega_p(m,n) \subseteq \Omega_p(h,k)$ iff every (m,n)-admissible set $V \subseteq \{0,1\}^k$ is (h,k)-admissible.*

From results in [17, Proposition 3.3] and Theorem 3, it follows that for $\frac{m}{n} < \frac{2}{3}$ and any h, k, we have $\Omega_p(m,n) = \Omega_p(h,k)$ iff $(m,n) = (h,k)$. Thus, in this case there are only trivial equalities. From a result of Kinber [11] (see [17, Fact 3.12]) it follows that $\Omega_p(n, n+1) = \Omega(2,3)$ for all $n \geq 2$. An explicit characterization of all equalities for $\frac{m}{n}, \frac{h}{k} \geq \frac{2}{3}$ is not known, but Kinber conjectures that $\Omega_p(m,n) = \Omega_p(h,k)$ iff $n - m = k - h$.

3 A Characterization of the Inclusion Problem

The following definition slightly strengthens Definition 1 and will lead to a combinatorial characterization of the inclusion problem for $\Omega(m,n)$ classes. The condition was found by looking for a property such that Dëgtev's construction in the proof of Theorem 2 can be carried out for all $m \geq 1$.

Definition 4. A set $V \subseteq \{0,1\}^s$ of vectors is strongly (m,n)-admissible if for every suitable t, every projection of V on n–t distinct components is $(m$–t, n–$t)$-admissible via a vector *with at least m zeros*. Formally, for every t with $0 \leq t < m$ and $n-t \leq s$, and all $x_1 < \ldots < x_{n-t} \in \{1, \ldots, s\}$ there is $(b_1, \ldots, b_{n-t}) \in \{0,1\}^{n-t}$ such that

- $|\{j : b_j = 0\}| \geq m$ and
- $|\{j : b_j = v[x_j]\}| \geq m-t$ for all $v \in V$.

Note that every $(1,n)$-admissible set $V \subseteq \{0,1\}^k$ with $0^k \in V$ is also strongly $(1,n)$-admissible. Thus, every $(1,n)$-admissible set can be transformed into a strongly $(1,n)$-admissible set by interchanging 0 and 1 in some columns. But this does not hold in general: Kinber [11] found a $(3,5)$-admissible set which is not $(4,6)$-admissible. On the other hand, every strongly $(3,5)$-admissible set V is also $(4,6)$-admissible: every projection $V[X]$ onto three coordinates is $(1,3)$-admissible via 0^3, thus no $v \in V$ contains more than two 1s and V is $(4,6)$-admissible via 0^6.

The following theorem establishes a necessary condition for inclusions. Note that the case $m = 1$ already follows from Dëgtev's Theorem 2, (1).

Theorem 5. *If $\frac{m}{n} \leq \frac{1}{2}$ and if there is a strongly (m,n)-admissible set $V \subseteq \{0,1\}^k$ which is not (h,k)-admissible then $\Omega(m,n) \not\subseteq \Omega(h,k)$.*

Proof. Let m, n, h, k, V be as in the hypothesis of the theorem. The proof is based on Dëgtev's interval construction [8]. $\{I_j\}_{j \in \omega}$ denotes a partition of ω into the intervals $I_j = \{jk+1, jk+2, \ldots, jk+k\}$ of length k and $A(x,s)$ is a 2-r.e. approximation of the value $A(x)$ where A is the witness for $\Omega(m,n) \not\subseteq \Omega(h,k)$. We satisfy the following requirements P_i for all $i \in \omega$ to ensure that $A \notin \Omega(h,k)$.

$$(P_i) \qquad A \text{ is not } (h,k)\text{-recursive via } \varphi_i.$$

Construction
Stage 0: Let $A(x,0)=0$, for all x. Let $p(i,0)=i$, for all i.
Stage $s{+}1$: If there is an i such that

- $p(i,s) < s$,
- $\varphi_{i,s}(p(i,s)k+1, p(i,s)k+2, \ldots, p(i,s)k+k) = (b_1, b_2, \ldots, b_k)$ and
- $|\{l : A(p(i,s)k+l) = b_l\}| \geq h$,

then we let the interval $I_{p(i,s)}$ for the least such i *act*, i.e., we define

- $A(p(i,s)k+l, s{+}1) = v[l]$ for some $v \in V$ with $|\{l : v[l] = b_l\}| < h$; note that v exists because V is not (h,k)-admissible.
- $A(x, s{+}1) = 0$ for all $x > p(i,s)k+k$. ("Reset action")
- $p(j, s{+}1) = s+j$ for all $j > i$; note that the intervals $I_{p(j,s+1)}$ for $j > i$ are by now unused.

