On the Structure of Degrees of Inferability

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Abstract

Degrees of inferability have been introduced to measure the learning power of inductive inference machines which have access to an oracle. The classical concept of degrees of unsolvability measures the computing power of oracles. In this paper we determine the relationship between both notions.

1 Introduction

We consider learning of classes of recursive functions within the framework of inductive inference [21]. A recent theme is the study of inductive inference machines with oracles ([8, 10, 11, 17, 24] and tangentially [12]; cf. [10] for a comprehensive introduction and a collection of all previous results.) The basic question is how the information content of the oracle (technically: its Turing degree) relates with its learning power (technically: its inference degree—depending on the underlying inference criterion).

In this paper a definitive answer is obtained for the case of recursively enumerable oracles and the case when only finitely many queries to the oracle are allowed. It turns out that the learning power of an oracle is (almost) identical with its information content, and is independent from the particular inference criterion (as e.g. EX-, BC-, or $Team$-inference). This extends (and partially corrects) previous results from [8] in an unexpected way. The technique can also be used to give a simplified proof of a result of Slaman and Solovay that characterizes the trivial inference degrees [24].

Adleman and Blum [1] characterized the oracles A that allow one to EX-infer all recursive functions. This is the case iff A has *high* information content (i.e., $K' \leq_T A'$). We extend this result by showing that it already requires a high oracle to EX-infer all

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classes from EX^1. Similar results are obtained for $Team$-inference with n machines versus $Team$-inference with $n+1$ machines.

Further we obtain two partial characterizations of the general EX-degree structure. As a consequence we show that there exists a minimal EX-degree, answering a question of Slaman and Solovay [24].

We also consider language learning with oracles and show that for recursively enumerable oracles learning power equals information content. There is an interesting difference between learning from text and learning from informant: Any nonrecursive oracle strictly increases the learning power w.r.t. informant, but not necessarily w.r.t. text. The investigation of the $InfEx$-degrees also leads to new results on the structure of the Turing degrees below $\mathbf{0}'$.

2 Definitions and notation

Notation 2.1 We denote the natural numbers by $\omega = \{0, 1, 2, \ldots\}$. $A \oplus B = \{2x : x \in A\} \cup \{2x+1 : x \in B\}$, the disjoint union of A and B. For $f : \omega \to \omega$ we denote by $f \restriction n$ the restriction of f to arguments less than n. χ_A is the characteristic function of $A \subseteq \omega$. We often write A for χ_A and $A \restriction x$ for $\chi_A \restriction x$. $\exists^\infty x$ means "for infinitely many x". $\forall^\infty x$ means "for almost all x".

We denote strings from $\{0,1\}^*$ by σ, τ, η. $|\sigma|$ is the length of σ. $\sigma \preceq \tau$ ($\sigma \preceq f$) means that σ is an inital segment of the string τ (the function f). If $a \in \omega$ then we use σa^ω to denote the total function that has initial segment σ and is then all a's. $\sigma \cdot \tau$ is the concatenation of σ and τ. $\sigma \cdot S = \{\sigma \cdot \tau : \tau \in S\}$ for $S \subseteq \{0,1\}^*$. $\mathrm{const}_0(\sigma)$ is a canonical index of $\sigma 0^\omega$ such that $\mathrm{const}_0(\sigma) = \mathrm{const}_0(\sigma 0)$ for all σ

We are using an acceptable programming system $\varphi_0, \varphi_1, \ldots$; the function computed by the e-th program within s steps is denoted by $\varphi_{e,s}$. REC is the set of all total recursive functions. $REC_{0,1}$ is the set of all $0,1$-valued total recursive functions. $W_e = \{x : \varphi_e(x)\downarrow\}$ is the domain of the e-th program. $K = \{e : \varphi_e(e)\downarrow\}$ is the halting problem. If A is an r.e. set then A_s denotes the set of elements enumerated before step s in some fixed recursive enumeration of A.

Turing reducibility is denoted by $\leq_T$. If A is a set then A' is the halting problem relative to A, that is $\{e \mid \varphi_e^A(e)\downarrow\}$. A is *high* if $K' \leq_T A'$. (This differs slightly from the definition in [27] since we do not require that $A \leq K$.) A is *low* if $A' \leq_T K$.

A total function f is *increasing* iff $f(x) \leq f(x+1)$ for all x. A total function g is said to *dominate* a set S of total functions iff for every $f \in S$ there is x_0 such that $f(x) \leq g(x)$ for all $x \geq x_0$.

A set A is 1-*generic* iff for each r.e. set $W \subseteq \{0,1\}^*$ there is a string $\sigma \preceq \chi_A$ such that either $\sigma \in W$ or $\tau \notin W$ for all $\tau \succeq \sigma$.

Remaining recursion theoretic notation is from Soare's book [27].

Definition 2.2 [7] An *inductive inference machine* (IIM) M is a total Turing machine that is trying to learn recursive functions f from their initial segments $\sigma \preceq f$. M BC-infers f, if for almost all $\sigma \preceq f$, $M(\sigma)$ is an index of a program for f. If furthermore almost all values $M(\sigma)$, $\sigma \preceq f$, are the same, then M EX-infers f. The indices output by IIMs are relative to the acceptable programming system $\{\varphi_i\}_{i\in\omega}$.

Let $S \subseteq REC$. Then $S \in EX$ (BC) if there exists an IIM M such that for all $f \in S$, M EX-infers f (BC-infers f).

Let $a, b \geq 1$ be such that $a \leq b$. A set of recursive functions S is in $[a,b]EX$ (concept from [26], notation from [22]) if there exist b IIMs M_1, M_2,…, M_b such that for every $f \in S$, there are a machines $M_{i_1}, M_{i_2}, \ldots, M_{i_a}$, which all EX-infer f, with $1 \leq i_1 < \cdots < i_a \leq b$. If $a = 1$ then in the literature this is referred to as inferring S by a *team of b IIMs*. $[a,b]BC$ is defined similarly.

$S \in EX^n$ ($S \in EX^*$) if there exists an IIM M such that for all $f \in S$, when M is run on initial segments of f, almost all the programs output are the same, and the function computed by that program differs from f on at most n numbers (on some finite set of numbers).

M $LIMEX$-infers $S \subseteq REC$ (concept from [6]) if for all $f \in S$, M converges on input f to a K-recursive program for f (or, equivalently to a limiting recursive program for f).

Definition 2.3 [10] An *oracle inductive inference machine* (OIIM) M^A is a Turing machine with access to an oracle A. M^A is total for any oracle $A \subseteq \omega$. Let $use(M^A, \sigma)$ denote the maximal element that is queried by M^A on input σ. We define M^A $EX[A]$-infers f ($BC[A]$-infers f) similar to our definition of M EX-infers f (BC-infers f). For a set S of recursive functions, $S \in EX[A]$ ($BC[A]$) is defined similar to $S \in EX$ (BC).

Note that in the definition of $EX[A]$ (and the other classes, except $LIMEX$) we are inferring indices for recursive functions *not* indices for recursive-in-A functions.

Let S be a set of recursive functions. $S \in EX[A*]$ if there exists an OIIM M^A such that (1) S is $EX[A]$-inferred by M^A, and (2) for every $f \in S$, during the inference of f by M^A, only finitely many queries to A are made, i.e., $(\exists k)(\forall \sigma \preceq f)\ [use(M^A, \sigma) \leq k]$. $BC[A*]$, $LIMEX[A*]$, $[a,b]EX[A*]$, $[a,b]BC[A*]$ are defined similarly. For instance, $S \in [1,b]EX[A*]$ iff there exist n IIMs M_1^A, M_2^A,…, M_n^A such that for every $f \in S$, there is a machine M_i^A which $EX[A]$-inferred f and, during the inference of f, the machine makes only finitely many queries to A.

A is of EX-degree below B, iff $EX[A] \subseteq EX[B]$. A and B are of the same EX-degree, iff $EX[A] = EX[B]$. If $REC \in EX[A]$ then there is no greater EX-degree and this degree is called the *omniscient* EX-degree. $\emptyset$ defines the *trivial* EX-degree.

In the same way it is possible to define BC-degrees, $EX*$-degrees and $BC*$-degrees, etc.

Definition 2.4 [13] $\mathcal{E}$ denotes the class of all r.e. subsets of ω, a set $L \in \mathcal{E}$ is called a *language*. A *text* for L is any mapping T from ω to $\omega \cup \{\star\}$ such that L is the set of natural numbers in the range of T. T is called *recursive text* if T is a recursive function.

An IIM M $TxtEx$-infers L iff $M(T)$ converges for *every* text T for L to an r.e. index of L, i.e., $(\forall T \text{ text for } L)(\exists e)(\forall^\infty x)[M(T \upharpoonright x) = e \ \wedge \ W_e = L]$. Let $\mathcal{L}$ be a class of languages. $\mathcal{L} \in TxtEx$ iff there is an IIM M that $TxtEx$-infers all $L \in \mathcal{L}$. The classes $TxtEx[A]$, $TxtEx[A*]$ are defined in the usual way.

Let $\mathcal{L}$ be a class of languages. $\mathcal{L} \in InfEx$ ($\mathcal{L}$ is learnable from informant) iff there is an IIM M that EX-infers from input χ_L an r.e. index of L for all $L \in \mathcal{L}$, i.e., $(\exists e)(\forall^\infty x)[M(\chi_L \upharpoonright x) = e \ \wedge \ W_e = L]$. The classes $InfEx[A], InfEx[A*]$ are defined in the usual way.

Note that $TxtEx[A] \subseteq InfEx[A]$. Both $TxtEx$- and $InfEx$-degrees are a refinement of the EX-degrees (consider functions as graphs).

3 Summary of facts and previous results

The following basic characterization (due to D. A. Martin) gives us an effective handle on non-high sets [27, Theorem XI.1.3]:

Fact H: *A is non-high iff for every total A-recursive function f there is a total recursive function g such that $f(x) < g(x)$ for infinitely many x. In other words, REC is not dominated by any A-recursive function.*

Notation: $S_0 = \{\varphi_e : \varphi_e \in REC_{0,1} \wedge 1^e0 \preceq \varphi_e\}$ and $S_1 = \{\sigma 0^\omega : \sigma \in \{0,1\}^*\}$.

In many of our proofs the task is to find a suitable generalization of the following result of Blum and Blum [5, p. 133]:

Nonunion Theorem: *$S_0 \in EX$ and $S_1 \in EX$, but $S_0 \cup S_1 \notin EX$.*

It is well-known (see [27, p. 97]) that for every set $B \geq_T K$ there is a 1-generic set A with $A' \equiv_T A \oplus K \equiv_T B$.

The following facts hold for all of the considered inference criteria though we state them only for EX. See [10] for the proofs; the $LIMEX$-case can be proved as the EX-case.

- A 1-generic $\Rightarrow EX[A] = EX[A*]$.
- $A \leq_T B' \;\Rightarrow\; EX[A*] \subseteq EX[B]$.
- For every A there is a 1-generic set G with $EX[A*] = EX[G*] = EX[G]$.
- $EX[A*] = EX$ iff $A \leq_T K$.
- $REC \in EX[A*]$ iff $K' \leq_T A$.
- $EX[A] = EX$ iff $A \leq_T K$ and either A is recursive or A has 1-generic degree. (The difficult "only if" direction is due to Slaman and Solovay [24]. We shall later present a simplified proof.)

If infinitely many queries are allowed then the omniscient degrees are no longer independent of the inference criterion. All of the following results, except the first, are from [10].

- $REC \in EX[A]$ iff A is high (Adleman and Blum [1]).
- $REC \in [a, b]EX[A]$ iff A is high.
- There is a low set A with $REC \in BC[A]$.
- For A r.e.: $REC \in BC[A]$ iff A is high.

Fact: *There is a low set A with $REC \in LIMEX[A]$.*
Proof: By the Low Basis Theorem (see [27, VI.5.13, XII.3.9]), there is a low set $A \subseteq \omega^2$ such that for all i, x:

$$\varphi_i(x) \downarrow \in \{0, 1\} \Rightarrow \chi_A(i, x) = \varphi_i(x).$$

Let $A^{[i]} = \{x : (i, x) \in A\}$. The following A-recursive algorithm $LIMEX$-infers any $f \in REC$: For the input $(f(0), \ldots, f(n))$ find the first number i such that $10^{f(0)}10^{f(1)}1 \ldots 10^{f(n)}1 \preceq A^{[i]}$, and (using $A \leq_T K$) supply a K-recursive index of $g_i = \lambda n.\ a_{i,n+1} - a_{i,n} - 1$, where $a_{i,n}$ is the n-th element of $A^{[i]}$. $\Box$

Pitt and Smith [22] proved that $[a, b]EX = [1, \lfloor \frac{b}{a} \rfloor]EX$ and similar for BC. The proof relativizes, hence for all A:

- $[a, b]EX[A] = [1, \lfloor \frac{b}{a} \rfloor]EX[A]$.
- $[a, b]BC[A] = [1, \lfloor \frac{b}{a} \rfloor]BC[A]$.

The $*$-version follows using the fact that for every A there is a 1-generic G with $EX[A*] = EX[G]$ and $BC[A*] = BC[G]$. Therefore, it suffices to consider the $[1, n]$-case of $Team$-inference.

4 Technical summary

This paper characterizes the structure of inference degrees for different notions of inference of recursive functions or r.e. sets. The main results are:

$*$-degrees We obtain a complete characterization: The $*$-degrees of all inference types ($EX*$, $BC*$, $LIMEX*$, $[a,b]EX*$, $[a,b]BC*$) have the same structure: A has omniscient $*$-degree iff $K' \leq_T A \oplus K$. If the $*$-degree of A is not omniscient then the $*$-degree of B is below the $*$-degree of A iff $B \oplus K \leq_T A \oplus K$.

trivial degrees A simplified proof of the Theorem of Slaman and Solovay [24] is obtained. This proof easily generalizes to other inference criteria.

r.e. inference degrees We obtain a complete characterization: For non-high r.e. A, B, $EX[A] \subseteq EX[B]$ iff $A \leq_T B$. Corresponding characterizations hold for BC-, $LIMEX$-, and $Team$-inference. In each of these cases, an r.e. oracle has omniscient degree iff it is high.

comparing different inference criteria Several classical theorems hold in relativized form, except in the obvious case (for omniscient oracles) where they fail. $EX^{n+1} \subseteq EX^n[A] \Rightarrow A$ is high. $BC \subseteq EX^*[A] \Rightarrow A$ is high. $[1,n+1]EX \subseteq [1,n]EX[A] \Rightarrow A$ is high. In particular, $EX[A] \subset EX^1[A] \subset \cdots \subset EX^*[A] \subset BC[A]$, and $EX[A] \subset [1,2]EX[A] \subset \cdots$ unless $REC \in EX[A]$.

further results on EX-degrees We obtain a partial characterization of the EX-degrees via the jump operator: $EX[A] \subseteq EX[B]$ implies either $A' \leq_T B'$ or $K' \leq_T B'$. In the latter case, B belongs to the omniscient EX-degree. We obtain a partial characterization of the EX-degrees in terms of a domination property and show that there is a minimal EX-degree.

language learning We obtain a complete characterization for $*$-degrees and r.e. degrees. *Learning from text:* $TxtEx[A] \subseteq TxtEx[B] \Rightarrow A' \leq_T B'$, hence there is no omniscient $TxtEx$-degree [15]. A has trivial $TxtEx$-degree iff A is 1-generic and low. $TxtEx[A*] \subseteq TxtEx[B*] \Leftrightarrow A \leq_T B \oplus K$. For r.e. oracles, $TxtEx[A] \subseteq TxtEx[B] \Leftrightarrow A \leq_T B$.

Learning from informant: A has omniscient $InfEx$-degree iff $A \geq_T K$. A has trivial $InfEx$-degree iff A is recursive. There is no omniscient $InfEx*$-degree. For r.e. oracles, $InfEx[A] \subseteq InfEx[B] \Leftrightarrow A \leq_T B$. There exist $A, B \leq_T K$, B r.e. such that $A \not\leq_T B$ and $InfEx[A] \subseteq InfEx[B]$.

As a related result on T-degrees we obtain: There is 1-generic T-degree $\mathbf{a} < \mathbf{0}'$ such that for all ω-r.e. T-degrees $\mathbf{b}$, $\mathbf{c}$, if $\mathbf{c} \leq \mathbf{a} \cup \mathbf{b}$ then $\mathbf{c} \leq \mathbf{b}$.

5 The structure of $*$-degrees

In [10] the trivial and omniscient $*$-degrees have been determined for EX-, BC-, and $Team$-inference. It turned out that A has trivial degree iff $A \leq_T K$, and A has omniscient degree iff $A \geq_T K'$. In this section we extend these results and give a complete characterization of the $*$-degrees. The main tools are the following lemmas which are extensions of [10, Theorem 5.8].

Lemma 5.1 *If A is not high and $S_0 \in BC[A*]$ via M^A then there is a string σ such that for any $\tau \succeq \sigma$, $use(M^A, \tau) < |\sigma|$ and $dom(\varphi_{M^A(\tau)}) \subseteq \{0, \ldots, |\tau| - 1\}$.*

Proof: This proof is a modification of the proof of Theorem 5.8 in [10]. For the convenience of the reader we supply all details.

Assume that A is not high and the conclusion of the lemma is not satisfied. If $dom(\varphi_{M^A(\eta)}) \not\subseteq \{0, \ldots, |\eta| - 1\}$ then there is $x \geq |\eta|$ such that $\varphi_{M^A(\eta)}(x)$ is defined. Hence there is $\tau \in \{0,1\}^*$ with $\eta \prec \tau$ and $\tau(x) \neq \varphi_{M^A(\eta)}(x) \downarrow$. Therefore we can reformulate the hypothesis on M^A as follows: Every σ has an extension τ satisfying the following *mindchange condition* $mc(\sigma, \tau, A)$:

$mc(\sigma, \tau, A) \Leftrightarrow$ There is some η, $\sigma \preceq \eta \prec \tau$, such that

- either $(\exists x < |\tau|)[\varphi_{M^A(\eta)}(x) \downarrow \neq \tau(x)]$
- or $M^A(\eta)$ queries some $x \geq |\sigma|$.

We will show that M^A does not $BC*$-infer S_0.

First we define two functions independent of A. Second we use these functions to construct some A-recursive function g. In the third part, g enables us to find some $\varphi_e \in S_0$ not inferred by M^A. We split it up this way so that we can use the functions constructed in the first part in both the second and third part.

First part: We define recursive functions ψ and T where ψ maps $\Sigma^* \times \omega$ to Σ^*, and T maps $\Sigma^* \times \omega$ to $\mathcal{P}^{fin}(\omega)$ (the set of finite subsets of ω). We will have

$$\begin{aligned} \sigma = {} & \psi(\sigma, 0) \preceq \psi(\sigma, 1) \preceq \cdots, \\ \{D : D \subseteq \{0, \ldots, |\sigma|\}\} = {} & T(\sigma, 0) \supseteq T(\sigma, 1) \supseteq \cdots. \end{aligned}$$

The idea is that we are (at first) looking for an extension τ of σ and a subset D of $\{0, 1 \ldots, |\sigma|\}$ such that $\varphi_{M^D(\tau)}$ is wrong on $|\tau|$. The parameter t bounds how long we can search for such an extension. We keep doing this, forming larger and larger extensions of σ and trying to make more and more of the sets D yield incorrect guesses.

We use finite sets D instead of the oracle A itself in the mindchange condition $mc(\cdot,\cdot,D)$, since we do not have direct access to A. So the strategy will be to diagonalize as many finite sets D of given length as possible. So we define ψ and T inductively. The base case is

$$\psi(\sigma,0)=\sigma, \qquad T(\sigma,0)=\{D : D\subseteq\{0,\dots,|\sigma|\}\}.$$

Assume $\psi(\sigma,t)$ and $T(\sigma,t)$ have been defined. If there exist a string $\tau\succ\psi(\sigma,t)$ with $|\tau|\le t$ and a set $D\in T(\sigma,t)$, satisfying the mindchange condition $mc(\sigma,\tau,D)$, then take the first such (D,τ) and let

$$\psi(\sigma,t+1)=\tau, \qquad T(\sigma,t+1)=T(\sigma,t)-\{D\}$$

else ψ,T remain unchanged ($\psi(\sigma,t+1)=\psi(\sigma,t)$, $T(\sigma,t+1)=T(\sigma,t)$).

There are at most $2^{|\sigma|+1}+1$ distinct values of $\psi(\sigma,t)$. This is because every time $\psi(\sigma,t)$ changes a set is removed from $T(\sigma,t)$, and there are $2^{|\sigma|+1}$ elements in $T(\sigma,0)$.

Second part: There is an A-recursive function g given by:

$$\begin{aligned}\hat g(\sigma) &= (\mu t>|\sigma|)\,[\,(A\restriction|\sigma|)\notin T(\sigma,t)].\\ g(n) &= \max\{\hat g(\sigma) : |\sigma|\le n\}.\end{aligned}$$

$\hat g$ and g are total by the hypothesis on M^A. Note that $mc(\sigma,\psi(\sigma,\hat g(\sigma)),A)$ holds and $|\psi(\sigma,\hat g(\sigma))|>|\sigma|$.

