# On Intersections of R.E. and Random Sets*

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**Abstract.** This paper investigates for various randomness notions for which r.e. sets $A$ it does hold that all r.e. coretraceable $B$ Turing below $A$ and all randoms $R$ under the given notion satisfy that $B$ is Turing reducible to the intersection of $R$ and $B$. In the case of many randomness notions including the Martin-Löf random sets, the so characterised $A$ are the Turing incomplete ones; in the case of Schnorr randomness, the so characterised $A$ are the non-high ones.

## 1 Introduction

Coretraceable r.e. sets have found much attention in recursion theory. Dekker and Myhill [6] studied them intensively and established that they are found in every r.e. Turing degree [5, 6]; furthermore, Jockusch [10] observed that every r.e. coretraceable set is semirecursive.

In the present work it is shown that intersections between coretraceable r.e. sets and random sets preserve the computational power of the r.e. set whenever this set is sufficiently incomplete; depending of the underlying randomness notion "sufficiently incomplete" means Turing incomplete for the notion of Martin-Löf random and non-high for the notion of Schnorr random. These results are in-line with the fact that the Turing complete and the high Turing degrees are those r.e. Turing degrees in which random sets of the corresponding randomness notions occur. Both classes, those of the Turing complete and the high r.e. degrees, have also various other characterisations in recursion theory, for example the Turing complete r.e. sets are characterised by Arslanov's completeness criterion (existence of a fixed-point free function in that degree) [2, 11, 13] and the high r.e. degrees are characterised as those which compute dominating functions [14] or permit to explanatorily learn the class of all recursive functions [1]. The reader may find further background on recursion theory and randomness in the usual textbooks of these fields [4, 12, 17, 19–21, 23, 26].

A martingale $M$ is a total function which maps strings to positive real numbers such that $M(\sigma) = 0.5 \cdot (M(\sigma 0) + M(\sigma 1))$. Furthermore, $M$ is recursive iff the undergraph of $M$ given as $G(M) = \{(\sigma, q) : \sigma \in \{0,1\}^* \wedge q \in \mathbb{Q} \wedge M(\sigma) > q\}$ is recursive and $M$ is r.e. iff $G(M)$ is an r.e. set. In the present work, the following randomness notions are more deeply investigated:

* F. Stephan is partially supported by NUS grant R252-000-420-112.

- A set $R$ is Martin-Löf random iff there is no r.e. martingale $M$ such that for every $n$ there is an $m$ with $M(R(0)R(1)\dots R(m)) > n$ [15];
- A set $R$ is strongly random iff it is Martin-Löf random and forms a minimal pair with the halting problem in the Turing degrees [7];
- A set $R$ is Kolmogorov random iff there is a constant $c$ such that the plain Kolmogorov complexity of $R(0)R(1)\dots R(n)$ takes infinitely often the value $n-c$ (this notion is equivalent to saying that $R$ is Martin-Löf random relative to the halting problem) [16, 18];
- A set $R$ is recursively random iff there is no recursive martingale $M$ such that for every $n$ there is an $m$ with $M(R(0)R(1)\dots R(m)) > n$ [23];
- A set $R$ is injectively recursively random (TIR) iff every set one-one reducible to $R$ is recursively random [3];
- A set $R$ is Schnorr random iff there is no recursive martingale $M$ and no recursive function $f$ such that for infinitely many $n$ it holds that $M(R(0)R(1)\dots R(f(n))) > n$ [23, 24].

Note that the recursive martingales in these conditions can be taken to have the savings-property, that is, $M(\sigma\tau) \geq M(\sigma) - 2$ for all $\sigma, \tau \in \{0,1\}^*$ [9].

## 2 Characterising Turing complete r.e. degrees

The following remark is used as a tool for various subsequent results.