Else do nothing.
End of construction

Every interval acts at most once, in this case all intervals with larger index are reset to zero. The intervals with smaller index are not changed. If $A(x,s+1)$ is reset from 1 to 0 then x is also thrown out of any diagonalization interval (by the way in which $p(j,s+1)$ is updated), thus $A(x,s')=0$ for all $s' > s$. This proves that A is a 2-r.e. set. By induction on i it is easy to show that each i is chosen at only finitely many stages and consequently (P_i) is satisfied.

It remains to construct the (m,n)-algorithm for A. For given $x_1 < \ldots < x_n$ let $s = x_n$. Determine j such that $x_m \in I_j$ and choose numbers $a \geq 0$, $b \geq 1$ such that $I_j \cap \{x_1, \ldots, x_n\} = \{x_{a+1}, \ldots, x_{a+b}\}$; let $c = n-a-b$. In this way we partition the input vector into three blocks: The initial segment of a components, the middle segment of b components which all belong to the interval which contains x_m, and the end segment of c components. Note that if A changes on one block in some stage $s' > s$ then all blocks to the right of it are definitely set to zero. The (m,n)-algorithm $f(x_1, \ldots, x_n)$ is defined according to one of the following three cases:

1.1 $b \geq n-m$ and $A(x_l, s) = 0$ for all $x_l \in I_j$.
As V is strongly (m,n)-admissible there is a vector $v \in \{0,1\}^b$ with at least m zeros such that $V[x_{a+1}-jk, x_{a+2}-jk, \ldots, x_{a+b}-jk]$ is $(m-a-c, b)$-admissible via v.
Let $f(x_1, \ldots, x_n) = (A(x_1,s), \ldots, A(x_a,s), v[1], \ldots, v[b], 0, \ldots, 0)$.
1.2 $b \geq n-m$ and $A(x_l, s) \neq 0$ for some $x_l \in I_j$.
Let $f(x_1, \ldots, x_n) = (A(x_1,s), \ldots, A(x_m,s), 0, \ldots, 0)$.
2 $b < n-m$, i.e., $a+c > m$.
Let $f(x_1, \ldots, x_n) = (A(x_1,s), \ldots, A(x_m,s), 0, \ldots, 0)$.

Let p be the least index such that I_p acts after stage s. Now it follows by case distinction that A is (m,n)-recursive via f.

For the converse we need the following combinatorial lemma which shows that if a certain extension of a set V – by appending a block of ones and including the all zero vector – is still (m,n)-admissible, then V is even strongly (m,n)-admissible.

Lemma 6. *Let $V \subseteq \{0,1\}^s$. If $W = \{1^n v : v \in V\} \cup \{0^{n+s}\}$ is (m,n)-admissible then V is strongly (m,n)-admissible.*

Proof. Consider any $t < m$ such that $n - t \leq s$ and any distinct $x_1, \ldots, x_{n-t} \in \{1,2,\ldots,s\}$. The projection $W[1,2,\ldots,t,n+x_1,n+x_2,\ldots,n+x_{n-t}] = \{0^n\} \cup 1^t V[x_1,\ldots,x_{n-t}]$ is (m,n)-admissible via a vector u. W.l.o.g. the x_i are ordered such that $u = 0^b 1^{t-b} 0^a 1^{n-t-a}$. If $a \geq m$ let $w = 0^a 1^{n-t-a}$; otherwise let $w = 0^m 1^{n-t-m}$. w has at least m zero components.