Third part: Since A is not high, there exists a recursive increasing h such that $(\exists^\infty n)[g(n)<h(n)]$. For given σ, let

$$h'(\sigma)=(\mu t\ge|\sigma|)\,[\psi(\sigma,t)=\psi(\sigma,h(t))\wedge t>|\psi(\sigma,t)|>|\sigma|\,].$$

h' is a recursive function. We define, for each e, a sequence of strings inductively. Let $\sigma_0=1^e0$, and let $\sigma_{n+1}=\psi(\sigma_n,h'(\sigma_n))$. Let f_e be defined as the limit of the sequence. Note that f_e is recursive. Assume, that f_e is $BC[A*]$-inferred by M^A. Therefore there are numbers m, s such that $|\sigma_m|\ge s$ and for all σ, $\sigma_m\preceq\sigma\preceq f_e$:

(a) $M^A(\sigma)$ queries no element greater than s.

(b) $(\forall x)[\varphi_{M^A(\sigma)}(x)\downarrow=f_e(x)]$.

Since $(\exists^\infty n)[g(n)<h(n)]$, there is some $n>|\sigma_{m+1}|$ such that $h(n)>g(n)$. Let k be the greatest number such that $|\sigma_k|\le n$. Now $|\sigma_m|<|\sigma_k|\le n<|\sigma_{k+1}|$. Let $t=h'(|\sigma_k|)$. So $\sigma_{k+1}=\psi(\sigma_k,t)$, $t\ge|\psi(\sigma_k,t)|\ge n$ and

$$\psi(\sigma_k,\hat g(\sigma_k))\preceq\psi(\sigma_k,h(n))\preceq\psi(\sigma_k,h(t))=\psi(\sigma_k,t)\preceq\sigma_{k+1}.$$

Therefore $mc(\sigma_k, \sigma_{k+1}, A)$ holds in contradiction to the conditions (a) and (b).

So M^A does not infer any f_e with only finitely many queries. By the recursion theorem, there is an index e_0 satisfying $\varphi_{e_0} = f_{e_0}$. f_{e_0} is a witness for S_0 not being inferred by M^A. ▮

Lemma 5.2 *If A is not high and $S_0 \in EX[A*]$ via M^A then there is a string σ such that for any $\tau \succeq \sigma$, $M^A(\tau) = M^A(\sigma)$ and $use(M^A, \tau) < |\sigma|$. The same holds for $LIMEX*$ instead of $EX*$.*

Proof: The only part of Lemma 5.1 that is specific for $BC*$ is the mindchange condition. We replace it by:

$mc(\sigma, \tau, A) \Leftrightarrow$ There is some η, $\sigma \preceq \eta \prec \tau$, such that

- either $M^A(\eta) \neq M^A(\sigma)$,
- or $M^A(\eta)$ queries some $x \geq |\sigma|$.

So we obtain a proof of the current lemma. ▮

Finally we need to lift Lemma 5.1 to obtain a version suitable for $Team$-inference.

Definition 5.3 Let $\pi_i(f) = \lambda x.f(\langle i, x\rangle)$ for any function f and $i \in \omega$.
$\pi_i(\sigma) = \lambda x.\, \sigma(\langle i, x\rangle)$ for strings σ,
$F_n(\sigma) = \{\tau \succeq \sigma : (\forall i > n)[\pi_i(\tau) \in \pi_i(\sigma) \cdot \{0\}^*]\}$ and
$\tilde{S}_n(\sigma) = \{f \succ \sigma : (\exists \tau \in \{0,1\}^*)(\exists e)(\exists i, 1 \leq i \leq n)[\pi_i(f) = \tau 0 1^e 0^\omega \wedge f = \varphi_e]\}$.

Note that $\tilde{S}_n(\sigma) \in [1,n]EX$: the k-th machine $M_k(\sigma)$ just outputs the number of consecutive 1s in the last block of 1s in $\pi_k(\sigma)$. Clearly $F_{n-1}(\sigma) \subseteq F_n(\sigma)$ and $\tilde{S}_{n-1}(\sigma) \subseteq \tilde{S}_n(\sigma)$.

Lemma 5.4 *If A is not high and $\tilde{S}_n(\eta) \in [1,n]BC[A*]$ via $M_1^A, \ldots, M_n^A$ then there is a string $\sigma \in F_n(\eta)$ such that for any $\tau \in F_0(\sigma)$, the use of $M_1^A(\tau), \ldots, M_n^A(\tau)$ is bounded by $|\sigma|$ and $x < |\tau|$ for all $x \in dom(\varphi_{M_1^A(\tau)}) \cup \cdots \cup dom(\varphi_{M_n^A(\tau)})$ with $(x)_1 = 0$.*

Proof: (Induction on n) Assume the lemma holds for $n-1$ (if $n = 1$ we assume nothing). We prove it for n. Fix A, η, and a team $M_1^A, \ldots, M_n^A$ such that the hypothesis of the lemma is satisfied. Let $T_k(\sigma)$ be the set of all $\tau \succeq \sigma$ such that use$(M_k^A, \tau) < |\sigma|$ and $x < |\tau|$ for all $x \in dom(\varphi_{M_k^A(\tau)})$ with $(x)_1 = 0$. Hence we need to show that $(\exists \sigma \in F_n(\eta))(\forall k, 1 \leq k \leq n)[F_0(\sigma) \subseteq T_k(\sigma)]$. For the inductive step, we distinguish the following two cases:

(a) $(\exists \sigma \in F_n(\eta))(\exists k, 1 \leq k \leq n)[F_{n-1}(\sigma) \subseteq T_k(\sigma)]$:

If $n = 1$ then case (a) is just a restatement of the conclusion of the lemma and we are done. Now assume that $n > 1$. By a renumbering of the machines we may assume that $k = n$. Then the team of $n-1$ machines $M_1^A, \ldots, M_{n-1}^A$ $BC*$-infers $\tilde{S}_{n-1}(\sigma)$. By the induction hypothesis there is a string $\sigma' \in F_{n-1}(\sigma)$ such that $F_0(\sigma') \subseteq T_i(\sigma')$ for $i = 1, \ldots, n-1$. By the hypothesis of case (a) we get $F_0(\sigma') \subseteq T_n(\sigma')$. The conclusion of the lemma follows.

(b) $(\forall \sigma \in F_n(\eta))(\forall k, 1 \leq k \leq n)[F_{n-1}(\sigma) \not\subseteq T_k(\sigma)]$:
We show that this case cannot happen, which completes the inductive step. As in the proof of Lemma 5.1 we construct a function $f \in \tilde{S}_n(\sigma)$ that is not inferred by $M_1^A, \ldots, M_n^A$. First adapt the mindchange condition such that only extensions in $F_{n-1}(\sigma)$ are considered and all n machines are diagonalized.

$mc(\sigma, \tau, A) \Leftrightarrow \tau \in F_{n-1}(\sigma)$ and for all $i = 1, \ldots, n$ there is η_i, $\sigma \preceq \eta_i \prec \tau$ such that

- either $(\exists x < |\tau|, (x)_1 = 0)[\varphi_{M_i^A(\eta_i)}(x) \downarrow \neq \tau(x)]$
- or $M_i^A(\eta_i)$ queries some $x \geq |\sigma|$.

Note that, by the hypothesis of case (b), every $\sigma \in F_n(\eta)$ has an extension τ such that $mc(\sigma, \tau, A)$ holds.

Now we proceed as in Lemma 5.1. Only the third part has to be modified: We need to define a suitable starting point σ_0. Uniformly in e choose $\sigma_0 \in F_n(\eta)$ such that $\pi_n(\sigma_0) = \pi_n(\sigma)01^e$. As in the original proof we obtain a recursive function f_e with $\pi_n(f_e) = \pi_n(\sigma)01^e0^\omega$. Here we are using that by the mindchange condition only extensions in $F_{n-1}(\sigma_0)$ are allowed. Now f_e is not $BC*$-inferred by the team $M_1^A, \ldots, M_n^A$. By the recursion theorem, there is e_0 such that $f_{e_0} \in \tilde{S}_n(\eta)$. Hence the team does not infer $\tilde{S}_n(\eta)$ which contradicts the hypothesis. ∎

Inference with finitely many queries to A is equivalent to inference with finitely many queries to $A \oplus K$ (a relativization of $EX[K*] = EX$, cf. [11], [10, Note 4.10, Lemma 5.6]). Hence for a characterization of the $*$-degrees it suffices to consider oracles A with $K \leq_T A$.

Theorem 5.5 *The following statements are equivalent for all sets $A, B \geq_T K$ and all $n \geq 1$:*
(1) $A \leq_T B$, *or* $K' \leq_T B$;
(2) $EX[A*] \subseteq EX[B*]$;
(3) $BC[A*] \subseteq BC[B*]$;
(4) $LIMEX[A*] \subseteq LIMEX[B*]$;
(5) $[1, n]EX[A*] \subseteq [1, n]EX[B*]$;
(6) $[1, n]BC[A*] \subseteq [1, n]BC[B*]$.

Proof: $(1 \Rightarrow 2-6)$ If $K' \leq_T B$ then the $*$-degree of B is omniscient, since already $REC \in EX[B*] \subseteq BC[B*]$, $LIMEX[B*]$, $[1,n]EX[B*]$, $[1,n]BC[B*]$. If $A \leq_T B$ then any $*$-inference algorithm for A can be translated into one for B.

$(2 \Rightarrow 1)$ Assume that $EX[A*] \subseteq EX[B*]$ and $K' \not\leq_T B$. Consider the following set:

$$S_A = \{1^x 0 \cdot f : (f \in S_0 \wedge x \in A \oplus \overline{A}) \vee (f \in S_1 \wedge x \notin A \oplus \overline{A})\}.$$

It is easy to see that $S_A \in EX[A*]$. By the Friedberg Completeness Criterion there is C such that $B \equiv_T C' \equiv_T C \oplus K$. Since $B \not\geq_T K'$, C is not high. $S_A \in EX[A*] \subseteq EX[B*] = EX[(C \oplus K)*] = EX[C*]$. Thus there is an OIIM M^C that infers S_A.

If $x \notin A \oplus \overline{A}$ every $\sigma \in 1^x 0 \cdot \{0,1\}^*$ has an extension τ such that $M^C(\sigma) \neq M^C(\tau)$ (since M^C infers both $\sigma 10^\omega$ and $\sigma 00^\omega$).

If $x \in A \oplus \overline{A}$ then M^C infers $1^x 0 \cdot S_0$ and by Lemma 5.2 there is $\sigma \in 1^x 0 \cdot \{0,1\}^*$ such that for any $\tau \succeq \sigma$, $M^C(\tau) = M^C(\sigma)$ and $\mathrm{use}(M^C, \tau) < |\sigma|$.

Therefore we obtain:

$$\begin{array}{rcl} x \in A \oplus \overline{A} & \Leftrightarrow & \text{there are } \sigma \in 1^x 0 \cdot \{0,1\}^* \text{ and } e \text{ such that} \\ & & (\forall \tau \succeq \sigma)[\mathrm{use}(M^C, \tau) < |\sigma|] \\ & & \text{and} \quad (\forall \tau \succeq \sigma)[M^C(\tau) = e]. \end{array}$$

It follows that $A \oplus \overline{A}$ is r.e. in $C \oplus K$ which implies that A is recursive in $C \oplus K$, hence $A \leq_T B$.

$(3 \Rightarrow 1)$ Assume that $BC[A*] \subseteq BC[B*]$ and $K' \not\leq_T B$. Choose S_A and C as above. We have $S_A \in EX[A*] \subseteq BC[A*] \subseteq BC[B*] = BC[C*]$. Take any OIIM M^C that $BC*$-infers S_A.

If $x \notin A \oplus \overline{A}$ every $\sigma \in 1^x 0 \cdot \{0,1\}^*$ has an extension τ such that $M^C(\tau)$ is the index of a total function.

If $x \in A \oplus \overline{A}$ then M^C infers $1^x 0 \cdot S_0$. Hence by Lemma 5.1 there is $\sigma \succ 1^x 0$ such that for any $\tau \succeq \sigma$, $\mathrm{use}(M^A, \tau) < |\sigma|$, and $dom(\varphi_{M^A(\tau)}) \subseteq \{0, \ldots, |\tau| - 1\}$.

Therefore we obtain:

$$\begin{array}{rcl} x \in A \oplus \overline{A} & \Leftrightarrow & \text{there is some } \sigma \in 1^x 0 \cdot \{0,1\}^* \text{ such that} \\ & & (\forall \tau \succeq \sigma)[\mathrm{use}(M^C, \tau) < |\sigma|] \\ & & \text{and} \quad (\forall \tau \succeq \sigma)(\forall y \geq |\tau|)(\forall t)[\varphi_{M^C(\tau),t}(y) \uparrow]. \end{array}$$

So $A \oplus \overline{A}$ is r.e. in $C \oplus K$, hence $A \leq_T B$ as above.

$(4 \Rightarrow 1)$ This is proved identically to $(2 \Rightarrow 1)$, since that proof only exploits mind-changes and does not refer to convergence of the inferred program.

$(5, 6 \Rightarrow 1)$ Here also the proof follows mainly the proofs of $(2, 3 \Rightarrow 1)$. Consider the following set:

$$S_A = \{1^x 0 \cdot f : (1^x 0 \cdot f \in \tilde{S}_n(1^x 0) \wedge x \in A \oplus \overline{A}) \vee (f \in S_1 \wedge x \notin A \oplus \overline{A})\}.$$

$S_A \in [1, n]EX[A*]$, since $\tilde{S}_n(\lambda) \in [1, n]EX$. On the other hand, assume that also $S_A \in [1, n]BC[B*]$ and $B \equiv_T C' \equiv_T C \oplus K$ for some non-high set C. Then $S_A \in [1, n]BC[C*]$ via a team of n OIIMs $M_1^C, \ldots, M_n^C$.

If $x \notin A \oplus \overline{A}$ then every $\sigma \in 1^x 0 \cdot \{0, 1\}^*$ has an extension $\tau \in F_0(\sigma)$ such that for some k, $M_k^C(\tau)$ is an index of a total recursive function (since $\sigma 0^\omega$ is inferred).

If $x \in A \oplus \overline{A}$ then $M_1^C, \ldots, M_n^C$ $[1, n]BC$-infer $\tilde{S}_n(1^x 0)$ and by Lemma 5.4 there exists $\sigma \in F_n(1^x 0)$ such that for all machines M_k^C and all $\tau \in F_0(\sigma)$ the use of $M_1^C(\tau), \ldots, M_n^C(\tau)$ is bounded by $|\sigma|$ and $y < |\tau|$ for all $y \in dom(\varphi_{M_1^C(\tau)}) \cup \cdots \cup dom(\varphi_{M_n^C(\tau)})$ with $(y)_1 = 0$.

Therefore we obtain:

$$\begin{array}{lll} x \in A \oplus \overline{A} & \Leftrightarrow & \text{there is some } \sigma \in F_n(1^x 0) \text{ such that} \\ & & (\forall \tau \in F_0(\sigma))(\forall k, 1 \leq k \leq n)[\text{use}(M_k^C, \tau) < |\sigma|] \quad \text{and} \\ & & (\forall \tau \in F_0(\sigma))(\forall k, 1 \leq k \leq n)(\forall y \geq |\tau|, (y)_1 = 0)(\forall t)[\varphi_{M_k^C(\tau), t}(y) \uparrow]. \end{array}$$

From this description it is easy to see that $A \oplus \overline{A}$ is r.e. in $C \oplus K$. Hence $A \leq_T B$ as above. ∎

Remark: 1) Note that the proof of $(3 \Rightarrow 1)$ implies that $S_0 \cup S_1 \in BC[A*]$ only if the $*$-degree of A is already omniscient.
2) Using cylindrifications as in [10, Fact 6.27] one can transfer Theorem 5.5 to EX^n, EX^* and BC^n.

Corollary 5.6 *The structures of the non-omniscient $EX*$-, $BC*$-, $LIMEX*$-, and $Team*$-inference degrees and the Turing degrees above $\mathbf{0}'$ and not above $\mathbf{0}''$ are the same. Their first order theory is undecidable.*

Proof: The first part is just a restatement of Theorem 5.5. The second result is a consequence of known results on T-degrees as we now explain. The undecidability of the partial ordering $\mathcal{D}$ of the T-degrees follows from the fact that all finite distributive lattices are isomorphic to an initial segment of $\mathcal{D}$ (see [20, Theorem VI.4.6]). The latter result relativizes to $\mathbf{0}'$, i.e., for every finite distributive lattice $\mathcal{L}$ there is a degree $\mathbf{a}$ with $\mathbf{0}' \leq \mathbf{a}$ such that $\mathcal{D}[\mathbf{0}', \mathbf{a}]$ is isomorphic to $\mathcal{L}$. As $\mathcal{L}$ is finite, $\mathbf{a}$ is not above $\mathbf{0}''$. Hence, with the argument from the proof of [20, Theorem VI.4.6], it follows that the partial ordering of the T-degrees above $\mathbf{0}'$ and not above $\mathbf{0}''$ is undecidable. ∎

In particular, Corollary 5.6 allows to deduce all previous results about $*$-degrees (as e.g. those obtained in [8, 17]) from known properties of Turing degrees. An interesting new result is the existence of minimal $*$-degrees (represented by any Turing degree which is a minimal cover for $\mathbf{0}'$).

6 The trivial inference degrees

Let $\mathcal{G}$ denote the set of all $A \leq_T K$ such that A is either recursive or is of 1-generic T-degree. Slaman and Solovay [24] proved that $EX[A] = EX$ iff $A \in \mathcal{G}$. The difficult part is the "only if" direction (and in [24] only a sketch of proof appeared). They use a rather complicated finite injury argument to show that if $A \leq_T K \ \wedge \ EX[A] = EX$ then there is a 1-generic set $G \leq_T K$ such that $A \leq_T G$. Then they conclude that $A \in \mathcal{G}$ using a result of Haught [14] who proved that the nonrecursive predecessors of a 1-generic degree $< \mathbf{0}'$ are all 1-generic. Haught's proof is also a rather complicated finite injury argument.

In this section we provide a simplified proof of the Theorem of Slaman and Solovay. We still need Haught's result but the rest of the proof is just a suitable generalization of the fact that $S_0 \cup S_1 \notin EX$ which does not need a priority argument. This proof easily generalizes to other inference criteria. It has been claimed earlier in [10] that these generalizations can be obtained by a modification of the construction of Slaman and Solovay. However, our proofs are much simpler.

The crucial tool is the following result of Haught. Call a recursive sequence of strings $\{\alpha_s : s \in \omega\}$ a Σ_1-*correct approximation* of A if $(\forall x)[\lim_s \alpha_s(x) = \chi_A(x)]$ and for any infinite r.e. set $T \subseteq \omega$ there exists $t \in T$ such that $\alpha_t \preceq A$.

Theorem 6.1 [14, p. 771, l. 37–38] *If A has a Σ_1-correct approximation then $A \in \mathcal{G}$.*

We will need the following corollary which reformulates Theorem 6.1 in terms of a domination property.

Theorem 6.2 *Let $A \leq_T K$ and let $c_A(x) = \min\{s \geq x : A_s \upharpoonright x = A \upharpoonright x\}$ where A_s is some recursive approximation of A. If $A \notin \mathcal{G}$ then for any recursive function g there is an increasing recursive function h dominated by c_A but not by g; i.e., $(\forall x)[h(x) \leq c_A(x)]$ and $(\exists^\infty x)[h(x) > g(x)]$.*

Proof: Assume that $A \leq_T K \ \wedge \ A \notin \mathcal{G}$. By Theorem 6.1 there is no Σ_1-correct approximation of A. Let g be any recursive function. We define a recursive sequence of strings as follows: α_s is the longest common prefix of $A_s, A_{s+1}, \ldots, A_{g(s)}$. Formally:

$$\alpha_s(x) = \begin{cases} \chi_{A_s}(x) & \text{if } x < s \text{ and } (\forall t, s \leq t \leq g(s))(\forall y \leq x)[\chi_{A_s}(y) = \chi_{A_t}(y)]; \\ \uparrow & \text{otherwise, in particular for } x \geq s. \end{cases}$$

Clearly $\lim_s \alpha_s = A$. Since $\{\alpha_s : s \in \omega\}$ is not Σ_1-correct there is an infinite recursive set T such that $\alpha_s \not\preceq A$ for any $s \in T$. By the definition of α, if $s \in T$ then $A_t \upharpoonright s \not\preceq A$ for $t = s, \ldots, g(s)$. So $c_A(s) > g(s)$ for $s \in T$.

Let $h(x) = g(y(x)) + 1$, where $y(x) = \max(T \cap \{0, \ldots, x\})$ ($\max(\emptyset) = 0$). Since c_A is an increasing function and $c_A(x) \geq c_A(y(x)) \geq g(y(x)) + 1 = h(x)$, it follows that c_A majorizes h. On the other hand, $h(x) > g(x)$ for all $x \in T$, hence g does not dominate h. ∎

Theorem 6.3 [24] *If $A \leq_T K$ and $A \notin \mathcal{G}$ then A has nontrivial EX-degree.*

Proof: Assume that $A \leq_T K$ and $A \notin \mathcal{G}$. First a set $S_A \in EX[A]$ is defined and then it is shown that $S_A \notin EX$. S_A is given via an A-recursive "tree" $T_A \subseteq \{0,1\}^*$ which is defined as the smallest set of strings satisfying the following two conditions:

(I) [Base case] $\{1\}^* \subseteq T_A$
(II) [Inductive step] $\sigma 1 0^s \in T_A$ for all $\sigma \in T_A$ and $s \leq c_A(|\sigma|)$.