**Remark 1.** Let an r.e. Turing incomplete set $B$ and a coretraceable set $A \leq_T B$ and a partial-recursive function $U$ with $U(n)$ being the time to enumerate $n$ into the halting problem whenever $n$ is a member of the halting problem. Now the following exists:

1. An approximation $A_s$ to $A$ such that there is a $B$-recursive function telling from which stage $s$ onwards all approximations $A_t(x)$ for $t > s$ are correct for an input $x$;
2. An approximation $a_{m,s}$ to the $m$-th nonelement $a_m$ of $A$ such that $a_{m,s}$ is the $m$-th nonelement of $A_s$ where the least nonelement of $A$ and $A_s$ are $a_0$ and $a_{0,s}$, respectively;
3. A partition $I_n$ of the integers into intervals indexed by the finite strings such that for each $n$, $I_n$ has $2 \cdot n$ elements and $\min(I_n) \geq n$;
4. A recursive sequence $t_0, t_1, \dots$ of stages such that at stage $t_n$ the following conditions hold for all $m \leq \max(I_n)$: $a_{m,s} \leq t_n$ and the retraceing function produces within $t_n$ steps from $a_{m,t_n}$ the values $a_{0,t_n}, a_{1,t_n}, \dots, a_{m-1,t_n}$, that is, $A_{t_n}$ and the retraceing function are compatible up to $a_{m,t_n}$;
5. Sets $J_n$ where $J_n = \{a_{m,t_n+U(n)} : m \in I_n\}$ for all $n \in dom(U)$.

The first item is well-known and follows from the fact that one can choose $A_s = \{x : \varphi_e^{B_s}(x)$ halts in $s$ steps with output $1\}$ where $B_s$ is the subset of $B$ of those elements which are enumerated with $s$ steps and $e$ is chosen such that $A = \varphi_e^B$; it is easy to see that one can find using the oracle $B$ on input $x$ an upper bound $s$ on the computation time of $\varphi_e^B(x)$ such that in addition all $u \in B$ which were queried are also in $B_s$.

The second and third item are straightforward to satisfy. The fourth item can be satisfied by searching for $t_n$ once all $t_m$ with $m < n$ are found; this search has to terminate as for almost all $t$ it holds that $a_{m,t} = a_m$ for all $m \leq \max(I_n)$ and that the corresponding retraceing function terminates within $t$ steps on these $a_m$. The fifth item is a mere definition; note that one can compute a list of all the $2 \cdot n$ elements of $J_n$ whenever $U(n)$ is defined and note that for different $n, n' \in dom(U)$ the sets $J_n$ and $J_{n'}$ are disjoint.

The next two theorems characterise when an r.e. Turing degree is complete by looking at when the coretraceable r.e. sets in it are Turing reducible to their intersections with sufficiently random sets.

**Theorem 2.** *Every set $A$ which is r.e., Turing incomplete and coretraceable and every injectively recursively random set $R$ satisfy $A \leq_T A \cap R$.*

**Proof.** Indeed, one can show the result even for all $A$ which are coretraceable and Turing reducible to a Turing incomplete r.e. set. So let such an $A$ be given and let $A_s$, $U$, $I_n$, $t_0, t_1, \ldots$, $J_n$ as in Remark 1. Now let $R$ be an arbitrary injectively recursively random set.

Note that there are infinitely many $n \in dom(U)$ such that $a_{m,t_n+U(n)} = a_m$ for all $m \in I_n$. The reason is that the partial function $n \mapsto U(n)$ is not majorised by any function of incomplete Turing degree. So there are infinitely many $n$ such that $U(n)$ is defined and larger than the function which tells when all the $a_{m,s}$ with $m \leq \max(I_n)$ have converged to their final values without leaving them again. So $J_n$ contains for infinitely many $n \in dom(U)$ only values in the complement of $A$; note that these values are above $n$.

Now let $g^{A\cap R}(n)$ be the first $p \in dom(U)$ found such that $p \geq n$ and $J_p \cap (A \cap R) = \emptyset$. Now one conjectures that $n \in A$ iff the least element $a$ of $J_{g^{A\cap R}(n)}$ has the property that the retraceing function starting from $a$ only provides elements different from $n$. This Turing reduction is total as $J_p$ contains for every $p \in dom(U)$ only elements on which the retraceing function is defined. Assume now by way of contradiction that the Turing reduction is incorrect for infinitely many $n$.