By hypothesis, for each $v \in V[x_1,\ldots,x_{n-t}]$, at least m components of $1^{m-t}v$ and u agree. Thus at least $m-(t-b)$ components of v and $0^a 1^{n-t-a}$ agree. If $a \geq m$, then $w = 0^a 1^{n-t-a}$ has at least $m-t$ correct components. If $a < m$ then w differs from $0^a 1^{n-t-a}$ in $m-a$ components and at least $m-(t-b)-(m-a) = a+b-t$ are correct. Since u agrees with 0^n in at least m components, $a+b \geq m$, i.e., $a+b-t \geq m-t$.

Theorem 7. *If every strongly (m,n)-admissible $V \subseteq \{0,1\}^k$ is (h,k)-admissible then $\Omega(m,n) \subseteq \Omega(h,k)$.*

Proof. Assume that every strongly (m,n)-admissible set $V \subseteq \{0,1\}^k$ is (h,k)-admissible. Let A be (m,n)-recursive via an (m,n)-algorithm f. We define the tree T of characteristic functions that are consistent with f, and then we construct a recursive (h,k)-algorithm from T in a nonuniform way. This idea goes back to Trakhtenbrot [20], but McNicholl [18] realized that it can also be fruitful for frequencies below $\frac{1}{2}$. It is not known whether the nonuniformity can be eliminated in this case.

$$\begin{aligned} T &= \{\sigma \in \{0,1\}^* : \sigma \text{ is consistent with } f\} \\ &= \{\sigma \in \{0,1\}^* : (\forall x_1 < x_2 < \ldots < x_n < |\sigma|) \\ &\qquad [f(x_1,x_2,\ldots,x_n) = (b_1,b_2,\ldots,b_n) \;\Rightarrow\; |\{i : \sigma[x_i] = b_i\}| \geq m]\}. \end{aligned}$$

T is a recursive tree and A is an infinite branch of T. Trakhtenbrot [20] observed that if all infinite branches are a finite variant of A, then all infinite branches are recursive, in particular A is (h,k)-recursive.

Now assume that there is an infinite branch B which differs from A in at least n places, say w.l.o.g. $1^n \preceq A$ and $0^n \preceq B$. In particular, for all $x > n$ there is $\tau \in T$ with $0^n \preceq \tau$ and $|\tau| = x$. We define an (h,k)-algorithm g for A as follows:

Given $x_1,\ldots,x_k$ such that $n \leq x_1 < x_2 < \ldots < x_k$, let $x = x_k + 1$. Take any string $\tau \in T$ with $0^n \preceq \tau$ and $|\tau| = x$, and let

$$V = \{\sigma[x_1,x_2,\ldots,x_k] : \sigma \in T \wedge 1^n \preceq \sigma \wedge |\sigma| = x\}.$$

By definition of T, the set $1^n V \cup \{0^n \tau[x_1,x_2,\ldots,x_k]\}$ is (m,n)-admissible and contains $A[x_1,x_2,\ldots,x_k]$. Let U be obtained from V by interchanging 0 and 1 in all columns i with $\tau(x_i) = 1$. Then $W = 1^n U \cup \{0^{n+k}\}$ is (m,n)-admissible and hence, by Lemma 6, U is strongly (m,n)-admissible. Thus, by hypothesis, U is (h,k)-admissible. Then also V is (h,k)-admissible, say via w. Let $g(x_1,\dots,x_k) = w$.

If $x_1 < x_2 < \dots < x_l < n \leq x_{l+1}$, then let $w = g(x_{l+1},\dots,x_k,x_k+1,\dots,x_k+l)$ and define $g(x_1,\dots,x_k) = 1^l w[1,\dots,k-l]$.