T_A has the following further properties:

(III) $\sigma 1 \tau \in T_A$ for all $\sigma \in T_A$ and τ with $|\tau| \leq c_A(|\sigma|)$.
(IV) $1^e 0 \preceq \sigma \preceq \tau \wedge \tau \in T_A \ \Rightarrow \ \sigma \in T_A$.
(V) $\sigma 1 0^{c_A(|\sigma|)+1} \notin T_A$ for all σ.
(VI) No function $f \in S_1$ is an infinite branch of T_A.

Condition (III) can be obtained by iterating (II) and using the fact that c_A is increasing.

Informally, a branch of T_A can have arbitrarily long runs of 1s, but if it wants to have a run of 0s there is a restriction on how long they can be, based on $c_A(|\sigma|)$ where you want to begin the run of 0s after $\sigma \cdot 1$.

Further let

$$S_A = \{\varphi_e : 1^e 0 \preceq \varphi_e \wedge \varphi_e \text{ is total } \wedge (\forall \sigma \preceq \varphi_e)[\sigma \in T_A]\} \cup S_1.$$

Informally S_A is the union of

- functions which are almost everywhere zero and
- infinite recursive branches of T_A whose index is coded as an initial segment.

S_A is $EX[A]$-inferred via the following algorithm N^A (here we are using property (VI)):

$$N^A(\sigma) = \begin{cases} e & \text{if } 1^e 0 \preceq \sigma \text{ and } \sigma \in T_A; \\ \text{const}_0(\sigma) & \text{otherwise.} \end{cases}$$

Suppose for a contradiction that $S_A \in EX$ via an IIM M. Let $mc(\sigma, \tau)$ denote the fact that a mindchange occurs between σ and τ, i.e.,

$$mc(\sigma, \tau) \Leftrightarrow (\exists \eta, \sigma \prec \eta \preceq \tau)[M(\sigma) \neq M(\eta)].$$

If for infinitely many $\sigma \preceq f$, there is some $\tau \preceq f$, $\sigma \preceq \tau$ such that $mc(\sigma, \tau)$ holds then M does not infer f. Now define

$$\begin{aligned} \hat{g}(\sigma) &= \min\{|\eta| : mc(\sigma, \sigma \cdot \eta)\}, \\ g(n) &= \max\{\hat{g}(\sigma) : |\sigma| \leq n+1\}. \end{aligned}$$

Note that M infers both $\sigma 00^\omega$ and $\sigma 10^\omega$. Hence there is $\eta \in \sigma \cdot \{0,1\} \cdot \{0\}^*$ such that $M(\eta) \neq M(\sigma)$, i.e., $mc(\sigma, \tau)$ for all $\tau \succeq \eta$. Therefore $\hat{g}$ and g are total recursive functions.

By Theorem 6.2 there is an increasing recursive function h dominated by c_A but not by g. Using h it is possible to construct a family of recursive functions $\{f_e\}_{e\in\omega}$ such that no f_e is inferred by M and for some e, $f_e \in S_A$. The construction will be uniform in e. For each e we construct f_e as follows.

Starting with $\sigma_0 = 1^e 0$, the function f_e is the limit of an inductively defined sequence σ_n of strings: If there exists $\eta \in \{0,1\}^{h(|\sigma_n|)}$ with $mc(\sigma_n, \sigma_n \cdot 1 \cdot \eta)$ then let $\sigma_{n+1} = \sigma_n \cdot 1 \cdot \eta$ else let $\sigma_{n+1} = \sigma_n \cdot 1$.

Note that $\sigma_{n+1} \in \sigma_n \cdot 1 \cdot \{0,1\}^s$ for some $s \leq h(|\sigma_n|) \leq c_A(|\sigma_n|)$. In particular, $\sigma_n \in T_A \Rightarrow \sigma_{n+1} \in T_A$ for all σ_n (by property (III)). f_e is an infinite branch of T_A, since $\sigma_0 \in T_A$.

Suppose for a contradiction that M infers f_e. Then there exists σ_{n_0} such that no mindchange occurs beyond σ_{n_0} during the inference of f_e. In particular, $\sigma_{n+1} = \sigma_n \cdot 1$ for all $n \geq n_0$, i.e., $\sigma_{n_0+k} = \sigma_{n_0} \cdot 1^k$. Also, since the first case in the definition of σ_{n+1} did not apply, it follows that $h(|\sigma_n|) < \hat{g}(\sigma_n \cdot 1)$ for $n \geq n_0$. Thus we get $g(|\sigma_{n_0}| + k) = g(|\sigma_{n_0+k}|) \geq \hat{g}(\sigma_{n_0+k} \cdot 1) > h(|\sigma_{n_0+k}|) = h(|\sigma_{n_0}| + k)$, i.e., g dominates h which contradicts the hypothesis on h. Thus M does not infer f_e.

By the recursion theorem, there is a fixed point e_0 of f with $\varphi_{e_0} = f_{e_0}$ and $f_{e_0} \in S_A$. Hence M does not infer S_A, a contradiction. This proves that $S_A \notin EX$. ▮

We now adapt the above proof for $LIMEX$, BC and $Team$-inference. First we prove the easier cases $LIMEX$ and BC, then we generalize the argument for $Teams$.

Theorem 6.4 *If $A \leq_T K$ and $A \notin \mathcal{G}$ then*
(1) $LIMEX[A] \not\subseteq LIMEX$,
(2) $BC[A] \not\subseteq BC$.

Proof: (1) The proof for $LIMEX$ is the same as that for EX since the latter only uses that M converges for different functions from S_A to different indices. It does not use that these indices are programs.

(2) For BC, we use a slightly modified mindchange condition:

$$mc(\sigma, \tau) \Leftrightarrow (\exists \eta, \sigma \prec \eta \preceq \tau)(\exists x < |\tau|)[\varphi_{M(\eta),|\tau|}(x) \downarrow \neq \tau(x)].$$

If $mc(\sigma, \tau)$ holds then the program computed by $M(\eta)$ does not compute any function $f \succeq \tau$. Now the whole proof easily generalizes. To see that $\hat{g}(\sigma)$ converges note that since M infers $\sigma 0^\omega$ there is some $\eta \in \sigma \cdot 0^*$ such that $M(\eta)$ is an index of $\sigma 0^\omega$. Then $mc(\sigma, \tau)$ holds for any sufficiently long $\tau \succeq \eta 1$. ∎

Theorem 6.5 *If $A \leq_T K$ and $A \notin \mathcal{G}$ then $(\forall n)[EX[A] \not\subseteq [1,n]BC]$.*
In particular, $(\forall n)$ $[[1,n]EX[A] \not\subseteq [1,n]EX]$ and $(\forall n)$ $[[1,n]BC[A] \not\subseteq [1,n]BC]$.

Proof: The idea of this proof is to iterate the construction of T_A: on each $\sigma 10^s$ with $s > c_A(|\sigma|)$ ($\sigma 10^s$ is therefore outside T_A) we implant a copy of T_A. Formally let

$$\hat{T}_A = \{\sigma 10^s : \sigma \in \{0,1\}^* \wedge s \leq c_A(|\sigma|)\}.$$

Then $\hat{T}_A$ also satisfies

(I) $\sigma 0^{c_A(|\sigma|)+1} \notin \hat{T}_A$ for all σ.
(II) If $\sigma \notin \hat{T}_A$ then $\sigma 0 \notin \hat{T}_A$.
(III) If $f \in S_1$ then $\sigma \notin \hat{T}_A$ for almost all $\sigma \prec f$.
(IV) $\sigma \cdot 1 \cdot T_A \subseteq \hat{T}_A$ for all $\sigma \notin \hat{T}_A$.

Now let

$$\hat{S}_A = \{f : (\exists \tau \notin \hat{T}_A)[\tau 1^{e+1} 0 \preceq \varphi_e \wedge f = \varphi_e \wedge (\forall \sigma \preceq f, \sigma \succ \tau)[\sigma \in \hat{T}_A]]\} \cup S_1.$$

$\hat{S}_A \in EX[A]$ via the following OIIM N^A:

$$N^A(\sigma) = \begin{cases} e & \text{if } \sigma \in \tau 1^{e+1} 0 \cdot \{0,1\}^* \text{ where } \tau \text{ is the maximal initial} \\ & \text{segment of } \sigma \text{ with } \tau \notin \hat{T}_A \text{ (note that } \lambda \notin \hat{T}_A); \\ \text{const}_0(\sigma) & \text{otherwise (in particular for } \sigma \notin \hat{T}_A). \end{cases}$$

Assume that $\hat{S}_A \in [1,n]BC$ via a team $M_1, M_2, \ldots, M_n$ of n IIMs. Define the mind-change condition $mc(M_i, \sigma, \tau)$ for a single machine M_i as in the case of BC. Let $MC(\sigma) = \{i : (\exists \tau \succ \sigma)[mc(M_i, \sigma, \tau)]\}$. There is a string σ' such that $MC(\sigma')$ is minimal in the following sense: There is no $\sigma'' \succ \sigma'$ with $MC(\sigma'') \subset MC(\sigma')$. If $MC(\sigma') = \emptyset$ then at most n of the functions $\sigma' 1^i 0^\omega$ are inferred by the team hence $\hat{S}_A$ is not inferred. Thus $MC(\sigma') \neq \emptyset$.

Let $\tau' \succ \sigma'$ be some string not in $\hat{T}_A$. Now one adapts the definition of $\hat{g}$ and g from Theorem 6.3. $\hat{g}(\sigma)$ is defined only for $\sigma \succeq \tau'$, but g is a total recursive function:

$$\begin{aligned} \hat{g}(\sigma) &= \min\{|\eta| : (\forall i \in MC(\sigma'))[mc(M_i, \sigma, \sigma \cdot \eta)]\}, \\ g(x) &= \max\{\hat{g}(\sigma) : \tau' \preceq \sigma \wedge |\sigma| \leq x + 1 + |\tau'|\}. \end{aligned}$$

By Theorem 6.2, there is a recursive increasing function $h \leq c_A$ (i.e., a function $h \in REC$ with $(\forall x)[h(x) \leq c_A(x)]$) which is not dominated by g. Starting with

$\sigma_0 = \tau' 1^{e+1} 0$, one constructs as in Theorem 6.3 a function $f_e = \lim_s \sigma_s$ as follows: If there is an $\eta \in \{0,1\}^{h(|\sigma_m|)}$ with $mc(M_i, \sigma_m, \sigma_m \cdot 1 \cdot \eta)$ for all $i \in MC(\sigma')$, then let $\sigma_{m+1} = \sigma_m \cdot 1 \cdot \eta$ else let $\sigma_{m+1} = \sigma_m \cdot 1$.

Again it follows that f_e is an infinite branch of the copy $\tau' \cdot 1 \cdot T_A$ of T_A above $\tau' \cdot 1$. Further the mindchange conditions are satisfied above any initial segment of f_e. Thus no machine M_i with $i \in MC(\sigma')$ infers f_e. By the recursion theorem, there are infinitely many functions f_e with $\varphi_e = f_e$ and these are in $\hat{S}_A$. Since the machines M_i with $i \notin MC(\sigma')$ do not have any mindchanges above σ they can infer only finitely many of these f_e. Therefore some $f_e \in \hat{S}_A$ is not inferred by them, and hence is not inferred by the team. Thus the team does not infer $\hat{S}_A$, i.e., $\hat{S}_A$ is a witness for $EX[A] \not\subseteq [1,n]BC$. ∎

7 The structure of the r.e. degrees of inferability

In [8, Corollary 57] it is shown that the structure of the low r.e. EX- and BC-degrees is equal to the structure of the low r.e. Turing degrees. The following theorem extends this result as far as possible and completely characterizes the r.e. inference degrees. It refutes a wrong claim in [8, Theorem 62] and shows that the r.e. inference degrees are much more well-behaved than was previously believed.

Theorem 7.1 *The following conditions are equivalent for all r.e. sets A, B and all $n \geq 1$:*
(1) $A \leq_T B$*, or* B *is high;*
(2) $EX[A] \subseteq EX[B]$;
(3) $BC[A] \subseteq BC[B]$;
(4) $LIMEX[A] \subseteq LIMEX[B]$;
(5) $[1,n]EX[A] \subseteq [1,n]EX[B]$;
(6) $[1,n]BC[A] \subseteq [1,n]BC[B]$.

Proof: $(1 \Rightarrow 2-6)$: If $A \leq_T B$ then obviously the inference degree of A is less than or equal the inference degree of B for all of the above notions of inference. If B is high then the inference degree of B is omniscient, i.e., $EX[A] \subseteq EX[B]$ for all A.

$(2 \Rightarrow 1)$: This and the following proofs are an adaption of the construction from the previous section. For an r.e. set A let $c_A(n) = \min\{s : A_s \upharpoonright n = A \upharpoonright n\}$ where A_s is a recursive enumeration of A. T_A and S_A are constructed from c_A as in Theorem 6.3. Since c_A can be approximated from below, there is a recursive enumeration $t_0, t_1, \ldots$ of T_A. $S_A \in EX[A]$ via the following M^A:

$$M^A(\sigma) = \begin{cases} e & \text{if } 1^e 0 \preceq \sigma \text{ and } \sigma \in T_A; \\ \text{const}_0(\sigma) & \text{otherwise.} \end{cases}$$

Let B be any r.e. non-high set and assume that $S_A \in EX[B]$ via OIIM N^B. For $\sigma \in T_A$, N^B infers both $\sigma 00^\omega$ and $\sigma 10^\omega$. For one of these functions $N^B(\sigma)$ is not a correct index. Therefore σ has an extension τ such that a mindchange occurs between σ and τ; formally $mc(\sigma, \tau)$ holds where:

$$mc(\sigma, \tau) \Leftrightarrow (\exists \eta \prec \tau)[\sigma \preceq \eta \wedge N^B(\sigma) \neq N^B(\eta)].$$

Assume that there is some string $\sigma \in T_A$ such that every mindchange above σ occurs outside T_A. Then the B-recursive function $g(n) = \min\{|\tau| : \sigma 1^{n+1} \preceq \tau \wedge mc(\sigma, \tau)\}$ dominates c_A. Therefore $A \leq_T B$.

Now suppose for a contradiction that such a σ does not exist. We show that N^B does not infer S_A: Consider the set $\tilde{T}_A = \{t_s 1^s : s \in \omega\} \subseteq T_A$. Any $\sigma \in T_A$ has an extension in $\tilde{T}_A$. Furthermore, $\tilde{T}_A$ is recursive. We can B-recursively compute an extension $\tau(\sigma) \in \tilde{T}_A$ such that $mc(\sigma, \tau(\sigma))$ holds. There is a recursive approximation $\lambda i, \sigma.\ \tau_i(\sigma)$ of τ such that:

(a) The modulus of convergence $\hat{m}$ is B-recursive on $\tilde{T}_A$.

(b) For all $\sigma \in T_A$ and all i: $\tau_i(\sigma) \succ \sigma \wedge \tau_i(\sigma) \in \tilde{T}_A$.

The first condition follows from the Modulus Lemma [27, III.3.2] the second can be obtained by using the first $j > i$ satisfying $\tau_j(\sigma) \succ \sigma \wedge \tau_j(\sigma) \in \tilde{T}_A$ instead of i itself. Such a j exists since $\tau(\sigma) = \lim_i \tau_i(\sigma)$ satisfies this condition.

Let $m(n) = \max\{\hat{m}(\sigma) : \sigma \in \tilde{T}_A \ \wedge\ |\sigma| \leq n\}$. Since B is non-high there is, by Fact H, an increasing recursive function h which is not dominated by m. Now construct for each e a function $f_e = \lim_n \sigma_n$ starting with $\sigma_0 = 1^e 0$ via induction:

$$\text{Let } \sigma_{n+1} = \tau_{h(t)}(\sigma_n) \text{ for the first } t \text{ such that } t \geq |\tau_{h(t)}(\sigma_n)|.$$

Each function f_e is a recursive infinite branch of T_A. For given x there is some $y > x + |\sigma_1|$ such that $h(y) > m(y)$. Let n be the largest number such that $|\sigma_n| \leq y$. $\sigma_n \in \tilde{T}_A$, since $n \geq 1$. Now let $t \geq |\sigma_{n+1}|$. Then $h(t) \geq h(y) > m(y) \geq \hat{m}(\sigma_n)$ and therefore $\sigma_{n+1} = \tau_{\hat{m}(\sigma_n)}(\sigma_n) = \tau(\sigma_n)$. So $mc(\sigma_n, \sigma_{n+1})$ holds for infinitely many n and N^B does not infer f_e. By the recursion theorem, there is a fixed point $\varphi_{e_0} = f_{e_0}$. Thus, S_A contains some f_{e_0} that is not inferred by N^B.

($3 \Rightarrow 1$): This is a modification of the proof ($2 \Rightarrow 1$) using the same set $S_A \in EX[A] \subseteq BC[A]$. In order to show that $S_A \notin BC[B]$ for r.e. non-high $B \not\geq_T A$ one has to replace the mindchange condition by

$$mc(\sigma, \tau) \Leftrightarrow (\exists \eta \prec \tau)(\exists x < |\tau|)[\sigma \preceq \eta \wedge \varphi_{N^B(\eta)}(x) \downarrow \neq \tau(x)].$$

The proof can then be adapted to the new mc.

$(4 \Rightarrow 1)$: This proof is identical to the proof of $(2 \Rightarrow 1)$ since that proof never evaluates $\varphi_e(x)$ for some x and $e = M^B(\sigma)$. So it does not use that the index is a recursive index and therefore goes through for K-recursive indices as well.

$(5, 6 \Rightarrow 1)$: Here again $\hat{T}_A$ and $\hat{S}_A$ from Theorem 6.5 are used. By the choice of c_A, $\hat{T}_A = \{t_0, t_1, \ldots\}$ is an r.e. set of nodes. $\hat{S}_A \in EX[A]$ via the following M^A:

$$M^A(\sigma) = \begin{cases} e & \text{if } \sigma \in \tau 1^{e+1} 0 \cdot \{0,1\}^* \text{ where } \tau \text{ is the maximal initial} \\ & \text{segment of } \sigma \text{ with } \tau \notin \hat{T}_A \text{ (note that } \lambda \notin \hat{T}_A); \\ \mathrm{const}_0(\sigma) & \text{otherwise.} \end{cases}$$

Thus $\hat{S}_A \in [1,n]EX[A] \subseteq [1,n]BC[A]$. Assume that B is r.e., not high, and $\hat{S}_A \in [1,n]BC[B]$. We need to show that $A \leq_T B$. Assume that a team $N_1^B, N_2^B, \ldots, N_n^B$ of n machines infers $\hat{S}_A$. Define the mindchange condition $mc(N_i^B, \sigma, \tau)$ for a single machine N_i^B as in the case of BC. Now let $MC(\sigma) = \{i : (\exists \tau \succ \sigma)[mc(N_i^B, \sigma, \tau)]\}$. There is a string σ' such that $MC(\sigma')$ is minimal in the following sense: There is no $\sigma'' \succ \sigma'$ with $MC(\sigma'') \subset MC(\sigma')$. If $MC(\sigma') = \emptyset$ then at most n of the functions $\sigma' 1^i 0^\omega$ are inferred by the team hence $\hat{S}_A$ is not inferred. Thus $MC(\sigma') \neq \emptyset$. Now define the following mindchange condition for strings $\tau \succ \eta \succeq \sigma'$:

$$\overline{mc}(\eta, \tau) \Leftrightarrow (\forall i \in MC(\sigma))[mc(N_i^B, \eta, \tau)]$$

and let $\tau' \succ \sigma'$ be some string not in $\hat{T}_A$. For any σ, let $\hat{T}_A(\sigma)$ denote the largest subset of $\hat{T}_A \cap (\sigma \cdot \{0,1\}^*)$ such that if $\tau \in \hat{T}_A(\sigma)$ then $\eta \in \hat{T}_A(\sigma)$ for all η with $\sigma \cdot 1 \preceq \eta \preceq \tau$. If there is a string $\sigma \in \hat{T}_A(\tau')$ such that every mindchange above σ occurs outside $\hat{T}_A(\tau')$ then it follows as above that $A \leq_T B$. Otherwise as in the cases for EX and BC we construct functions f_e with $\tau' 1^{e+1} 0 \preceq f_e$ where $\hat{T}_A(\tau')$ plays the role of T_A. We obtain infinitely many $f_e = \varphi_e \in \hat{S}_A$ such that all N_i with $i \in MC(\sigma)$ have infinitely many mindchanges on φ_e. Thus the team does not infer $\hat{S}_A$, a contradiction. ∎

For $k \geq 0$ let EX_k denote the class of all sets which can be EX-inferred by IIMs with at most k mindchanges (cf. [10, Definition 6.30]). The previous result can be strengthened to:

Theorem 7.2 *If A, B are r.e. and non-high then $EX_1[A] \subseteq BC[B] \Leftrightarrow A \leq_T B$.*

Proof: This proof uses the same tree T_A as Theorem 6.3 and Theorem 7.1, parts $(2, 3 \Rightarrow 1)$, but a modified set $\tilde{S}_A$ replaces S_A:

$$\tilde{S}_A = \{\varphi_e : 1^e 0 \preceq \varphi_e \wedge \varphi_e \text{ is total } \wedge (\forall \sigma \preceq \varphi_e)[\sigma \in T_A]\} \cup \{\sigma 0^\omega : \sigma \in T_A\}.$$

The proof for $S_A \notin BC[B]$ is actually a proof for $\tilde{S}_A \notin BC[B]$. To show the remaining part $\tilde{S}_A \in EX_1[A]$, note that the machine M^A from $(2 \Rightarrow 1)$ above can easily be modified such that it infers $\tilde{S}_A$ with at most one mindchange. This mindchange occurs when the initial segment of f leaves T_A. ∎

Note, that in this theorem $EX_1[A]$ cannot be replaced by $EX_0[A]$. Indeed, $EX_0[A] \subseteq EX[A*]$, since only finitely many queries are made until the first guess and since from then on no further access to the oracle is necessary. But $EX[A*] \subseteq EX$, since $A \leq_T K$. Thus $EX_0[A] \subseteq BC[B]$ for all $A \leq_T K$.

8 When different notions of inference coincide

It is well-known that $EX \subset BC$. Is there a non-high set A such that $EX[A] = BC[A]$? Our next theorem shows that this cannot happen. In fact, if BC (unrelativized!) is contained in $EX[A]$ then A must already be high. As a byproduct one obtains an easier proof of the Adleman-Blum Theorem [1]. We also consider other inference criteria C for which it is known that $EX \not\subseteq C$ and show that $EX \subseteq C[A]$ only if $REC \in C[A]$.

Theorem 8.1 *If $EX^1 \subseteq EX[A]$ then A is high.*

Proof: We construct a family $\{\varphi_{g(i)}\}_{i\in\omega}$ ($g \in REC$) with the following properties:

(a) $1^i0 \preceq \varphi_{g(i)}$.

(b) $\varphi_{g(i)}$ is defined for all x with at most one exception $a_i > i$.

(c) If $\varphi_{g(i)} \subseteq \varphi_j$, φ_j is total, and W_i is finite, then $j \geq |W_i|$.

The family $S = \{f \in REC_{0,1} : (\exists i)[\varphi_{g(i)} \subseteq f]\}$ is obviously in EX^1. It will be shown that $S \in EX[A]$ implies that A is high. (Note that S is even in EX_0^1, so this proof actually shows "$EX_0^1 \subseteq EX[A] \Rightarrow A$ is high".)

Now we present the details. First the construction of $\varphi_{g(i)}$ in stages $s \geq i+1$ where we are trying to diagonalize against all indices that are $\leq |W_i|$:

Stage i+1: Let $\varphi_{g(i)} = 1^i0$, $J(i) = \emptyset$ and $a_i = i+1$.

Stage $s >$ i+1: If there is a $j \leq |W_{i,s}|$ such that $j \notin J(i)$ and $\varphi_{j,s}(a_i)\downarrow$, then choose $b \in \{0,1\} - \{\varphi_j(a_i)\}$, let $\varphi_{g(i)}(a_i) = b$, let $a_i = s$, and enumerate j into $J(i)$. Otherwise let $\varphi_{g(i)}(s) = 0$. $\Box$

By stage $i+1$, the property (a) is satisfied. After stage $s > i+1$ the function $\varphi_{g(i)}$ is defined for all $x \leq s$ except a_i, thus (b) holds. Assume now that W_i is finite and $j \leq |W_i|$ is an index of a total function. Since only values below $|W_i|$ go into $J(i)$, the marker a_i moves only finitely often. When a_i reaches its final place there will be some stage $s > a_i$ such that $\varphi_{j,s}(a_i)$ is defined. Since a_i does not move, j must already be in $J(i)$, and therefore $\varphi_j(x)\downarrow \neq \varphi_{g(i)}(x)\downarrow$ for some $x < a_i$. So φ_j does not extend $\varphi_{g(i)}$.

In the second part of the proof we show that $S \in EX[A] \Rightarrow FIN \leq_T A'$, where $FIN = \{i : W_i \text{ is finite}\}$ is the index set of the finite sets. It is well-known that $FIN \equiv_T \emptyset''$. By the Low Basis Theorem relativized to A, there is a set $B \geq_T A$ such that $B' \equiv_T A'$ and $\{H_i\}_{i\in\omega}$ is a uniformly B-recursive family of $\{0,1\}$-valued total recursive functions with $\varphi_{g(i)} \preceq H_i$ for all i. (The Low Basis Theorem is used as in [27, Ex. XII.3.9].) Note that

$$H_i = \varphi_j \;\Rightarrow\; [i \in FIN \Leftrightarrow |W_i| \leq j]$$

by condition (c) on $\varphi_{g(i)}$. Now assume that $S \in EX[B]$ via an OIIM M^B. Since $H_i \in S$, M^B converges on input H_i to an index $F(i) = \lim_n M^B(H_i \restriction n)$. Thus we have $i \in FIN \;\Leftrightarrow\; |W_i| \leq F(i)$. F is a B'-recursive total function. The test whether $|W_i| \leq j$ is recursive in K (since it is equivalent to $(\forall s)[|W_{i,s}| \leq j]$) and so, $FIN \leq_T B' \oplus K \equiv_T B'$. Hence B is high. Since $A' \equiv_T B'$, A is high, too. ∎

There is a also a direct proof of the second part that avoids the use of the Low Basis Theorem.

Define an A-recursive function $\text{fin}(i,s)$ as follows: Let σ_0, σ_1 be the two initial segments of length $s+1$ that are obtained from $\varphi_{g(i),s}$ by inserting 0 and 1 respectively for the missing value at $a_{i,s}$. Now let

$$\text{fin}(i,s) = \begin{cases} 1, & \text{if } |W_{i,s}| \leq M^A(\sigma_0) \text{ and } |W_{i,s}| \leq M^A(\sigma_1); \\ 0, & \text{otherwise.} \end{cases}$$

It is straightforward to show that $\lim_{s\to\infty} \text{fin}(i,s) = \chi_{FIN}(i)$. Thus, by the Limit Lemma [27, III.3.3], $FIN \leq_T A'$, i.e., A is high.

However, it is not clear whether this proof works for the following generalization.

Theorem 8.2 *If $EX^n \subseteq [1,n]EX[A]$ then A is high.*

Proof: Let S be the set from Theorem 8.1 and let $S^n = \{f_1 \oplus f_2 \oplus \ldots \oplus f_n : f_1, \ldots, f_n \in S\}$. Clearly $S^n \in EX^n$ by combining the n indices obtained for $f_1, \ldots, f_n$. Furthermore let the recursive function π_k compute an index of the k-th component of the n-function-join, i.e., if $\varphi_e = f_1 \oplus \ldots \oplus f_n$ then $\varphi_{\pi_k(e)} = f_k$.

Assume that $S^n \in [1,n]EX[A]$. Choose $B \geq_T A$ and the uniformly B-recursive family $\{H_i\}_{i\in\omega}$ as in the proof of Theorem 8.1. $S^n \in [1,n]EX[B]$ via a team of n OIIMs $M_1^B, \ldots, M_n^B$. For each n-tuple $(i_1, \ldots, i_n)$, let

$$\begin{aligned}
F_j(i_1,\ldots,i_n) &= \begin{cases} \lim_s M_j^B(H_{i_1} \oplus \ldots \oplus H_{i_n} \restriction s) & \text{if this limit exists;} \\ \uparrow & \text{otherwise;} \end{cases} \\
G_j(i_1,\ldots,i_n) &= \begin{cases} (b_1,\ldots,b_n) & \text{if } F_j(i_1,\ldots,i_n) \downarrow = e; \\ \uparrow & \text{if } F_j(i_1,\ldots,i_n) \uparrow; \end{cases} \\
\text{where } b_k &= \begin{cases} 1 & \text{if } |W_k| \leq \pi_k(e); \\ 0 & \text{otherwise.} \end{cases}
\end{aligned}$$

By the Limit Lemma, F_j and hence G_j are partial B'-recursive functions. Since at least one OIIM M_j^B infers $H_{i_1} \oplus \cdots \oplus H_{i_n}$, $e = F_j(i_1,\ldots,i_n)$ is an index of $H_{i_1} \oplus \cdots \oplus H_{i_n}$

and $\pi_k(e)$ is an index of H_{i_k}. By the choice of S, $\pi_k(e) \geq |W_{i_k}|$ iff W_{i_k} is finite. Therefore $G_j(i_1, \ldots, i_n) = (\chi_{FIN}(i_1), \ldots, \chi_{FIN}(i_n))$ and the n-fold characteristic function of FIN is computable via the n partial B'-recursive functions $G_1, \ldots, G_n$. By the Nonspeedup Theorem [3, Theorem 9], $FIN \leq_T B'$. So $FIN \leq_T A'$ and A is high. ∎

Note that Theorem 8.2 also provides an alternative and shorter proof for the fact that "$REC \in [1, n]EX[A] \Leftrightarrow A$ is high" which was originally proved in [10, Theorem 6.19].

Corollary 8.3 *A is high, if one of the following conditions hold:*
$EX^{n+1} \subseteq EX^n[A]$ for some n,
$[1, n+1]EX \subseteq [1, n]EX[A]$ for some n,
$BC \subseteq EX^*[A]$.

Proof: Smith [26, Theorem 6.1] showed, that $EX^n \subseteq [1, n+1]EX$. So $S^n \in [1, n+1]EX - [1, n]EX[A]$ for the set S^n from Theorem 8.2 and any non-high oracle A. Since Smith's result relativizes we get $S^n \notin EX^{n-1}[A] \subseteq [1, n]EX[A]$ for any non-high oracle A. The cylindrification $S' = \{f' : (\exists f \in S)(\forall x, y)[f'(\langle x, y\rangle) = f(x)]\}$ gives the last result (cf. [10, Fact 6.28]): If $S \in EX^1 - EX[A]$ then $S' \in BC - EX^*[A]$. ∎

For any A, there is a 1-generic set B with $B' \equiv_T B \oplus K \equiv_T A \oplus K$. Then $EX[B] = EX[B*] = EX[A*]$. Thus $EX^1 \subseteq EX[A*] \Leftrightarrow EX^1 \subseteq EX[B] \Leftrightarrow B' \geq_T K' \Leftrightarrow A \oplus K \geq_T K'$. Similar results hold for the notions $EX^n[A*]$ and $[1, n]EX[A*]$. The following theorem summarizes the results of the first part of this section:

Theorem 8.4 *If A has not omniscient EX-degree, i.e., if A is not high, then:*
$EX[A] = [1, 1]EX[A] \subset [1, 2]EX[A] \subset [1, 3]EX[A] \subset \ldots,$
$EX[A] \subset EX^1[A] \subset EX^2[A] \subset \ldots \subset EX^*[A] \subset BC[A].$
If A has not omniscient $EX*$-degree, i.e., if $A \oplus K \not\geq_T K'$, then:
$EX[A*] = [1, 1]EX[A*] \subset [1, 2]EX[A*] \subset [1, 3]EX[A*] \subset \ldots,$
$EX[A*] \subset EX^1[A*] \subset EX^2[A*] \subset \ldots \subset EX^*[A*] \subset BC[A*].$

Recall that $S_0 = \{\varphi_e : \varphi_e \in REC_{0,1} \wedge\ 1^e0 \preceq \varphi_e\}$ and $S_1 = \{\sigma 0^\omega : \sigma \in \{0, 1\}^*\}$. The following lemma shows that we cannot EX-infer $S_0 \cup S_1$ unless we have an omniscient oracle.

Lemma 8.5 *$S_0 \cup S_1 \notin EX[A]$ for non-high sets A.*

Proof: As in Theorem 8.1 construct a family of functions $\varphi_{g(i,j)}$ such that

(a) $1^j0 \preceq \varphi_{g(i,j)}$.

(b) $\varphi_{g(i,j)}$ is defined for all x with at most one exception $a_i > j$.

(c) If $\varphi_{g(i,j)} \subseteq \varphi_k$, φ_k is total, and W_i is finite then $k \geq |W_i|$.

By the recursion theorem, there is a recursive function h such that $\varphi_{g(i,h(i))} = \varphi_{h(i)}$ for all i. As in Theorem 8.1 it follows $S = \{f \in REC_{0,1} : (\exists i)[\varphi_{g(i,h(i))} \subseteq f]\} \notin EX[A]$ for all non-high oracles A. Now assume that $f = \varphi_{h(i)}$ is total. Then $1^{h(i)}0 \preceq f$ and $f \in S_0$. If otherwise $\varphi_{h(i)}$ is not total then $\varphi_{h(i)}(x) \downarrow= 0$ for all $x > a_i$ and so any $\{0,1\}$-valued total extension f of $\varphi_{h(i)}$ is a member of the set S_1. Therefore $S \subseteq S_0 \cup S_1$ and so $S_0 \cup S_1 \notin EX[A]$. ∎

A total inference machine M is called *reliable* if for all $f \in REC$, $M(f)$ converges iff it EX-identifies f. Let REX denote the set of all $S \subseteq REC$ such that S is EX-inferred by a reliable total inference machine. $REX[A]$ is defined as usual. If A is high then $REC \in REX[A]$. By a well-known result of Minicozzi (see [5]), $REX[A]$ is closed under union. Since $S_1 \in REX$ it follows that: $S_0 \in REX[A] \Rightarrow S_0 \cup S_1 \in REX[A]$. Hence we get from Lemma 8.5:

Corollary 8.6 $EX \subseteq REX[A] \;\Rightarrow\; A$ *is high.*

An inference machine M *consistently* infers S if $S \in EX$ via M and for every initial segment σ of a function from S we have $\sigma \preceq \varphi_{M(\sigma)}$. The corresponding inference criterion is denoted by $CONS$ and (in relativized form) by $CONS[A]$. If A is high then $REC \in CONS[A]$. It is well-known that $CONS \subset EX$ (see [21]).

Theorem 8.7 $EX \subseteq CONS[A] \;\Rightarrow A$ *is high.*

Proof: Let $S_2 = \{\sigma 2^\omega : \sigma \in \{0,1,2\}^*\}$. It is easy to see that $S = S_0 \cup S_2 \in EX$. Suppose that $S \in CONS[A]$ via M^A where A is not high. Since S is dense in $I = \{0,1,2\}^*$, i.e., since every $\sigma \in I$ has an extension in S, it follows that $\sigma \preceq \varphi_{M(\sigma)}$ for all $\sigma \in I$. We define an A-recursive function as follows:

$$g(x) = \min\{s : (\forall \sigma \in \{0,1,2\}^{\leq 2x})[\sigma \preceq \varphi_{M^A(\sigma),s}]\}.$$

Since A is not high there is a recursive function h such that $g(x) < h(x)$ for infinitely many x. Using the recursion theorem we define a recursive function φ_e as the limit of the following recursive sequence of strings: $\sigma_0 = 1^e 0$; if there is an $i < n$ and $b \in \{0,1\}$ such that $\sigma_n b \preceq \varphi_{i,h(n)}$ then choose the least such i and let $\sigma_{n+1} = \sigma_n \cdot (1-b)$; otherwise let $\sigma_{n+1} = \sigma_n 0$.

Obviously, $f = \varphi_e \in S_0$. Hence M^A infers f, i.e., there exists i and $m > e+1$ such that $\varphi_i = f \;\wedge\; M^A(\sigma) = i$ for all σ with $|\sigma| \geq m$ and $\sigma \preceq f$. Note that $|\sigma_n| = e+1+n < 2n$ for all $n \geq m$. By the definition of g it follows that $\sigma_n \cdot (f(|\sigma_n|)) \preceq \varphi_{i,g(n)}$ for all $n \geq m$. Since there are infinitely many $n > m$ with $h(n) > g(n)$ there are infinitely many n such that i satisfies the condition in the definition of σ_{n+1}. Since each $j < i$ is chosen at most once, there is an n_0 such that i is chosen in the definition of σ_{n_0+1}. Hence $\sigma_{n_0+1} \not\preceq f$, a contradiction. ∎

In the proof of Theorem 8.1 and in the last two proofs we took the well-known standard examples S which are used in the literature to witness the noninclusions $EX^1 \not\subseteq EX$, $EX \not\subseteq REX$, $CONS \not\subseteq EX$. We showed that the noninclusions can be lifted when the right-hand side is relativized to any non-omniscient oracle. The techniques used for the "lifting" part appear to be quite general and are applicable in several other cases as well.

9 Further results on EX-degrees

In this section we investigate the general structure of the EX-degrees in more detail. The methods exploit specific properties of EX-inference and do not work for BC or $LIMEX$. Also, the two main results of this section, Theorems 9.1, 9.4 are definitely false for BC and $LIMEX$ since they would contradict the fact that there are low omniscient BC- and $LIMEX$-degrees.

First we give a necessary condition for $EX[A] \subseteq EX[B]$ in terms of the jump operator.

Theorem 9.1 *If A, B are not high and $EX[A] \subseteq EX[B]$ then $A' \leq_T B'$.*

Proof: For any fixed set A consider the following class of total recursive functions:

$$S_A = \{1^x 0 : x \in A' \oplus \overline{A'}\} \cdot S_0 \cup \{1^x 0 : x \notin A' \oplus \overline{A'}\} \cdot S_1.$$

It is easy to see, that $S_A \in EX[(A')*] \subseteq EX[A]$.

Assume that A, B satisfy the hypothesis of the theorem. Therefore $S_A \in EX[B]$ via some OIIM M^B. By Lemma 8.5, it follows that $\{1^x 0 \cdot f : f \in S_0 \cup S_1\} \notin EX[B]$ for all x. Thus if M^B infers $1^x 0 \cdot S_0$ then it does not infer $1^x 0 \cdot S_1$ and vice versa. Indeed M^B infers $1^x 0 \cdot S_1$ iff $x \notin A' \oplus \overline{A'}$. As in [10, Lemma 6.15], M^B can be modified such that for any function $\sigma 0^\omega \in S_1$, M^B converges to some index e (not necessarily a correct one). The set $\{x : M^B$ does not infer $1^x 0 \cdot S_1\}$ is r.e. in B': M^B does not infer $1^x 0 \cdot S_1$ iff there are $\sigma \in 1^x 0 \cdot \{0,1\}^*$, e and y such that

(a) $(\forall k)[M^B(\sigma \cdot 0^k) \downarrow = e]$,

(b) $\varphi_e(y) \neq (\sigma \cdot 0^\omega)(y)$.

Since M^B is total recursive in B, condition (a) is recursive in B'; condition (b) is obviously recursive in K. Thus $A' \oplus \overline{A'}$ is r.e. in B' and so $A' \leq_T B'$. ∎

Remark: The converse of Theorem 9.1 does not hold; for a counterexample let A be an r.e. nonrecursive low set, and let $B = \emptyset$.

However, the converse holds for A 1-generic: If $A' \leq_T B'$ then we get $EX[A*] = EX[(A \oplus K)*] \subseteq EX[(B')*] \subseteq EX[B]$. For 1-generic A we get $EX[A] = EX[A*]$, thus $EX[A] \subseteq EX[B]$.

Theorem 9.1 allows to deduce some properties of EX-degrees from known facts about Turing degrees and the jump operator. For instance it follows that the EX-degrees are upward dense which was left open in [8].
The next theorem shows that EX- and $EX*$-degrees need not be compatible:

Theorem 9.2 *There exist sets A, B such that: $EX[A] \subset EX[B]$ and $EX[A*] \supset EX[B*]$.*

Proof: Let A, B be sets such that $B <_T K <_T A' <_T B' <_T K'$ and A is 1-generic. Then $EX[B] \not\subseteq EX[A]$ by Theorem 9.1. Furthermore, $EX[A] = EX[A*] \subseteq EX[(B')*] \subseteq EX[B]$. On the other hand, by Theorem 5.5 $EX[B*] \subset EX[A*]$, since $B \oplus K \equiv_T K <_T A' \equiv_T A \oplus K$. ∎

A set B is called 1-*generic over* A, if for each A-r.e. set W_e^A there is a string $\sigma \preceq B$ such that either $\sigma \in W_e^A$ or $\tau \notin W_e^A$ for all $\tau \succeq \sigma$ (see [16, p. 125]).

Theorem 9.1 implies that any non-omniscient EX-degree contains at most countably many sets (the omniscient degree contains of course continuum many sets). By a suitable relativization of the fact that $EX[A] = EX$ for A 1-generic we obtain the following:

Theorem 9.3 *If B is 1-generic over A and $B \leq_T A'$ then $EX[A \oplus B] = EX[A]$. Furthermore, the number of Turing degrees within every non-omniscient EX-degree is countably infinite.*

Proof: Modify the proofs of [10, Lemma 4.19, Theorem 4.21] in the following way: Consider A-recursive inference machines $M^{()}$ instead of recursive ones. As in the proof of [10, Lemma 4.19] it follows that if B is 1-generic over A then any $S \in EX[B]$ can be inferred by an A-recursive inference machine with only finitely many queries to B. If $B \leq_T A'$ then $EX[B*] \subseteq EX[A]$ (see Section 3). Now the first part of the Theorem follows.