These results can be summarized as follows:

Theorem 8. *$\Omega(m,n) \subseteq \Omega(h,k)$ iff $\frac{m}{n} > \frac{1}{2}$ or every strongly (m,n)-admissible set $V \subseteq \{0,1\}^k$ is (h,k)-admissible.*

Corollary 9. [McNicholl, 1994] *The inclusion problem for frequency classes is decidable.*

4 The Inclusion Problem for R.E. Sets

We now present a combinatorial characterization of the inclusion problem restricted to r.e. sets, i.e., of $\{(m,n,h,k) : \Omega(m,n) \cap \mathcal{E} \subseteq \Omega(h,k)\}$ where $\mathcal{E}$ is the class of all r.e. sets. This is based on a further strengthening of the notion strongly admissible which we call *special (m,n)-admissible*. Intuitively, a set V was called *strongly admissible* if it is admissible via vectors which contain a certain amount of zeros. For *special admissible* it is required that they contain in addition a certain amount of ones. One is led to this condition by looking for a property which allows the set A in the construction of Theorem 5 to be made r.e. (instead of merely 2-r.e.).

Definition 10. A set $V \subseteq \{0,1\}^k$ is called special (m,n)-admissible iff for every t with $0 \leq t < m$ and $n-t \leq k$, for all $a,c \geq 0$ with $a+c=t$ and for all $x_1 < \dots < x_{n-t} \in \{1,\dots,k\}$ there is $(b_1,\dots,b_{n-t}) \in \{0,1\}^{n-t}$ such that

- $|\{j : b_j = 0\}| \geq m-a$,
- $|\{j : b_j = 1\}| \geq m-c$ and
- $|\{j : b_j = v[x_j]\}| \geq m-t$ for all $v \in V$.

Note that every special (m,n)-admissible set is also strongly (m,n)-admissible. Furthermore, if V is special (m,n)-admissible so is $V \cup \{0^k, 1^k\}$.

Theorem 11. *Let $\frac{m}{n} \leq \frac{1}{2}$. $\Omega(m,n) \cap \mathcal{E} \subseteq \Omega(h,k)$ iff every special (m,n)-admissible set $V \subseteq \{0,1\}^k$ is (h,k)-admissible.*

Sketch. We restrict ourselves to the direction ($\Rightarrow$). We show the contrapositive. Let m,n,h,k,V be as in the hypothesis of the theorem, i.e., $m \leq \frac{n}{2}$ and V is special (m,n)-admissible but not (h,k)-admissible. The proof is again based on Dëgtev's interval construction [8]. In fact, the case $m=1$ already follows from Theorem 2, (2). We use almost the same construction as in Theorem 5. Only the "Reset action" is modified: this time we reset to ones (instead of zeros) so that

$A(x, s)$ never changes from one to zero, i.e., A is r.e. Formally, with the notation from the proof of Theorem 5, the "Reset action" is done as follows:

$$A(x, s{+}1) = 1 \text{ for } x = p(i, s)k + k + 1, p(i, s)k + k + 2, \ldots, sk.$$

As before it follows by induction on i that each requirement (P_i) is satisfied. Thus A is an r.e. set not in $\Omega(h, k)$. Similariy as before it can be shown that $A \in \Omega(m, n)$.

5 Explicit Inclusions and Noninclusions

In this section we present some explicit results on inclusions and noninclusions for frequencies below $\frac{1}{2}$. These are obtained by applying the combinatorial characterizations of the previous sections.
Dëgtev [8] noted the following three easy inclusions: $\Omega(m, n) \subseteq \Omega(m, n{+}1)$; $\Omega(m{+}1, n{+}1) \subseteq \Omega(m, n)$; $\Omega(m, n) \cap \Omega(h, k) \subseteq \Omega(m + h, n + k)$. The next result shows that there are also inclusions which are nontrivial:

Theorem 12. *$\Omega(2, 4) \subseteq \Omega(3, 6)$, $\Omega(2, 5) \subseteq \Omega(3, 8)$ and $\Omega(3, 6) \subseteq \Omega(4, 9)$.*

We have obtained these inclusions by showing that every $(2, 4)$-admissible set is $(3, 6)$-admissible and so on. Further it is possible to show that every class $\Omega(m, 2m)$ is contained in almost all classes $\Omega(h, 2h)$; therefore we conjecture that $\Omega(m, 2m) \subseteq \Omega(h, 2h)$ for all $h \geq m$.