The second part follows at once from the well-known fact that there are countably infinitely many T-degrees that are 1-generic over A and below A': By a relativization of the proof of the Friedberg Completeness Criterion there is a set B which is 1-generic over A such that $(A \oplus B)' = A'$. Now by iterating this construction with $B_0 = \emptyset$, one can obtain a sequence $B_1, B_2, \ldots$, such that $(A \oplus B_{n+1})' = (A \oplus B_n)' = A'$ and B_{n+1} is 1-generic over $(A \oplus B_n)$ and therefore 1-generic over A. So there are infinitely many different degrees $A \oplus B$ such that B is 1-generic over A and $B \leq_T A'$. ∎

We now characterize the inclusion-relation of EX-degrees below K. This characterization uses domination-properties of families of recursive functions; it fails for the case of BC and $LIMEX$ since there are non-high omniscient oracles below K. A corollary implies the Adleman-Blum Theorem and the proofs can be seen as a combination of the proof of Theorem 7.1 with the proof of Adleman-Blum.

Theorem 9.4 *Let $A \leq_T K$ and let $c_A(x) = min\{s \geq x : A \upharpoonright x \preceq A_s\}$ for some recursive approximation A_s of A. Then $EX[A] \subseteq EX[B]$ iff some B-recursive function dominates the family R_A, where $R_A = \{f \in REC : (\forall x)[f(x) \leq c_A(x)]\}$.*

Proof: ($\Leftarrow$) For any OIIM there is an equivalent one, which is monotone in the following sense: $M^X(\sigma) \leq M^X(\tau)$ for all oracles X and strings σ, τ with $\sigma \preceq \tau$. This follows easily using the padding lemma. Furthermore we may assume that use$(M^X, \sigma) < |\sigma|$.

If M^A infers $f \in REC$ then the total function

$$F_f(n) = \min\{s \geq n : M^{A_s}(f \upharpoonright n) = M^A(f \upharpoonright n)\}$$

is recursive since $M^A(f \upharpoonright n)$ is constant almost everywhere. Since $M^{A_s}(f \upharpoonright n)$ queries only numbers less than n it follows that $F_f(n) \leq c_A(n)$. So $F_f \in R_A$ for all f inferred by M^A. By assumption, some B-recursive function g dominates R_A.

Consider the inference algorithm $N^B(\sigma) = \min\{M^{A_s}(\sigma) : |\sigma| \leq s \leq g(|\sigma|)\}$. Further let f be an arbitrary function inferred by M^A, say $M^A(f) = e$. Since g dominates F_f we have $N^B(\sigma) \leq M^A(\sigma) = e$ for almost all $\sigma \preceq f$. Let $\eta \preceq f$ be the first string such that $M^A(\eta) = e$ and choose t such that $M^{A_s}(\eta) = e$ for all $s \geq t$. Since $M^{()}$ is monotone we get for all $\sigma \preceq f$ with $|\sigma| \geq t$:

$$N^B(\sigma) \geq \min\{M^{A_s}(\sigma) : s \geq t\} \geq \min\{M^{A_s}(\eta) : s \geq t\} \geq e.$$

Hence $N^B(f) = e = M^A(f)$. It follows that $EX[A] \subseteq EX[B]$.

($\Rightarrow$) Let $A \leq_T K$ be given and let $c_A(x) = \min\{s \geq x : A_s \upharpoonright x = A \upharpoonright x\}$, where A_s is a recursive approximation of A. Assume that $EX[A] \subseteq EX[B]$ and that R_A is not dominated by any B-recursive function. Define T_A and S_A as in the proof of Theorem 6.3. Choose an OIIM $M^{()}$ such that M^B infers S_A. Now we define suitable B-recursive functions as in the proof of the Adleman-Blum Theorem:

$$\begin{aligned} \hat{g}(\sigma) &= \min\{s \geq |\sigma| : M^B(\sigma 0^s) \neq M^B(\sigma) \vee (\forall x \leq |\sigma|)[\varphi_{M^B(\sigma),s}(x) \downarrow]\}, \\ g(x) &= \max\{\hat{g}(\sigma) : |\sigma| \leq x\}. \end{aligned}$$

The functions $g, \hat{g}$ are total since M^B infers S_A.

Since $g \leq_T B$ we could take some recursive function in R_A that is not dominated by g. But we would not have any control over *where* that function is larger than g. Instead we define a function $G \leq_T B$ that is based on (and bigger than) g, take an $h \in R_A$ that is not dominated by G, and we will have some knowledge of where h is bigger than g.

Let $G(x) = g(4g(2x))$ (G is actually taken of that form in order to have Claim 1 below). G is B-recursive and increasing. By assumption, there is a function $h \in R_A$ not dominated by G; since c_A is increasing, h can be taken increasing as well.

For each e we construct a recursive function f_e as follows. Starting with $\sigma_0 = 1^e0$, the function f_e is the limit of an inductively defined sequence σ_n of strings:

Let $\sigma_{n+1} = \sigma_n 1 a_0 a_1 \ldots a_n 1 0^{h(k_n)}$ where $k_n = |\sigma_n|$, $a_i = 0$ if $\varphi_{i,h(k_n)}(k_n + i + 1) \downarrow > 0$ and $a_i = 1$ otherwise. Note that $k_{n+1} = k_n + n + 3 + h(k_n)$.
Note that $\sigma_n \in T_A$ because $h(k_n) \leq c_A(k_n)$ and $n + 1 \leq c_A(|\sigma_n|)$.

Claim 1 There are infinitely many n such that $h(k_n) > g(2k_n)$.
Proof: There are infinitely many $x, n > 0$ such that $k_n < x \leq k_{n+1}$ and $h(x) > G(x)$. Assume that $g(2k_n) \geq h(k_n)$. Then we verify that $h(k_{n+1}) > g(2k_{n+1})$ which proves the claim: $g(2k_{n+1}) \leq g(2(k_n + n + 3 + h(k_n))) \leq g(2(k_n + n + 3 + g(2k_n))) \leq g(4g(2k_n))$, since $k_n + n + 3 \leq g(k_n + n + 3) \leq g(2k_n)$; finally $g(4g(2k_n)) = G(k_n) \leq G(x) < h(x) \leq h(k_{n+1})$. □

Claim 2 M^B does not infer f_e for any e.
Proof: Assume for a contradiction that M^B infers f_e. Then there exist i, m such that $M^B(f_e) = i = M^B(f_e \upharpoonright n)$ for all $n \geq m$. By Claim 1 there is an $n \geq i + m + 1$ with $h(k_n) > g(2k_n)$. Now let $a_0, \ldots, a_n$ be as in the construction of σ_{n+1} and $\sigma = \sigma_n 1 a_0 \ldots a_n 1$; note that $|\sigma| \leq 2k_n$. Since $\sigma 0^{g(2k_n)} \preceq \sigma 0^{h(k_n)} \preceq \sigma_{n+1}$, $M^B(\sigma 0^{\hat{g}(\sigma)}) = M^B(\sigma)$ and therefore $\varphi_{M^B(\sigma), \hat{g}(\sigma)}(k_n + i + 1) = \varphi_{i, \hat{g}(\sigma)}(k_n + i + 1) \downarrow$. But then $\varphi_{i,h(k_n)}(k_n + i + 1) \neq a_i$ which contradicts the hypothesis on i. □

Now we can complete the proof in the usual way. All f_e are infinite branches of T_A and the construction is uniform in e. Thus, by the recursion theorem, there is a fixed point $\varphi_{e_0} = f_{e_0}$. Hence S_A contains some f_{e_0} that is not inferred by M^B, which contradicts the hypothesis that M^B infers S_A. ▮

Note that Theorem 6.3 can be obtained directly from the previous result and Theorem 6.2. For the proof of ($\Rightarrow$) it is not essential that A is recursive in K but only that there is a set $R \subseteq REC$ which is dominated by an A-recursive function and is not dominated by any B-recursive function. Hence we obtain the following corollary.

Corollary 9.5 *If $EX[A] \subseteq EX[B]$ then every set $R \subseteq REC$ that is dominated by an A-recursive function is dominated by a B-recursive function.*

However, the converse does not hold. Even if we require in addition that $A' \leq_T B'$ then $EX[A] \subseteq EX[B]$ does not follow. We present a counterexample: By a well-known result of Jockusch and Soare (see [27, VI.5.15, XII.3.9]) there is a set $A \subseteq \omega^2$ of hyperimmune-free degree such that for all e, x: $\varphi_e(x) \downarrow \in \{0,1\} \Rightarrow \chi_A(e,x) = \varphi_e(x)$. Let $A^{[e]} = \{x : (e,x) \in A\}$. Consider the inference algorithm given by $M^A(\sigma) = e$, if $1^e0 \preceq \sigma \preceq A^{[e]}$, and $\mathrm{const}_0(\sigma)$ otherwise. M^A EX-infers a dense superset S of S_0. Since A has hyperimmune-free degree, every set $R \subseteq REC$ dominated by an A-recursive function is already dominated by a recursive function. Now choose a 1-generic set B with $A' \equiv_T B'$. B is not high since A is not high. As S is a dense superset of S_0 it follows from Lemma 5.2 that $S \notin EX[B*]$. Since B is 1-generic we have $EX[B*] = EX[B]$, hence $EX[A] \not\subseteq EX[B]$.

The growth condition on R_A is rather implicit. We get a more explicit condition if "$A \leq_T K$" is replaced by "A r.e.", giving the following corollary:

Corollary 9.6 *If A is r.e. then $EX[A] \subseteq EX[B] \Leftrightarrow A \leq_T B \vee K' \leq_T B'$.*

Proof: The "if" direction is obvious. For the other direction assume that B is not high and $A \not\leq_T B$. Further define c_A by a recursive enumeration A_s, not by an arbitrary approximation; $c_A(x)$ is recursively approximated by $\tilde{c}_A(x,t)$ from below, w.l.o.g. $\tilde{c}_A(x,t) \leq \tilde{c}_A(x',t')$, if $x \leq x'$ and $t \leq t'$.

Let g be any B-recursive function. Since $A \not\leq_T B$, g does not dominate c_A. Let $G(x)$ be the first t such that $\tilde{c}_A(y,t) > g(y)$ for some $y \geq x$. Note that $G \leq_T B$. Since B is not high, there is an increasing recursive function H not dominated by G. The function $h(x) = \tilde{c}_A(x, H(x))$ is dominated by c_A. Consider now some x with $H(x) > G(x)$. Then there is some $y \geq x$ with $\tilde{c}_A(y, G(x)) > g(y)$ and $h(y) = \tilde{c}_A(y, H(y)) \geq \tilde{c}_A(y, H(x)) \geq \tilde{c}_A(y, G(x)) > g(y)$, thus g does not dominate h. Therefore the family R_A is not dominated by any B-recursive function and by Theorem 9.4, $EX[A] \not\subseteq EX[B]$. ∎

Corollary 9.7 *If A is r.e. nonrecursive and B does not have omniscient $EX*$-degree then $EX[A] \not\subseteq EX[B*]$.*

Proof: It is mentioned in Section 3 that for every B there is a 1-generic set C such that $EX[C] = EX[B*]$. Assume that $REC \notin EX[B*]$. Then C is not high, by the Adleman-Blum Theorem. Jockusch and Posner [27, VI.3.8(c)] showed that no 1-generic Turing degree bounds an r.e. nonrecursive Turing degree. Thus $C \not\geq_T A$ for any r.e. nonrecursive set A. By Corollary 9.6, $EX[A] \not\subseteq EX[C] = EX[B*]$. ∎

The EX-degrees of A and B form a *minimal pair*, if any set C with $EX[C] \subseteq EX[A]$ and $EX[C] \subseteq EX[B]$ has trivial EX-degree. A set A has *minimal EX-degree*, if $EX[A]$ is nontrivial, and any set B with $EX[B] \subset EX[A]$ has trivial EX-degree. Slaman and Solovay [24] asked whether there exist minimal EX-degrees. The following results give a positive answer. Recall that a set is called 2-*generic* if it is 1-generic over K.

Lemma 9.8 *Let A be 2-generic and let R be a set of total K-recursive functions. If R is dominated by an A-recursive function then R is dominated by a recursive function.*

Proof: Let $\{e\}^A$ be the A-recursive function dominating R and let $U = \{\sigma : (\exists x)(\forall \tau \succ \sigma)[\{e\}^\tau(x) \uparrow]\}$. U is r.e. in K and does not contain any initial segment of A, since $\{e\}^A$ is total. As A is 2-generic there is some $\eta \preceq A$ such that no $\sigma \succeq \eta$ belongs to U. W.l.o.g. η is the empty string and $U = \emptyset$.

Therefore the sets $V(\sigma, x) = \{y : (\exists \tau \succeq \sigma)[\{e\}^\tau(x) \downarrow = y]\}$ are nonempty and uniformly r.e. Let $g(\sigma, x)$ be the first element which is enumerated in $V(\sigma, x)$. Now let $G(x) = \max\{g(\sigma, x) : |\sigma| \leq x\}$. G is a recursive function which dominates R as we shall show. Consider any $f \in R$. By definition there is some value x_0 such that $\{e\}^A(x) \geq f(x)$ for all $x \geq x_0$. Now let $W = \{\tau : (\exists x > x_0)[\{e\}^\tau(x) \downarrow < f(x)]\}$. Since no initial segment of A belongs to W and W is r.e. in K, there is some $\sigma \preceq A$ such that no $\tau \succeq \sigma$ belongs to W. Thus $y \geq f(x)$ for any $y \in V(\sigma, x)$ and $x \geq x_0$. Therefore $\lambda x.\, g(\sigma, x)$ dominates f and so does G, i.e., G dominates R. ∎

Lemma 9.9 *There is a minimal pair of EX-degrees.*

Proof: Let A be a low r.e. nonrecursive set and let B be 2-generic. Assume that $EX[C] \subseteq EX[A] \cap EX[B]$. Then $C' \leq_T A'$ and therefore C is below K. Now let $R_C = \{f \in REC : (\forall x)[f(x) \leq c_C(x)]\}$. By Theorem 9.4 ($\Rightarrow$), R_C is dominated by a B-recursive function and by Lemma 9.8, R_C is dominated by a recursive function. Thus $EX[C] = EX$ by Theorem 9.4 ($\Leftarrow$). So the EX-degrees of A and B form a minimal pair. ∎

Theorem 9.10 *There is a minimal EX-degree.*

Proof: Kumabe [18] proved that there is a 1-generic degree which bounds a minimal degree. Relativizing this result to computations relative to K, one gets two sets U, V such that $K <_T U \oplus K \leq_T V \oplus K$, $U \oplus K$ is a minimal cover of K and V is 1-generic over K, i.e., V is 2-generic. Choose a 1-generic set A such that $A' \equiv_T A \oplus K \equiv_T U \oplus K$. We claim that A has minimal EX-degree.

$EX[A]$ is nontrivial since A is not low. Consider now any set C with $EX[C] \subseteq EX[A] = EX[A*]$. By Theorem 9.1, $C' \leq_T A' \equiv_T A \oplus K$ and thus $C \oplus K \leq_T A \oplus K$. Since $A \oplus K$ is a minimal cover of K, either $C \oplus K \equiv_T A \oplus K$ or $C \oplus K \equiv_T K$. In the first case, by Theorem 5.5, $EX[C*] = EX[A*] = EX[A]$ and so $EX[A] \subseteq EX[C]$. In the second case, $C \leq_T K$. Since V is 2-generic and $EX[C] \subseteq EX[A] = EX[A*] \subseteq EX[V*] = EX[V]$ it follows as in the proof of Lemma 9.9 (with $B = V$) that C has trivial EX-degree. ▮

A perhaps more intuitive definition of minimal pair would be the following. We say that the EX-degrees of A and B form a *strong minimal pair* iff they are nontrivial and $EX[A] \cap EX[B] \subseteq EX$. However, we will now show that strong minimal pairs do not exist.

Theorem 9.11 *$EX[A*] \cap EX[B*] = EX$ iff $A \leq_T K$ or $B \leq_T K$.*

Proof: The "if" direction is obvious as $EX[A*] = EX$ iff $A \leq_T K$. For the other direction suppose that $EX[A*] \cap EX[B*] = EX$. Since $EX[A*] = EX[(A \oplus K)*]$ we may assume that $A, B \geq_T K$. Consider the following class $S_{A,B}$ of total recursive functions:

$$\begin{aligned} S_{A,B} &= \{1^x 0 : (x)_1 \in A \oplus \overline{A} \wedge (x)_2 \in B \oplus \overline{B}\} \cdot S_0 \\ &\cup \{1^x 0 : (x)_1 \notin A \oplus \overline{A} \wedge (x)_2 \notin B \oplus \overline{B}\} \cdot S_1. \end{aligned}$$

It is easy to see that $S_{A,B} \in EX[A*] \cap EX[B*]$. By the hypothesis there is an IIM M which EX-infers $S_{A,B}$. Let $C = \{x : M \text{ does not infer } 1^x 0 \cdot S_1\}$. By the definition of $S_{A,B}$ we get:

$$(1) \quad x \in C \;\Rightarrow\; (x)_1 \in A \oplus \overline{A} \;\text{ or }\; (x)_2 \in B \oplus \overline{B}.$$

Since $S_0 \cup S_1 \notin EX$ we get:

$$(2) \quad x \in (A \oplus \overline{A}) \times (B \oplus \overline{B}) \;\Rightarrow\; x \in C.$$

Arguing as in the proof of Theorem 9.1 it follows that C is r.e. in K. For every pair (y, z) there exist $a, b \in \{0, 1\}$ such that $x = \langle 2y + a, 2z + b\rangle \in (A \oplus \overline{A}) \times (B \oplus \overline{B})$, i.e., $x \in C$ by property (2). Therefore, there is a K-recursive function f such that for every pair (y, z), $f(y, z) = \langle a, b\rangle$, $a, b \in \{0, 1\}$, and $\langle 2y + a, 2z + b\rangle \in C$.

By property (1), either $2y + a \in A \oplus \overline{A}$, i.e., $a = 1 - \chi_A(y)$, or $2z + b \in B \oplus \overline{B}$, i.e., $b = 1 - \chi_B(z)$. Hence we get:

$$f(y, z) \neq \langle \chi_A(y), \chi_B(z)\rangle \text{ for all } y, z.$$

As in [4, Theorem 3.3] and [19, Theorem 9] one can now show that the T-degrees of A and B are comparable:

Assume that there is y such that $(f(y,z))_2 = 1 - \chi_B(z)$ for all z. Then $B \leq_T K \leq_T A$. Otherwise for each y there is a z_y such that $f(y,z_y) = \langle a_y, \chi_B(z_y)\rangle$. Then $a_y = 1 - \chi_A(y)$. Since such a z_y can be found using a B-oracle we get $A \leq_T B$. So either $EX[A*] \cap EX[B*] = EX[A*]$ or $EX[A*] \cap EX[B*] = EX[B*]$. By the hypothesis, it follows that $EX[A*] = EX$ or $EX[B*] = EX$, hence either $A \leq_T K$ or $B \leq_T K$. ▮

Theorem 9.12 *If A, B have nontrivial EX-degree then $EX[A] \cap EX[B] \not\subseteq EX$.*

Proof: Assume that A, B have nontrivial EX-degree. There are three cases:

(a) $A, B \not\leq_T K$:

Then by Theorem 9.11, $EX[A] \cap EX[B] \supseteq EX[A*] \cap EX[B*] \supset EX$.

(b) $A \leq_T K$, $B \not\leq_T K$:

Since $A \notin \mathcal{G}$ it follows from the proof of Theorem 6.3 that there is $\hat{S}_0 \subseteq S_0$ such that $\hat{S}_0 \cup S_1 \in EX[A]$ and $\hat{S}_0 \cup S_1 \notin EX$. Now let

$$S_{A,B} = \{1^x 0 : x \in B \oplus \overline{B}\} \cdot \hat{S}_0 \cup \{1^x 0 : x \notin B \oplus \overline{B}\} \cdot S_1.$$

$S_{A,B} \subseteq \{1\}^* 0 \cdot (\hat{S}_0 \cup S_1) \in EX[A]$ and $S_{A,B} \in EX[B]$. Assume for a contradiction, that $S_{A,B} \in EX$. Arguing as in the proof of Theorem 9.1 we find that $B \oplus \overline{B}$ is r.e. in K, which contradicts $B \not\leq_T K$.

(c) $A, B \leq_T K$:

Note that $A, B \notin \mathcal{G}$. A modification of the proof of Theorem 6.2 shows that $R_{A,B} = \{f \in REC : (\forall x)[f(x) \leq c_A(x) \wedge f(x) \leq c_B(x)]\}$ is not dominated by any recursive function. Using this we can modify the construction in the proof of 6.3 in a straightforward way to obtain a set $S \in (EX[A] \cap EX[B]) - EX$. ▮

10 Inference of r.e. sets from text

The properties of $TxtEx$-inference degrees are somewhat related to the EX-degrees. The most prominent difference was noted by Jain and Sharma [15] who proved that there is no omniscient $TxtEx$-degree. We determine the trivial $TxtEx$-degree and characterize the structure of the r.e. degrees and of the $*$-degrees. In contrast to the case of EX the proofs for $TxtEx$ are much easier.

In this section σ, τ denote strings from $(\omega \cup \{\star\})^*$. The following fact is well-known.

Fact 10.1 ([5], [21, Prop. 2.1A]) (1) *If M^A $TxtEx$-identifies L then there exists $\sigma \in (L \cup \{\star\})^*$ such that $M^B(\sigma \cdot \tau) = M^B(\sigma)$ for all $\tau \in (L \cup \{\star\})^*$.*
(2) $\{\omega\} \cup \{D \subseteq \omega : D$ *is finite*$\} \notin TxtEx[A]$ *for all* $A \subseteq \omega$.

The following result is a slightly refined version of [15, Theorem 1].

Theorem 10.2 *If $TxtEx[A] \subseteq TxtEx[B]$ then $A' \leq_T B'$.*

Proof: Suppose that $TxtEx[A] \subseteq TxtEx[B]$. Define

$$\begin{aligned}\mathcal{L}_A &= \{\{\langle i,x\rangle : x \in \omega\} : i \in A' \oplus \overline{A'}\} \\ &\cup \{\{\langle i,x\rangle : x \in D\} : i \notin A' \oplus \overline{A'} \wedge D \text{ finite}\}.\end{aligned}$$

It is easy to see, that $\mathcal{L}_A \in TxtEx[(A')*] \subseteq TxtEx[A]$. By hypothesis, $\mathcal{L}_A \in TxtEx[B]$ via some inference machine M^B. Fact 10.1, (2) implies that for all i, $i \in A' \oplus \overline{A'}$ iff M^B $TxtEx$-infers $N_i = \{\langle i,x\rangle : x \in \omega\}$. Using Fact 10.1, (1) we get that

$$A' \oplus \overline{A'} = \{i : (\exists \sigma \in (N_i \cup \{\star\})^*)(\forall \tau \in (N_i \cup \{\star\})^*)[M^B(\sigma) = M^B(\sigma \cdot \tau)]\}.$$

From this representation it follows at once that $A' \oplus \overline{A'}$ is r.e. in B' and so $A' \leq_T B'$. ▮

Corollary 10.3 [15] *There is no omniscient $TxtEx$-degree.*

Theorem 10.4 *If A is r.e. then $TxtEx[A] \subseteq TxtEx[B]$ iff $A \leq_T B$.*

Proof: The "if" direction is trivial. For the "only if" direction consider $\hat{\mathcal{L}}_A = \{A \cup \{x\} : x \in \omega\}$ and assume that A is r.e. and nonrecursive. It is shown in [21, Lemma 4.2.1C] that $\hat{\mathcal{L}}_A \notin TxtEx$. An inspection of the proof shows that $\hat{\mathcal{L}}_A \in TxtEx[B] \Rightarrow A \leq_T B$ as required. ▮

The following result shows that the trivial $TxtEx$-degree coincides with the trivial EX-degree.

Theorem 10.5 *$TxtEx = TxtEx[A]$ iff A has 1-generic degree and $A \leq_T K$.*

Proof: The "only if" direction follows from the corresponding result for EX-degrees [24] using the fact that if $S \subseteq REC$ then $S \in EX[A] \Leftrightarrow \{\text{graph}(f) : f \in S\} \in TxtEx[A]$.

The "if" direction follows by a modification of the proof of the corresponding result for EX-degrees [10, Theorem 4.15]: Suppose that G is 1-generic, $G \leq_T K$ and $\mathcal{L} \in TxtEx[G]$. Using the 1-genericity of G it follows from the proof of [10, Lemma 4.19] that there is an inference machine M^G that infers $\mathcal{L}$ from *recursive* text with only finitely many queries to the oracle. As $G \leq_T K$ it follows that $\mathcal{L} \in TxtEx$ with respect to recursive text. By a result of Blum and Blum [5], [21, Proposition 5.5.2B] this implies that $\mathcal{L} \in TxtEx$. ▮

Finally, we show that the $TxtEx*$-degrees are isomorphic with the T-degrees above $\mathbf{0}'$.

Theorem 10.6 *$TxtEx[A*] \subseteq TxtEx[B*]$ iff $A \oplus K \leq_T B \oplus K$.*

Proof: The "if" direction follows from the fact that $TxtEx[X*]$=$TxtEx[(X \oplus K)*]$ (which is analogous to $EX[X*] = EX[(X \oplus K)*]$). For the "only if" direction we first apply the proof of Theorem 10.2 to 1-generic sets A, B. The set $\mathcal{L}_A$ constructed there is actually in $TxtEx[(A')*]$. So $\mathcal{L}_A \in TxtEx[A*]$ by $A' \equiv_T A \oplus K$. If $\mathcal{L}_A \in TxtEx[B*] \subseteq TxtEx[B]$ then by Theorem 10.2 $A \oplus K \leq_T B' \equiv_T B \oplus K$. Since for given X there is a 1-generic A with $A \oplus K \equiv_T X \oplus K$, the "only if" direction follows for all sets A, B. ▮

Corollary 10.7
(1) *There is no omniscient $TxtEx*$-degree.*
(2) *$TxtEx[A*] = TxtEx$ iff $A \leq_T K$.*

11 Inference of r.e. sets from informant

The $InfEx$-degrees turn out to be markedly different from the EX-degrees and the $TxtEx$-degrees. Interestingly, the $InfEx*$ degrees and the $TxtEx*$-degrees coincide. But in contrast to $TxtEx$ there is an omniscient $InfEx$-degree, and the trivial $InfEx$-degree contains recursive sets only.

We start with some basic results with easy proofs. Then we prove a characterization of the $InfEx$-degrees below K which is more complex but analogous to Theorem 9.4 and which allows us to determine the trivial $InfEx$-degree and to obtain a couple of further results like the existence of a minimal $InfEx$-degree. We are also led to a problem concerning ω-r.e. T-degrees.

Definition 11.1 (Dekker, 1954) For any r.e. set A and any recursive 1-1 enumeration $a_0, a_1, a_2, \ldots$ of A we call $D = \{s : (\exists t > s)[a_t < a_s]\}$ a *deficiency set* for A.
Let $\mathcal{D}_A = \{A\} \cup \{C \subseteq \omega : C \text{ is cofinite}\}$.

Fact 11.2 (Dekker, see [27, Theorem V.2.5]) *Assume that A is r.e. and D is a deficiency set of A. D is r.e. and $A \equiv_T D$. If B is any set such that D is not hypersimple relative to B then $A \leq_T B$.*

Lemma 11.3 *Given any r.e. nonrecursive set A and any B. If D is a deficiency set of A then the following holds:*
(1) *$\mathcal{D}_D \in InfEx[A]$.*
(2) *If $\mathcal{D}_D \in InfEx[B]$ then $A \leq_T B$.*
(3) *$\mathcal{D}_D \notin InfEx[B*]$.*

Proof: (1) The inference machine guesses an index for D as long as it is consistent with the input σ. If σ is inconsistent with D it guesses an index of $\sigma 1^\omega$.

(2) Assume that $\mathcal{D}_D \in InfEx[B]$ via M^B. Then there is an index e of D and $\sigma_0 \preceq \chi_D$ such that $M^B(\sigma) = e$ for all σ with $\sigma_0 \preceq \sigma \preceq \chi_D$. We define a sequence of numbers $n_0, n_1, \ldots$ as follows: Let $n_0 = |\sigma_0|$, $n_{i+1} = n_i + 1 + k_i$, where $k_i = \min\{k : (\forall\sigma)[|\sigma| = n_i + 1 \Rightarrow M^B(\sigma 1^k) \neq e]\}$.
As M^B infers all cofinite sets and as W_e is not cofinite, it follows that n_i exists for all i. We claim that every interval $[n_i, n_{i+1} - 1]$ intersects $\overline{D}$: Choose $\sigma_1 \preceq \chi_D$ such that $|\sigma_1| = n_i + 1$. Then $M^B(\sigma_1 1^{k_i}) \neq e$, thus $\sigma_1 1^{k_i} \not\preceq \chi_D$, i.e., there is $x \notin D$ with $n_i \leq x \leq n_i + k_i$. Thus D is not hypersimple relative to B. By Fact 11.2, $A \leq_T B$.

(3) This follows by a modification of the proof of (2): Choose σ_0 such that in addition $\text{use}(M^B, \sigma) < |\sigma_0|$ for all σ with $\sigma_0 \preceq \sigma \preceq \chi_D$. In the definition of k_i we search for k such that $M^B(\sigma 1^k) \neq e \vee \text{use}(M^B, \sigma 1^k) \geq |\sigma_0|$. With these modifications we obtain that D is not hypersimple if $\mathcal{D}_D \in InfEx[B*]$. By Fact 11.2 this implies that A is recursive, a contradiction. ∎

It is now very easy to characterize the omniscient $InfEx$-degree.

Theorem 11.4 *$\mathcal{E} \in InfEx[A]$ iff $K \leq_T A$.*

Proof: The "if" direction is obvious. The other direction follows from Lemma 11.3, (2) if we choose D as a deficiency set of K. ∎

We obtain a partial characterization of the $InfEx$-degrees in terms of the jump-operator. It is similar though different from our previous results for EX and $TxtEx$.

Theorem 11.5 *If $InfEx[A] \subseteq InfEx[B]$ then $A' \leq_T B'$ or $K \leq_T B$.*

Proof: Suppose that $InfEx[A] \subseteq InfEx[B]$ and $K \not\leq_T B$. Let D denote a deficiency set of K. By Lemma 11.3, (2), we have $\mathcal{D}_D \notin InfEx[B]$. Let

$$\begin{aligned} \mathcal{L} &= \{\{\langle i, x\rangle : x \in D\} : i \in A' \oplus \overline{A'}\} \\ &\cup \ \{\{\langle i, x\rangle : x \in C\} : i \notin A' \oplus \overline{A'} \ \wedge \ C \text{ cofinite}\}. \end{aligned}$$

We have $\mathcal{L} \in InfEx[A]$. By hypothesis, $\mathcal{L} \in InfEx[B]$ via some M^B. Similar as in [10, Lemma 6.15] we can modify M^B such that it converges on any set from $\{\{\langle i, x\rangle : x \in C\} : C \text{ cofinite}\}$. (These sets can be put into a uniformly recursive listing $\mathcal{C}$. Now we can interleave the original inference algorithm with "identification by enumeration". I.e., if the original algorithm changes its mind we continue with "identification by enumeration" w.r.t. the listing $\mathcal{C}$; if this causes a mindchange we continue with the original algorithm, and so on.)

As $\mathcal{D}_D \notin InfEx[B]$ we get for all i: $i \in A' \oplus \overline{A'}$ iff there is a cofinite set C such that M^B does not infer $\{\langle i, x\rangle : x \in C\}$. Now it follows similarly as in the proof of Theorem 9.1 that $A' \leq_T B'$. ∎

From this theorem it follows that the structures of the $InfEx*$-degrees and the $TxtEx*$-degrees coincide.

Theorem 11.6 *$InfEx[A*] \subseteq InfEx[B*]$ iff $A \oplus K \leq_T B \oplus K$.*

Proof: The proof of this theorem is analogous to that of Theorem 10.6. The "if" direction follows from the fact that $InfEx[X*] = InfEx[(X \oplus K)*]$.

For the "only if" direction we first apply the proof of Theorem 11.5 to 1-generic sets A, B. The set $\mathcal{L}$ with parameter A constructed there is actually in $InfEx[(A')*]$. So $\mathcal{L} \in InfEx[A*]$ by $A' \equiv_T A \oplus K$. If $\mathcal{L} \in InfEx[B*] \subseteq InfEx[B]$ then by Theorem 11.5 $A \oplus K \leq_T B' \equiv_T B \oplus K$. Since for given X there is a 1-generic A with $A \oplus K \equiv_T X \oplus K$, the "only if" direction follows for all sets A, B. ∎

Corollary 11.7
(1) *There is no omniscient $InfEx*$-degree.*
(2) *$InfEx[A*] = InfEx$ iff $A \leq_T K$.*

Now we return to $InfEx$-degrees, but look on results for the special cases of r.e., ω-r.e. and $\mathbf{0}'$-recursive degrees.

Theorem 11.8 *If A, B are r.e. then $InfEx[A] \subseteq InfEx[B]$ iff $A \leq_T B$.*

Proof: This follows directly from Lemma 11.3. ∎

The next result shows that the trivial $InfEx$-degree is different from the trivial EX-degree.

Theorem 11.9 *There is a low 1-generic set A such that $InfEx[A] \supset InfEx$.*

Proof: Let B be a nonrecursive T-incomplete r.e. set. By [25, Theorem 2.1] there is a 1-generic set A such that $A \oplus B \equiv_T K$. Let $\mathcal{L} = \{B \oplus W : W \text{ r.e.}\}$. It follows that $\mathcal{L} \in InfEx[A]$. If we had $\mathcal{L} \in InfEx$ then it would follow that $\mathcal{E} \in InfEx[B]$, contradicting Theorem 11.4. ∎

The proof of the previous theorem can be easily extended to all 1-generic degrees $\mathbf{a} < \mathbf{0}'$ such that there exist r.e. degrees $\mathbf{b}, \mathbf{c}$ with $\mathbf{b} \leq \mathbf{c} \cup \mathbf{a}$ and $\mathbf{b} \not\leq \mathbf{c}$. However, we have shown that not all 1-generic degrees below $\mathbf{0}'$ have this property. The proof is included in Section 12. Nevertheless, using a different approach we shall show that

$InfEx[A]$ is trivial only if A is recursive. First we provide a characterization of the $InfEx$-degrees below K.

A total function f is called ω-r.e. iff there is a recursive function $g(x,s)$ such that $f(x) = \lim_{s\to\infty} g(x,s)$ and $|\{s : g(x,s) \neq g(x,s+1)\}| \leq x$ for all x. Let ω-REC denote the set of all ω-r.e. functions. A set A is called ω-r.e. iff $\chi_A \in \omega$-REC. A Turing degree is called ω-r.e. iff it contains an ω-r.e. set (see [9] for more information).

The following result is the analog of Theorem 9.4 for $InfEx$ degrees. We just need to replace REC by ω-REC in the definition of R_A.

Theorem 11.10 *Let $A \leq_T K$ and let $c_A(x) = \min\{s \geq x : A_s \upharpoonright x = A \upharpoonright x\}$ for some recursive approximation of A. Then $InfEx[A] \subseteq InfEx[B]$ iff there exists a B-recursive function that dominates $\tilde{R}_A$ where $\tilde{R}_A = \{f \in \omega\text{-}REC : (\forall x)[f(x) \leq c_A(x)]\}$.*

Proof: ($\Leftarrow$) The proof for Theorem 9.4 ($\Leftarrow$) generalizes almost verbatim. The only modification is that if M^A $InfEx$-infers the r.e. set W then the total function

$$\tilde{F}_W(n) = \min\{s \geq n : M^{A_s}(\chi_W \upharpoonright n) = M^A(\chi_W \upharpoonright n)\}$$

is ω-r.e. instead of recursive. This holds because the recursive enumeration of $W \upharpoonright n$ changes at most n times.

($\Rightarrow$) Though the general outline of this proof is the same as for Theorem 9.4 ($\Rightarrow$) more work is required here. First we need to define the analog of S_A.

Fix any recursive bijection $cod : \omega \to \{0,1\}^*$. For any $f : \omega \to \{0,1\}$ let $M_f = \{cod(u) : u \in \{0,1\}^* \ \wedge \ u \leq_{\text{lex}} f\}$ where $\leq_{\text{lex}}$ denotes the lexicographic ordering on $\{0,1\}^*$ induced by $0 < 1$. That is, $\sigma \leq_{\text{lex}} \tau$ iff $\sigma \preceq \tau$ or there exists a common prefix η of σ and τ such that $\eta \cdot 0 \preceq \sigma$ and $\eta \cdot 1 \preceq \tau$.

We say that f has a *monotone approximation* iff there is a recursive sequence of strings $\{\alpha_s\}_{s\in\omega}$ such that $\alpha_s \leq_{\text{lex}} \alpha_{s+1}$ for all s, and furthermore $\lim_s \alpha_s(x) = f(x)$ for all x.

If f has a monotone approximation then M_f is r.e. (and in addition semirecursive, but this is not important for the following).

We call f *admissible* iff there is a sequence of strings $\sigma_0, \sigma_1, \ldots$ such that $f = \lim_i \sigma_i$ and $\sigma_0 = 1^e0$ for some e, $\sigma_{n+1} = \sigma_n u_{n+1} v_{n+1} 1^{x_{n+1}} 0$ where u_{n+1}, v_{n+1} are strings with $|u_{n+1}| = (|\sigma_n|)^2$, $|v_{n+1}| = n+1$, and $0 \leq x_{n+1} \leq c_A(|\sigma_n|)$.

Now we are ready to define $\tilde{S}_A$ which plays in this proof the role of S_A:

$$\tilde{S}_A = \{M_f : f \text{ is admissible } \wedge \ (\exists e)[1^e0 \prec f \wedge \ M_f = W_e]\} \ \cup \ \{M_f : f =^* 1^\omega\}.$$

It is easy to see that there is a recursive operator which transforms f into χ_{M_f} and a recursive operator which transforms χ_{M_f} into f. Hence $\tilde{S}_A$ can be $InfEx$-inferred with B-oracle iff there is an OIIM M^B which on input f, with $M_f \in \tilde{S}_A$, converges to an index e with $W_e = M_f$. Note that any admissible f contains infinitely many zeros and that σ_n, u_n, v_n, x_n are effectively recoverable from f. Now it follows easily that $\tilde{S}_A \in InfEx[A]$.

Suppose for a contradiction that $\tilde{S}_A \in InfEx[B]$ and that $\tilde{R}_A$ is not dominated by any B-recursive function. By the remarks above there is an OIIM M^B such that for any f, with $M_f \in \tilde{S}_A$, $M^B(f)$ converges to an index of M_f. Define two B-recursive functions as follows:

$$\begin{aligned} \hat{g}(\sigma) &= \min\{s \geq |\sigma| : M^B(\sigma 1^s) \neq M^B(\sigma) \ \vee \\ &\qquad (\forall \tau, |\tau| \leq |\sigma|)[\tau \leq_{\text{lex}} \sigma \ \Rightarrow \ cod(\tau) \in W_{M^B(\sigma),s}]\}; \\ g(x) &= \max\{\hat{g}(\sigma) : |\sigma| \leq x\}. \end{aligned}$$

Since $M_f \in \tilde{S}_A$ for any $f = \sigma 1^\omega$ it follows that $g(\sigma)$ is defined for all σ. Hence g is an increasing total B-recursive function. Let $d(n) = 5n^2$ and $G(n) = g(d(2g(d(n))))$. G is an increasing B-recursive function. By hypothesis there is an ω-r.e. function $h \in \tilde{R}_A$ which is not dominated by G. Let $\hat{h}(x) = \max\{h(z) : z \leq x\}$. Note that $\hat{h}$ is increasing and $\hat{h}(x) \leq c_A(x)$ for all x. There is a recursive approximation $\hat{h}(x,s)$ such that $\lim_s \hat{h}(x,s) = \hat{h}(x)$ and $|\{s : \hat{h}(x,s) \neq \hat{h}(x,s+1)\}| \leq \Sigma_{z \leq x} z \leq x^2$.

We shall now construct, uniformly in e, an admissible function f_e via a monotone approximation $\{\alpha_s^e\}_{s\in\omega}$.

Fix e. In the construction we are using local variables c, u_n, v_n, x_n ($n \geq 1$). Furthermore let $\sigma_n = \sigma_{n-1} u_n v_n 1^{x_n} 0$ for $n \geq 1$, and $l_n = |\sigma_n|$, $q_n = (l_n)^2$. The current value of c indicates the largest n such that σ_n is defined.

Construction

Stage 0: $\sigma_0 = 1^e 0; c = 0; \alpha_0 = \sigma_0$.

Stage s+1:

1. Choose the least n such that either
   (a) $n = c+1$ or
   (b) $n \leq c \ \wedge \ \hat{h}(l_{n-1}, s) \neq \hat{h}(l_{n-1}, s+1)$.
2. In case (a) let $u_n = 0^{q_{n-1}}$. In case (b), if $u_n = 1^m 0^{q_{n-1}-m}$ let $u_n = 1^{m+1} 0^{q_{n-1}-m-1}$. (Note that this is well defined because $\hat{h}(l_{n-1}, s)$ changes at most q_{n-1} times.)
3. Define $v_n = a_0 \cdots a_n$ inductively for $i = 0, \ldots, n$:
   Let $a_i = 0$ if $cod(\sigma_{n-1} u_n a_0 \cdots a_{i-1}) \in W_{i,\hat{h}(l_{n-1},s+1)}$, else let $a_i = 1$.
4. Let $x_n = \hat{h}(l_{n-1}, s+1); c = n; \alpha_{s+1} = \sigma_n = \sigma_{n-1} u_n v_n 1^{x_n} 0$.