Theorem 13. *$(\forall m)(\forall^\infty h)[\Omega(m, 2m) \subseteq \Omega(h, 2h)]$.*

The next fact shows that every $(1, k)$-recursive set is "almost" $(h, 2h)$-recursive for large h.

Fact 2. [16, Fact 2.7(b)] *The quotient $q_k(n) = max\{\frac{m}{n} : \Omega(1, k) \subseteq \Omega(m, n)\}$ converges to $\frac{1}{2}$ for $n \to \infty$ and fixed k.*

Dëgtev showed that in general it is impossible to reduce the number of errors made by an (m, n)-algorithm:

Theorem 14. [8] *If $n - m > k - h$ and $\frac{m}{n} \leq \frac{1}{2}$ then $\Omega(m, n) \not\subseteq \Omega(h, k)$.*

This noninclusion is witnessed by the set $\{0, 1\}^{n-m}$ which is (h, k)-admissible iff $k - h \leq n - m$. Further noninclusions are given by the following theorem:

Theorem 15. *The set $V = \{0^n, 1^n\} \cup \{0^i 1 0^j : i{+}j = n{-}1\}$ is (strongly) (m, n)-admissible iff $\frac{m}{n} < \frac{1}{2}$.*

The next theorem generalizes Dëgtev's example of a $(1, 3)$-admissible set which is not $(2, 5)$-admissible [8].

Theorem 16. *The set* $V = \{0^{2h+1}, 1^{2h+1}, 10^{2h-1}1\} \cup \{0^i10^j : i + j = 2h\} \cup \{0^i110^j : i + j = 2h - 1\}$ *is strongly* $(m, 2m+1)$*-admissible for* $m < h$ *but not* $(h, 2h+1)$*-admissible.*

Corollary 17. $\Omega(m, n) \not\subseteq \Omega(h, k)$ *if* $\frac{m}{n} \leq \frac{1}{2}$ *and one of the following conditions holds:* (a) $n - m > k - h$*;* (b) $n - 2m > k - 2h$*;* (c) $n - 2m = k - 2h > 0$ *and* $m \neq h$.

Trakhtenbrot's Theorem and Corollary 17, (a), (b) yield an explicit solution of the equality problem for frequency classes:

Theorem 18. $\Omega(m, n) = \Omega(h, k)$ *iff* $(m, n) = (h, k)$ *or* $\frac{m}{n} > \frac{1}{2} \wedge \frac{h}{k} > \frac{1}{2}$.

Trivially, $\Omega(1, n) \subseteq \Omega(2, 2n)$. Is this optimal, i.e., does $\Omega(1, n) \not\subseteq \Omega(2, 2n - 1)$ hold? The case $n = 2$ follows from Trakhtenbrot's Theorem. Dëgtev proved it for $n = 3$. The case $n = 4$ is witnessed by the closure of {0000000, 0000011, 0000111, 0001101, 0011101, 1111111} under rotational shifts.

All noninclusions $\Omega(m, n) \not\subseteq \Omega(h, k)$ in this section are witnessed by special (m, n)-admissible sets which are not (h, k)-admissible, thus in each case there are r.e. sets $A \in \Omega(m, n) - \Omega(h, k)$. In particular the equality problem restricted to r.e. sets has the same solution as the general equality problem.

6 Conclusion

We have provided combinatorial characterizations for the inclusion problem of frequency computation. These conditions are derived from a local version of (m, n)-computation and look natural. In particular, it follows that the various inclusion problems are decidable and all computability issues are now clearly separated from the combinatorics. While from the point of view of computability theory the problem is completely solved, it would be desirable to obtain a better understanding of the underlying combinatorics which would lead to further explicit results. (An explicit solution for the full inclusion problem, say by a simple formula, is not to be expected, for similar reasons as in [5].) For instance, there is a connection with the "covering problem" from coding theory, cf. [15], but this seems to be rather superficial. Another obvious question is, whether the three criteria which we presented are really different for frequencies less than or equal to $\frac{1}{2}$. We conjecture that they are mutually distinct, but they agree on all values which we were able to test by now.

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