End of construction

If case (a) occurs in stage s+1 then we have $\alpha_s \prec \alpha_{s+1}$. In case (b) we get $\alpha_s <_{\text{lex}} \alpha_{s+1}$. Hence α_s is monotone with respect to $\leq_{\text{lex}}$. By induction on n we see that σ_n changes

only finitely often. Since $\sigma_n \prec \sigma_{n+1}$ there is a total function $f = f_e$ with $\lim_n \sigma_n = f$. $\{\alpha_s\}_{s\in\omega}$ is a monotone approximation of f and f is admissible.
Let k_n denote the final value of $l_n = |\sigma_n|$. We have $k_{n+1} = k_n+(k_n)^2+n+1+\hat{h}(k_n)+1 \le d(k_n) + \hat{h}(k_n)$.

Claim 1 There are infinitely many n such that $\hat{h}(k_n) > g(d(k_n))$.

Proof: There are infinitely many $x, n > 0$ such that $k_n < x \le k_{n+1}$ and $\hat{h}(x) > G(x)$. Assume that $\hat{h}(k_n) \le g(d(k_n))$. Then we verify that $\hat{h}(k_{n+1}) > g(d(k_{n+1}))$ which proves the claim: $g(d(k_{n+1})) \le g(d(d(k_n) + \hat{h}(k_n))) \le g(d(d(k_n) + g(d(k_n)))) \le g(d(2g(d(k_n)))) = G(k_n) \le G(x) < \hat{h}(x) \le \hat{h}(k_{n+1})$. $\Box$

Claim 2 $M^B(f)$ does not converge to an index of M_f.

Proof: Suppose for a contradiction that $M^B(f)$ converges to an index i of M_f. Choose m such that $M^B(f \restriction n) = i$ for all $n \ge m$. By Claim 1 there is $n \ge i + m$ with $\hat{h}(k_n) > g(d(k_n))$. Consider the stage of the construction when the final value of σ_{n+1} is defined: $\sigma_{n+1} = \sigma_n u_{n+1} a_0 \cdots a_n 1^{x_n} 0$. Let $\sigma = \sigma_n u_{n+1} a_0 \cdots a_n$. We have $|\sigma| = k_n + (k_n)^2 + n + 1 \le d(k_n)$. Since $\sigma 1^{g(d(k_n))} \preceq \sigma 1^{\hat{h}(k_n)} \preceq \sigma_{n+1} \prec f$ it follows that $M^B(\sigma 1^{\hat{g}(\sigma)}) = M^B(\sigma) = i$ and therefore by the definition of $\hat{g}$:

$$(*) \quad (\forall \tau, |\tau| \le |\sigma|)[\tau \le_{\text{lex}} \sigma \Rightarrow \mathit{cod}(\tau) \in W_{i,\hat{g}(\sigma)}].$$

Now we consider the value of a_i as defined in step 3 of the construction and obtain a contradiction. Let $\rho = \sigma_n u_{n+1} a_0 \cdots a_{i-1}$. If $a_i = 0$ then $\mathit{cod}(\rho 1) \in W_{i,\hat{h}(k_n)}$ and $f <_{\text{lex}} \rho 1$, hence $\mathit{cod}(\rho 1) \notin M_f$, i.e., $M_f \ne W_i$, a contradiction. If $a_i = 1$ then $\mathit{cod}(\rho 1) \notin W_{i,\hat{h}(k_n)}$, hence $\mathit{cod}(\rho 1) \notin W_{i,\hat{g}(\sigma)}$ since $\hat{h}(k_n) > g(d(k_n)) \ge \hat{g}(\sigma)$. This contradicts property $(*)$ with $\tau = \rho 1$. $\Box$

The construction is uniform in e and an index of M_{f_e} can be computed uniformly from the monotone approximation. Thus, by the recursion theorem, there is e such that $W_e = M_{f_e}$. Hence, by the hypothesis on M^B, $M^B(f_e)$ converges to an index of M_{f_e}, which contradicts Claim 2. This final contradiction shows that $InfEx[A] \not\subseteq InfEx[B]$ which completes the proof of the theorem. ∎

For the proof of $(\Rightarrow)$ it is not essential that $A \le_T K$ but only that there is a set of ω-r.e. functions which is dominated by an A-recursive function and is not dominated by any B-recursive functions. Hence we obtain the following corollary:

Corollary 11.11 *If $InfEx[A] \subseteq InfEx[B]$ then every set of ω-r.e. functions that is dominated by an A-recursive function is dominated by a B-recursive function.*
In particular $(\forall B)[InfEx[A] \subseteq InfEx[B] \Leftrightarrow A \le_T B]$ for all ω-r.e. sets A.

Now we are ready to determine the trivial $InfEx$-degree.

Theorem 11.12 *$InfEx[A] = InfEx$ iff A is recursive.*

Proof: We need to show the "only if" direction. Assume that $InfEx[A] = InfEx$. By Theorem 11.5 it follows that $A' \leq \emptyset'$, in particular $A \leq_T K$.

By Theorem 11.10 some recursive function g dominates $\tilde{R}_A$ based on a fixed approximation $\{A_s\}_{s\in\omega}$ of A. We define a recursive approximation of strings $\{\alpha_s\}_{s\in\omega}$ and a 2-r.e. function f such that $f \in \tilde{R}_A$ and for all s:

$$(P_s) \quad \alpha_s \prec A \ \vee \ (\exists t \geq s)[f(t) > g(t)].$$

First let α_s be the longest common prefix of $A_s, A_{s+1}, \ldots, A_{g(s)}$ as in Theorem 6.2:

$$\alpha_s(x) = \begin{cases} \chi_{A_s}(x) & \text{if } x < s \text{ and } (\forall t, s \leq t \leq g(s))(\forall y \leq x)[\chi_{A_s}(y) = \chi_{A_t}(y)]; \\ \uparrow & \text{otherwise, in particular for } x \geq s. \end{cases}$$

Now define f as follows:

$$f(s) = \begin{cases} 0 & \text{if } (\exists t > s)[\alpha_s \preceq \alpha_t]; \\ g(s)+1 & \text{otherwise, i.e., if } (\forall t > s)[\alpha_s \not\preceq \alpha_t]. \end{cases}$$

Since the condition in the first line of the definition is r.e., $f(s)$ is initialized as $g(s) + 1$ and then reset to 0 if there is some $t > s$ with $\alpha_t \succeq \alpha_s$, so f is a 2-r.e. function. Furthermore if $f(s) = g(s) + 1$, then $\alpha_s \not\preceq A$ and thus $A_t \upharpoonright s \not\preceq A$ for $t = s, s+1, \ldots, g(s)$. So $f(s) \leq c_A(s)$. Otherwise $f(s) = 0 \leq c_A(s)$ anyway. Therefore $f \in \tilde{R}_A$.

f satisfies the requirement P_s: If $\alpha_s \not\preceq A$, then only finitely many α_t extend α_s, thus there is a maximal t with $\alpha_t \succeq \alpha_s$. By the transitivity of $\preceq$, no $\alpha_{t'}$ with $t' > t$ extends α_t. So $f(t) = g(t) + 1$ and P_s holds.

Since g dominates f, there is some s such that $f(t) \leq g(t)$ for all $t \geq s$. So by P_t, $\alpha_t \preceq A$ for all $t \geq s$. Because the length of the α_t is not bounded by a constant, it follows that A is recursive. ▮

By Corollary 11.11 the $InfEx$-degrees coincide with Turing degrees when restricted to ω-r.e. sets. However, the next result shows that there exist sets below K of the same $InfEx$-degree but of different Turing degree. The proof is given in the next section.

Theorem 11.13 *There are sets A, B, $A \leq_T K$, B r.e. such that $A \not\leq_T B$ and $InfEx[B] = InfEx[A \oplus B]$.*

Since Lemma 9.8 applies to sets R of ω-r.e. functions we can easily transfer the proof of Theorem 9.10.

Theorem 11.14 *There is a minimal $InfEx$-degree.*

Theorem 11.15 *If A is 2-generic then $InfEx[A] = InfEx[A*]$.*

Proof: Note that χ_M is recursive in K if M is r.e. Then a modification of the proof of "A is 1-generic $\Rightarrow EX[A] = EX[A*]$" (see [10, Lemma 4.19]) yields the theorem. (In the proof of [10, Lemma 4.19] one replaces f by χ_M, and obtains that the corresponding W is in Σ_2 instead of Σ_1. The final step uses the 2-genericity of A.) ▮

Corollary 11.16 *For any 2-generic set A there is a set $B <_T A$ with $InfEx[A] = InfEx[B]$. In particular T-degrees and $InfEx$-degrees do not coincide.*

Proof: Jockusch [16, Theorem 5.1] proved that for every 2-generic set A there is a set $B <_T A$ such that A is r.e. in B. Therefore we get: $InfEx[A] = InfEx[A*] \subseteq InfEx[(B')*] \subseteq InfEx[B] \subseteq InfEx[A]$, which implies the corollary. ▮

12 On T-degrees below $\mathbf{0}'$ that do not help ω-r.e. degrees and related results

In this section we show that T-degrees and $InfEx$-degrees do not coincide below $\mathbf{0}'$. The proof is a $0''$-argument. It turns out that the techniques used here also solve some open questions in recursion theory. We first present those results in detail to illustrate the techniques, and then sketch the proof of the inference result. Throughout this section "degree" stands for "Turing degree".

Posner and Robinson [23, Corollary 6] showed that every nonrecursive degree $\mathbf{a}$ "helps" every degree $\mathbf{c}$ in the sense that either $\mathbf{c} \leq \mathbf{a}$ or there is a degree $\mathbf{b}$ such that $\mathbf{c} \not\leq \mathbf{b}$ and $\mathbf{c} \leq \mathbf{a} \cup \mathbf{b}$. The proof shows that this is also true for $\mathbf{a}, \mathbf{b}, \mathbf{c} \leq \mathbf{0}'$. What happens if $\mathbf{b}, \mathbf{c}$ are required to be r.e.? Jockusch observed that in this case no 2-generic degree $\mathbf{a}$ helps $\mathbf{c}$. But is there a nonrecursive (or even 1-generic) degree $\mathbf{a}$ below $\mathbf{0}'$ that does not help (w.r.t. r.e. degrees)? The following theorem gives a positive answer.

Theorem 12.1 *There is a 1-generic degree $\mathbf{a} < \mathbf{0}'$ such that for all r.e. degrees $\mathbf{b}, \mathbf{c}$, if $\mathbf{c} \leq \mathbf{a} \cup \mathbf{b}$ then $\mathbf{c} \leq \mathbf{b}$.*

Proof: We assume that the reader is familiar with the tree framework for presenting $0''$-arguments. See [27, Chap. XIV] whose terminology and notation we use.

Let $\{(U_e, V_e, W_e, \Phi_e)\}_{e \in \omega}$ be a recursive enumeration of all quadruples of r.e. sets U_e, V_e, W_e, and partial recursive functionals Φ_e. We will construct by full approximation a set A and enumerate for each e a functional Γ_e such that the following requirements are met:

$$(R_e) \quad U_e \neq \Phi_e^{V_e \oplus A} \ \vee \ U_e = \Gamma_e^{V_e}.$$

$$(S_e) \quad (\exists \sigma \prec A)[\sigma \in W_e \vee (\forall \tau \succeq \sigma)[\tau \notin W_e]].$$

A_s denotes the approximation of A at the end of stage s. It is convenient to identify it with a string of zeros and ones of length at most s. If we write $A_s \preceq A_t$ this means that the string A_s is a prefix of the string A_t.
The basic module for a single (R_e)-requirement is as follows:

BASIC MODULE:
Cycle x: Let $\overline{s} = s$ if cycle x is started in stage s.

1. Wait for stage $s_0 + 1 > \overline{s}$ such that $U_{e,s_0}(x) = \Phi_e^{V_e \oplus A}(x)[s_0]$ with use $\phi(e, x, s_0)$. (By definition the use is equal to 1+ the largest number actually queried during the computation with oracle $(V_e \oplus A)[s_0]$.)
2. Set $\Gamma_e^{V_{e,s_0}}(x) = U_{e,s_0}(x)$ with the same use $\phi(e, x, s_0)$.
3. Wait for a stage $s_1 + 1$, $s_1 > s_0$ such that either
   (a) $V_{e,s_0} \restriction \phi(e, x, s_0) \neq V_{e,s_1+1} \restriction \phi(e, x, s_0)$, or
   (b) $U_{e,s_0}(x) \neq U_{e,s_1}(x)$, and case (a) does not hold.
4. In case (a) let $\overline{s} = s_1 + 1$ and goto step 1. [Now $\Gamma_e^{V_{e,\overline{s}}}(x)$ is undefined.]
5. In case (b) reset $A_{s_1+1} = A_{s_0}$, set up the restraint $r_e = \phi(e, x, s_0)$, and stop all active cycles.
   [We now have $0 = \Phi_e^{V_e \oplus A}(x)[s_1 + 1] = \Phi_e^{V_e \oplus A}(x)[s_0] \neq U_{e,s_1}(x) = U_e(x) = 1$.]
6. Wait for a stage $s_2 + 1$, $s_2 > s_1$ such that $V_{e,s_0} \restriction \phi(e, x, s_0) \neq V_{e,s_2} \restriction \phi(e, x, s_0)$.
7. Reset $A_{s_2+1} = A_{s_1}$; let $\Gamma_e^X(x) = 1$ for all $X \supseteq V_{e,s_2}$, let $r_e = 0$, halt cycle x, and reactivate all stopped cycles. $\square$

We say that cycle x *requires attention* at stage $s + 1$
(1) if it is not yet started, or
(2) if it is waiting at step 1 [3 or 6, resp.], and at stage $s + 1$ the condition in step 1 [3 or 6, resp.] is satisfied.
If cycle x is allowed to *act* it will execute the next instructions until it reaches step 1, 3, 6, or 7.
Stage s: If no cycle x is waiting in step 6 then let the least cycle act that requires attention. Otherwise do nothing.
END OF BASIC MODULE

Note that at each stage at most one cycle x is waiting at step 6. During that period the definition of Γ_e is suspended. If cycle x is waiting forever then $U_e(x) = 1 \neq 0 = \Phi_e^{V_e \oplus A}(x)$. If cycle x reaches step 7 then the V_e-change allows us to reset $\Gamma_e^{V_e}(x) = 1$

forever. If cycle x moves from step 4 back to step 1 then $\Gamma_e^{V_e}(x)$ is undefined for the current value of V_e. If cycle x is waiting forever in step 3 then $\Gamma_e^{V_e}(x) = U_e(x)$.

(S_e) is the standard requirement to ensure that A is 1-generic (see [27, VI.3.7]). The strategy for a single (S_e)-requirement is as follows:

We are using a local variable σ_e. Assume that at stage $s+1$, σ_e is undefined or $\sigma_e \not\preceq A_s$, and there is $\sigma \in W_{e,s}$ with $A_s \restriction (r+e) \preceq \sigma$ where r is the restraint of the higher priority requirements. In that case we say that (S_e) requires attention at stage $s+1$. Then choose the first such σ (in order of enumeration) and set $\sigma_e = \sigma$; $A_{s+1} = \sigma$ with priority e. In that case we say that (S_e) acts at stage $s+1$.

As a preparation for the full construction consider how one (R)-requirement combines with infinitely many (S)-requirements in a finite injury construction with the priority ordering $R > S_0 > S_1 > S_2 > \cdots$:

At each stage $s+1$ we let the least cycle act that requires attention and then the strategy for the (S)-requirement of highest priority that requires attention. This defines the value A_{s+1}.

We claim that $A(x) = \lim_s A_s(x)$ exists. This set A is 1-generic and satisfies requirement (R). The key for the convergence of $A_s(x)$ is that each (S_e) acts at most finitely often. This is easy to see if there is some cycle x that is waiting forever at step 6. In that case there are only finitely many stages where the (R)-strategy acts. Now suppose every cycle which enters step 6 will eventually enter step 7.

By induction choose t_0 such that no $(S_{e'})$ with $e' < e$ acts at any stage $t \geq t_0$. Consider the first stage t_1+1 where (S_e) acts (if there is some). So $A_{t_1+1} = \sigma_e \in W_e$. Let $t_2+1 > t_1+1$ be the first stage with $A_{t_1+1} \not\preceq A_{t_2+1}$. At this stage some cycle x has reached step 6 and has reset A_{t_2+1} to a value $A_{s_0(x)}$ for $s_0(x) < t_1+1$. By assumption, there is a stage t_3+1 where cycle x enters step 7 and halts. In that stage there is no (R)-restraint and, by the reset action of cycle x in step 7, we again have $A_{t_1+1} \preceq A_{t_3+1}$. *The point is that (S_e) can be injured only by those cycles x that have been active before step t_1+1, and only once by each such x.* This is because we suspend the definition of Γ during the time when some x is waiting at step 6. Hence each time when in stage $s+1 > t_1+1$ a new agreement $U(x') = \Phi^{V\oplus A}(x')$ is established and $\Gamma^V(x')$ is set to $U(x')$ we have $A_{t_1+1} \preceq A_{s+1}$. So if cycle x' later enters step 5 its reset action will not injure (S_e).

It follows that $A_s(x)$ converges. Now it follows that each (S_e)-requirement is satisfied. If there are only finitely many cycles then (R) is satisfied because $U \neq \Phi^{V\oplus A}$. If there are infinitely many cycles then no cycle is waiting forever at step 6. If some cycle x is waiting at step 1 at infinitely many stages then $\Phi^{V\oplus A}(x)$ is undefined. In all other cases we have $U = \Gamma^V$. Thus requirement (R) is satisfied.

For later generalizations keep in mind that this argument goes through as long as we can guarantee that every cycle acts at most finitely often.

In the full construction we organize the basic modules on a tree $T = \{0,1\}^*$ with the usual orderings $\leq$ and $<_L$ induced by $0 < 1$ (see [27, Definition XIV.1.1]). For each requirement (R_e) we have 2^e strategies α with $\alpha \in \{0,1\}^e$. The strategy α works below the strategies $\alpha \restriction e'$ for $e' < e$ and guesses their outcomes. The final outcome of an α-module is 0 if the α-module acts infinitely often. Otherwise (i.e., if some cycle of the α-module is waiting forever in step 6) the final outcome is 1. The final outcomes are approximated by the usual Π_2-guessing. The interesting point in this argument is the convergence of A_s.

With each α ($|\alpha| = e$) we associate an α-module and below it an (S_α)-strategy. The α-module builds a functional Γ_α with the intention that $U_e = \Gamma_\alpha^{V_e}$.

The restraint of the α-module defined at the end of stage s is called $r_0(\alpha, s)$. The restraint for the sake of (S_e) that is set up below the α-module ($|\alpha| = e$) is called $r_1(\alpha, s)$. $R(\alpha, s) = \max(\{r_i(\alpha', s) : \alpha' < \alpha \ \wedge \ i = 0,1\} \cup \{r_0(\alpha, s)\})$.

To *initialize* α at stage s means to set $r_0(\alpha, s) = r_1(\alpha, s) = 0$, $\sigma_\alpha =\uparrow$, to reset $\mathrm{flag}(\alpha) = 0$, to cancel the current version of Γ_α, and to cancel all α-cycles. So each time when we initialize the α-module it will start to build a new functional Γ_α. It will turn out that if α is on the true path then the α-module is initialized only finitely often, i.e., there exists a final version of Γ_α. We will show that (R_e), $e = |\alpha|$, is satisfied if we let Γ_e be the final version of Γ_α.

Construction

Stage 0: Initialize all α.

Stage s+1:

1. Let $\alpha = \lambda; e = 0$.
2. If $\mathrm{flag}(\alpha) = 1$ then let $b = 0$, reset $\mathrm{flag}(\alpha) = 0$ and goto step 3.
   If there is some α-cycle waiting in step 6 and it does not require attention at stage $s + 1$ then let $b = 1$ and goto step 3.
   Otherwise choose the least cycle x of strategy α that requires attention at stage $s + 1$, and let cycle x act.
   If it performed step 5 then initialize all β with $\alpha \cdot 1 \leq \beta$ and goto stage $s + 2$.
   If it performed step 7 then initialize all β with $\alpha \cdot 1 \leq \beta$, let $\mathrm{flag}(\alpha) = 1$, and goto stage $s + 2$.
   In all other cases let $b = 0$ and goto step 3.
3. If σ_α is undefined or $\sigma_\alpha \not\preceq A_s$,
   and if there is $\sigma \in W_{e,s}$ such that $A_s \restriction (R(\alpha, s) + e) \preceq \sigma$,
   then choose the least such σ (in the order of enumeration) and do the following:
   Let $\sigma_\alpha = \sigma$, $r_1(\alpha, s + 1) = |\sigma|$, let $A_{s+1} = \sigma$,

initialize all β with $\alpha < \beta$, and goto stage $s+2$.

We say that (S_α) *acts* at stage $s+1$.

4. Let $\alpha = \alpha \cdot b$ and $e = e+1$.

If $e < s$ then goto step 2, else initialize all β with $\alpha <_L \beta$ and goto stage $s+2$.

End of Construction

Verification:

The true path f is defined inductively by $f(i) = b$ where $(f \upharpoonright i) \cdot b$ is the leftmost successor of $f \upharpoonright i$ eligible to act infinitely often.

Lemma 12.2 *Let $\alpha = f \upharpoonright e$.*

(1) *α is initialized only finitely often.*

(2) *If $f(e) = 0$ then there are infinitely many stages where some α-cycle acts.*

(3) *If $f(e) = 1$ then there is a unique α-cycle that is waiting forever in step 6. Let $t_1 + 1$ be the stage when it entered step 6 for the last time. Then $A_{t_1+1} \preceq A_t$ for almost all t.*

(4) *$\{R(\alpha, s) : s+1$ is an α-stage with outcome $f(e)\}$ is finite.*

(5) *(S_α) acts finitely often. If $\sigma = lim_s\ \sigma_\alpha[s]$ exists, then $\sigma \preceq A_s$ for almost all s.*

Proof: Assume by induction that (1-6) are true for all $e' < e$. Choose the least t_0 such that for all $t \geq t_0$ and all $\gamma \prec \alpha$:

(I) γ is not initialized at stage t.

(II) If $\gamma \cdot 1 \prec f$ then no γ-cycle acts at stage t.

(III) (S_γ) does not act at stage t.

It follows that α is not initialized at any stage $t \geq t_0$. Now (2) follows by inspection of the α-module.

(3) Assume that there is an α-cycle that is waiting forever in step 6. Let $t_1 + 1 \geq t$ be the stage when it performed step 5 and entered step 6 which it does not leave at all later stages. No α-cycle is allowed to act after stage $t_1 + 1$. Hence the α-module outputs 1 at all later stages. At each stage $t \geq t_1 + 1$ we say that (x, γ) *might injure* α if at the end of stage t the γ-cycle x is waiting at step 3 with $A_{t_1+1} \not\preceq A_{s_0(x)}$ (this cycle might later perform step 5 and reset A to $A_{s_0(x)}$). Let $\mathrm{inj}(\gamma, t)$ denote the set of all (x, γ) that might injure α.

$$\begin{aligned}
\mathrm{Inj}(\alpha, t) &= (\mathrm{inj}(\alpha \upharpoonright (e-1), t), \mathrm{inj}(\alpha \upharpoonright (e-2), t), \ldots, \mathrm{inj}(\alpha \upharpoonright 0, t)). \\
c(\alpha, t) &= (|\mathrm{inj}(\alpha \upharpoonright (e-1), t)|, |\mathrm{inj}(\alpha \upharpoonright (e-2), t)|, \ldots, |\mathrm{inj}(\alpha \upharpoonright 0, t)|).
\end{aligned}$$

We will now show that $\mathrm{Inj}(\alpha, t)$ converges to a finite limit. We call stage t *special* if there is no γ-cycle with $\gamma \cdot 0 \preceq \alpha$ waiting in step 6 at the end of stage t. Let

$z_0 < z_1 < \cdots$ be the sequence of all special stages $\geq t_1 + 1$ (note that $t_1 + 1 = z_0$). We claim that $A_{t_1+1} \preceq A_{z_i}$ and $\mathrm{inj}(\beta, z_i) = \emptyset$ for all i and all β with $\alpha \cdot 1 \leq \beta$. This is true for $i = 0$. (Note that each $\beta \geq \alpha \cdot 1$ is initialized at stage $t_1 + 1$.) Assume by induction that it is true for z_i. Suppose there is a least stage $s + 1$, $z_i < s + 1 \leq z_{i+1}$ with $A_{t_1+1} \not\preceq A_{s+1}$. Some γ-cycle x with $\gamma \cdot 0 \preceq \alpha$ and $(x, \gamma) \in \mathrm{inj}(\gamma, z_i)$ must have performed step 5 at stage $s + 1$. Then z_{i+1} is the next γ-stage where we reset $A_{z_{i+1}} = A_s$, so $A_{t_1+1} \preceq A_{z_{i+1}}$. Since all β with $\beta \geq \gamma \cdot 1$ are initialized it follows that $\mathrm{inj}(\beta, z_{i+1}) = \emptyset$ for all β with $\beta \geq \alpha \cdot 1$. (Note that no β with $\gamma \cdot 0 \preceq \beta$ was active after stage $s + 1$.) Furthermore,

$$\mathrm{inj}(\gamma, z_{i+1}) \subseteq \mathrm{inj}(\gamma, z_i) - \{(x, \gamma)\}, \text{ and}$$

$$\mathrm{inj}(\gamma', z_{i+1}) \subseteq \mathrm{inj}(\gamma', z_i), \text{for } \gamma \prec \gamma' \prec \alpha.$$

Hence $c(\alpha, z_{i+1}) <_{\mathrm{lex}} c(\alpha, z_i)$, where $<_{\mathrm{lex}}$ is the lexicographic ordering on ω^e. If $A_{t_1+1} \preceq A_{s+1}$ for all $s + 1$ with $z_i \leq s + 1 \leq z_{i+1}$ then $\mathrm{inj}(\gamma, z_{i+1}) \subseteq \mathrm{inj}(\gamma, z_i)$. (It is a proper inclusion if some γ-cycle x with $(\gamma, x) \in \mathrm{inj}(\gamma, z_i)$ goes back to step 1.) Therefore we have in both cases $c(\alpha, z_{i+1}) \leq_{\mathrm{lex}} c(\alpha, z_i)$. As $<_{\mathrm{lex}}$ is well-founded it follows that there is i_0 such that for all special stages z_i, $i > i_0$ we have

$$\mathrm{Inj}(\alpha, z_i) = \mathrm{Inj}(\alpha, z_{i_0}).$$

Let t_2 be the least α-stage after z_{i_0}. We claim that $A_{s+1} \preceq A_t$ for all $t \geq t_2$. Note that no $\gamma \prec \alpha$ will act such that $A_{s+1} \not\preceq A_t$ (by assumption on i_0). All β with $\alpha <_L \beta$ are initialized in stage t_2. So all β with $\alpha \cdot 1 \leq \beta$ satisfy $\mathrm{inj}(\beta, t) = \emptyset$ for all $t \geq t_2$. Therefore, no strategy will reset A to a value that does not extend A_{t_1+1}.

(4) Let $\hat{\alpha}$ denote the predecessor of α. By choice of t_0 we have $r_1(\hat{\alpha}, t) = r_1(\hat{\alpha}, t_0)$ for all $t \geq t_0$. Let $s + 1$ be an α-stage with $s + 1 > t_0$. If $f(e) = 0$ then $r_0(\alpha, s) = 0$. If $f(e) = 1$ then, by (3), we have $r_0(\alpha, s) = r_0(\alpha, t_1 + 1)$ for all $s \geq t_1 + 1$. Thus there are only finitely many different values for $r_1(\hat{\alpha}, s)$ and $r_0(\alpha, s)$. By the definition of the true path there are only finitely many α'-stages with $\alpha' <_L \alpha \cdot f(e)$. Together with the induction hypothesis this implies (4).

(5) This follows, using (4), by an analogous argument as in (3). ∎

Lemma 12.3 *$lim_s A_s(x)$ exists and (S_e) is satisfied for all e, x.*

Proof: By (4,5) of the previous lemma it follows that for all e with $W_e = \omega$, $\sigma_{f \restriction e}$ is defined and A_s extends $\sigma_{f \restriction e}$ for almost all s. Hence (A_s) converges pointwise to A. Using (4,5) again it follows that (S_e) is satisfied for all e. ∎

Lemma 12.4 *Let $\alpha = f \restriction e$. Then $U_e = \Gamma_\alpha^{V_e}$ or $U_e \neq \Phi_e^{V_e \oplus A}$.*

Proof: If $f(e)=0$ then there are several cases:
a) If some cycle x moves infinitely often from step 4 to step 1 then $U_e \neq \Phi_e^{V_e\oplus A}$ since $\Phi_e^{V_e\oplus A}(x)$ is undefined.
b) If some cycle x is waiting forever at step 1 then $U_e(x) \neq \Phi_e^{V_e\oplus A}(x)$.
c) Otherwise all cycles will end up in step 3 or step 7 and $U_e = \Gamma_\alpha^{V_e}$ where Γ_α is the final version produced by the α-strategy.
If $f(e)=1$ then some cycle x is waiting forever at step 6. By Lemma 12.2, (3) we get $U_e(x) = 1 \neq 0 = \Phi_e^{V_e\oplus A}(x)$. ∎

This concludes the proof of the theorem. ∎

What are the general conditions under which the above construction goes through? The key property is that each cycle x of the basic module changes A_s only finitely often. During the time when cycle x resets A_s all other cycles do not act. They will act only after the previous value A_s has been restored.

Theorem 12.5 *There is a* 1*-generic degree* $\mathbf{a} < \mathbf{0}'$ *such that for all* ω*-r.e. degrees* $\mathbf{b}, \mathbf{c}$*, if* $\mathbf{c} \leq \mathbf{a} \cup \mathbf{b}$ *then* $\mathbf{c} \leq \mathbf{b}$*.*

Proof sketch: We have the same formal requirements as in the proof of Theorem 12.1 but now $U_{e,s}, V_{e,s}$ refer to recursive approximations of ω-r.e. sets. We may assume that for each x, the value of $U_{e,s}(x)$ and $V_{e,s}(x)$ changes at most $x+1$ times.

By the remarks above it suffices to describe the algorithm for a single cycle x and to verify that it changes A only finitely often. In this construction it may happen that $V_{e,s}$ changes back to a previous value and restores an old Γ-computation. In order to make the bookkeeping more explicit we construct the functional $\Gamma_e^{()}$ by enumerating a set $AX = AX_e$ of axioms $\langle x, b, \nu\rangle$ where $x \in \omega$, $b \in \{0,1\}$ and $\nu \in \{0,1\}^*$. $\Gamma^B(x)$ is computed as follows: Enumerate AX until the first axiom $\langle x, b, \nu\rangle$ appears such that $\nu \prec B$. Then output b.

The basic module for (R_e) is similar as before but now uses the following cycles:

Cycle x: Let $\overline{s} = s$ if cycle x is started in stage s. Let $AX = \emptyset$.

1. Wait until a stage $s_0 + 1 > \overline{s}$ such that $U_e(x) = \Phi_e^{V_e\oplus A}(x)[s_0]$ with use $\phi(e,x,s_0)$ and one of the following conditions is satisfied
   (a) There is no axiom $\langle x, b, \nu\rangle \in AX$ with $\nu \preceq V_{e,s_0}$.
   (b) There is an axiom $\langle x, b, \nu\rangle \in AX$ with $b \neq U_{e,s_0}(x) \ \wedge \ \nu \preceq V_{e,s_0}$.
2. In case (a), enumerate $\langle x, U_{e,s_0}(x), V_{e,s_0} \restriction \phi(e,x,s_0)\rangle$ into AX, let $\overline{s} = s_0+1$, and goto step 1. [Now we have $\Gamma_e^{V_e}(x)[s_0+1] = U_{e,s_0}(x)$.]
3. In case (b), reset $A_{s_0+1} = A_t$ where $t+1$ is the stage when $\langle x, b, \nu\rangle$ was enumerated into AX. Set up the restraint $r_e = t$. Stop all active cycles.
   [Now we have $\Phi_e^{V_e\oplus A}(x)[s_0+1] = \Phi_e^{V_e\oplus A}(x)[t] = b \neq U_{e,s_0}(x)$.]

4. Wait until a stage $s_1+1 > s_0+1$ such that one of the following conditions holds:
   (c) $\nu \not\preceq V_{e,s_1}$ or
   (d) $U_{e,s_1}(x) \neq U_{e,s_0}(x)$.
   Then restore $A_{s_1+1} = A_{s_0}$, let $\overline{s} = s_1+1, r_e = 0$, reactivate all stopped cycles, and goto step 1. $\square$

Note that if $\langle x, b, \nu\rangle$ and $\langle x, 1-b, \mu\rangle$ belong to AX then ν and μ are incomparable. Now we verify that step 3 is executed only finitely often. Choose s' large enough such that $U_{e,s}(x) = U_e(x)$ for all $s \geq s'$. From this stage on we enumerate only axioms $\langle x, c, \nu\rangle$ with $c = U_e(x)$. Hence there are only finitely many axioms $\langle x, b, \nu\rangle$ in AX that may satisfy condition (b) in step 1. Assume that for some of these axioms we have $\nu \preceq V_e$. Choose $s'' > s'$ such that for all $\langle x, b, \nu\rangle \in AX$ with $\nu \preceq V_e$ we have $\nu \preceq V_{e,s}$ for all $s \geq s''$. If step 3 is executed after stage s'' then neither condition (c) nor condition (b) will be satisfied later on, i.e., cycle x will wait forever in step 4. Thus, cycle x will change A_s at most once after step s''.

Let us now verify the correctness of the basic module. If some cycle x is waiting forever at step 4 then $U_e(x) \neq \Phi_e^{V_e \oplus A}(x)$. Now assume that $U_e = \Phi_e^{V_e\oplus A}$. Consequently no cycle is waiting forever in step 4, so for every active cycle there are infinitely many stages when it is not stopped, i.e., when it is eligible to act. Now it is easily seen by induction on x that every x requires attention only finitely often and $\Gamma^{V_e}(x) = U_{e,s}(x)$.

The rest of the proof is analogous as in the previous construction and is therefore omitted. ▮

Theorem 12.6 *There are A, B with $A \leq_T K$ and B r.e. such that $A \not\leq_T B$ and $InfEx[A] \subseteq InfEx[B]$.*

Proof sketch: There is a recursive enumeration $\{f_e, \Phi_e\}_{e\in\omega}$ of pairs of recursive two-place functions and partial recursive functionals such that for every ω-r.e. function f and every partial recursive functional Φ there exists e with $f(x) = \lim_s f_e(x,s)$ for all x, and $\Phi = \Phi_e$. Furthermore, $|\{s : f_e(x,s) \neq f_e(x,s+1)\}| \leq x+1$ for any e, x. Let $\hat{f}_e$ denote the function $\lambda x. \lim_s f_e(x,s)$.

We shall construct, by full approximation, a set A and enumerate an r.e. set B together with a functional Ψ such that the following requirements are met:

$$(P_e) \quad (\exists x)[A(x) \neq \Phi_e^B(x)].$$

$$(R_e) \quad (\exists x)[c_A(x) < \hat{f}_e(x)] \ \vee (\forall^\infty x)[\hat{f}_e(x) \leq \Psi^B(x)].$$

In addition Ψ^B will be a total function. These requirements imply that $A \not\leq_T B$ and that Ψ^B dominates $\tilde{R}_A$. Hence, by Theorem 11.10, $InfEx[A] \subseteq InfEx[B]$.

The strategy for meeting a single requirement (P_e) is standard: Wait until $\Phi_{e,s}^{B_s}(x) = 0$ for some $x = \langle e, t\rangle$. Then let $A_{s+1}(x) = 1$ and protect the computation of $\Phi_e^B(x)$.

We approximate $c_A(x)$ by $c_A(x,s) = \min\{t \leq s : A_t \restriction x \preceq A_s\}$. The functional Ψ is chosen in advance as follows: $\Psi^X(x) = \min\{s : \langle x, s\rangle \notin X\}$. The strategy for meeting a single requirement (R_e) uses the following cycles:

Cycle x: Let $s(x)$ be the stage in which cycle x is started. Let $\overline{s} = s(x)$.

1. If $c_A(x, \overline{s}) > \Psi^{B_{\overline{s}}}(x)$ then enumerate all $\langle x, t\rangle$ with $t \leq \overline{s}$ into B.
   [Now we have $c_A(x, s(x)) \leq \Psi^B(x)$.]
2. Wait until a stage $s_0 > \overline{s}$ such that $\Psi^{B_{s_0}}(x) < f_e(x, s_0) \leq c_A(x, s_0)$.
3. Let $A_{s_0+1} = A_{s(x)}$ and set up the restraint $r_e = s(x)$. Stop all active cycles.
   [Now we have $c_A(x, s_0 + 1) \leq \Psi^{B_{s(x)}}(x) \leq \Psi^{B_{s_0}}(x) < f_e(x, s_0)$.]
4. Wait until a stage $s_1 > s_0 + 1$ such that $f_e(x, s_1) \leq c_A(x, s_1)$.
5. Restore $A_{s_1+1} = A_{s_0}$, let $r_e = 0, \overline{s} = s_1 + 1$, reactivate all stopped cycles,
   and goto step 2. $\square$

Note that at each stage at most one cycle is waiting at step 4. If it is waiting forever then $\hat{f}_e(x) > c_A(x)$ and (R_e) is satisfied. Since $f_e(x,s)$ changes at most $x+1$ times every cycle x will execute step 4 only finitely often. If cycle x will end up in step 2 then for all x either $c_A(x) < \hat{f}_e(x)$ or $\hat{f}_e(x) \leq \Psi^B(x)$. In any case, (R_e) is satisfied.

The strategies are put on a tree as in the previous construction. In this proof we have to slightly modify the strategy for meeting (P_α), $|\alpha| = e$: We believe in a $\Phi_{e,s}^{B_s}(x)$ computation with use u only if for all $\gamma \prec \alpha$ with $\gamma \cdot 0 \preceq \alpha$ every γ-cycle x with $x \leq u$ has already executed step 1 before stage s. If α is on the true path this will only cause some delay of the action of the (P_α)-strategy. It is needed to guarantee that (P_α) will act only finitely often.

If a strategy for (R_e) is initialized in step s then it is allowed to activate only cycles x with $x > s$. Thus for all x only finitely many strategies are allowed to change $\Psi^B(x)$. This will guarantee that Ψ^B is total.

The rest of the proof is as in the previous constructions and hence is omitted. ▮

Finally we outline the proof of Theorem 11.13 which states that there are A, B with $A \leq_T K$ and B r.e. such that $A \not\leq_T B$ and $InfEx[B] = InfEx[A \oplus B]$.

Proof sketch: We have to satisfy the requirements $(\tilde{R}_e)$ instead of (R_e) where c_A is replaced by $c_{A\oplus B}$.

One can use a slightly modified cycle x where in step 1 we enumerate all $\langle y, t\rangle$ with $t \leq \overline{s}$ and $x \leq y \leq \overline{s}$ into B. Hence, if in stage $s+1$ we get $c_{A\oplus B}(y,s) < c_{A\oplus B}(y, s+1)$ for some $y \leq s$ due to a B-change then we immediately make sure that $\Psi^{B_{s+1}}(y) \geq c_{A\oplus B}(y, s+1)$. Of course, if $B_{s+1} \restriction z \neq B_s \restriction z$ then we must update the $s(z)$-values in all active cycles z. (In the full construction on the tree it may happen that some β-cycle acts with β below α and this causes a change of $s_\alpha(z)$.)

We also modify the α-strategy for $(\tilde{R}_e)$: It will start cycle x in stage $s+1$ only if for all $\gamma \prec \alpha$ with $\gamma \cdot 0 \preceq \alpha$ every γ-cycle z with $z \leq x$ has already executed step 1 before stage s. If α is on the true path this will only delay the action of the α-module. This is needed to guarantee that $c_{A\oplus B}(x, s')$ does not increase by a B-change (which would be irreversible) during the time when cycle x is waiting in step 4 at stage s'. Also it makes sure that during the time when some γ with $\gamma \cdot 0 \preceq \alpha$ is waiting in step 4 the injury sets $\mathrm{inj}(\gamma', -)$ for $\gamma \prec \gamma' \prec \alpha$ do not change. With these modifications the rest of the proof is as before. ∎

13 Conclusion and Open Problems

We have studied the structure of the degrees of inferability for various inference criteria. The case when only finitely many queries to the oracle are allowed is completely settled and shows a close relationship to the T-degrees. Similarly for r.e. inference degrees. For EX-type inference degrees we could extend several results and techniques from the r.e. case to the general case. In particular, partial characterizations using domination properties were possible. The existence of low omniscient sets for BC and $LIMEX$ shows that these techniques do not work for BC, $LIMEX$, and variants thereof. Hence one obvious open problem is to extend the structural results to these inference types. For instance, is there any recursion theoretic characterization of the BC-degrees below $\mathbf{0}'$? We do not expect that. Are the structure of the EX-degrees and the BC-degrees isomorphic? We do not even know whether there is a minimal BC-degree. It is also open whether the $BC^n[A]$ hierarchy collapses only if $REC \in BC[A]$.

But there are also some interesting questions left concerning EX-type inference degrees. It is not known whether the EX-degrees form an upper semilattice. The usual join-operator does not suffice since there exist A, B of trivial EX-degree such that $A \oplus B$ has omniscient EX-degree. (One may take any two 1-generic sets below K which join up to K; they exist by [25, Theorem 2.1].)

In all of our proofs for $EX[A] \not\subseteq EX[B]$ this noninclusion was witnessed by a set $S \in EX[A] - EX[B]$ which belonged to $[1,2]EX$. Is this true for all noninclusions? Another intriguing question is whether the EX-degrees and the $Team$-EX degrees coincide, i.e., whether $EX[A] \not\subseteq EX[B]$ implies that $[1,n]EX[A] \not\subseteq [1,n]EX[B]$. For EX^n instead of $[1,n]EX$ the other direction is open, i.e., does $EX[A] \subseteq EX[B]$ imply that $EX^n[A] \subseteq EX^n[B]$?

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