Under this assumption, there must be infinitely many $n$ such that $\min(J_{g^{A\cap R}(n)}) \in A$ as only then the retraceing produces wrong values below $a$; note that every $p$ with $\min(J_p) \in A$ also satisfies that $J_p \subseteq A$. There is a recursive one-one enumeration of all $p$ with $p \in dom(U)$ and $J_p \subseteq A$; let $p_0, p_1, \ldots$ be that enunmeration and let $x_0, x_1, \ldots$ be the enumeration which enumerates the elements of $J_{p_0}$, then those of $J_{p_1}$, then those of $J_{p_2}$ and so on. Let $P = \{n : x_n \in R\}$. $P$ is one-one reducible to $R$ and thus, by assumption, recursively random.

But one can now define a martingale $M$ to be the sum of submartingales $M_p$ with the following property. Each submartingale $M_p$ starts with the initial value $2^{-p-1}$ and, in the case that $p = p_n$ for some $n$, the submartingale bets for all numbers $m$ with $x_m \in J_{p_n}$ all capital on $m \notin P$. From then onwards the submartingale abstains from betting and preserves its capital. If $p \neq p_n$ for all $n$ then the submartingale $M_p$ remains constant at $2^{-p-1}$. Note that the starting value of $M = \sum_p M_p$ is 1; furthermore, there are $p \notin dom(U)$ and for those $M_p$ remains constant $2^{-p-1}$ and so $M$ is on all inputs positive. By assumption, $J_{p_n} \cap (A \cap R) = \emptyset$ for infinitely many $n$; this implies that $J_{p_n} \cap R = \emptyset$ for the same $n$. For these $n$, the submartingales $M_{p_n}$ start with $2^{-|p_n|-1}$ and make $2|p_n|$ correct bets where they double their capital so that they eventually reach $2^{-|p_n|-1+2|p_n|} = 2^{|p_n|-1}$. So $M$ accumulates an arbitrary amount of money in the limit. Hence $P$

would not be recursively random and so $R$ would not be injectively recursively random. Thus the given Turing is almost everywhere correct and the finitely many errors can be amended in the program of the reduction. Hence $A \leq_T A \cap R$. ∎

**Theorem 3.** *For every oracle $X$ there is a Turing complete r.e. and coretraceable set $A$ and a set $R$ which is Martin-Löf random relative to $X$ such that $A \not\leq_T A \cap R$.*

**Proof.** Let $R$ be Martin-Löf random relative to $X'$; then $R$ is in particular Kolmogorov random, that is, Martin-Löf random relative to halting problem. Combining the arguments of Figueira, Miller and Nies [8] throughout their paper, one can show that there exists a Turing complete infinite r.e. set $A$ such that $(A \oplus A)$ is coretraceable and $R \cap (A \oplus A)$ is Martin-Löf random. Now $R$ is the join of two halves $R_1, R_2$. By van Lambalgen's theorem, the sets $R_1 \cap A$ and $R_2 \cap A$ are Martin-Löf random relative to each other. Assume now that $A \leq_T R_1 \cap A$. Then $R_2 \cap A$ is disjoint to an infinite set, namely the complement of $A$, which is recursive relative to $R_1 \cap A$; this contradicts that $R_2 \cap A$ is Martin-Löf random relative to $R_1 \cap A$. Hence both sets, $R_1 \cap A$ and $R_2 \cap A$, do not compute $A$. ∎

**Corollary 4.** *The following is equivalent for all r.e. sets $B$:*

- *$B$ is Turing incomplete;*
- *$A \leq_T A \cap R$ for all coretraceable r.e. $A \leq_T B$ and all Martin-Löf random sets $R$;*
- *$A \leq_T A \cap R$ for all coretraceable $A \leq_T B$ and all Martin-Löf random sets $R$;*
- *$A \leq_T A \cap R$ for all coretraceable r.e. $A \equiv_T B$ and all Martin-Löf random sets $R$;*
- *$A \leq_T A \cap R$ for all coretraceable $A \equiv_T B$ and all Martin-Löf random sets $R$.*

*The same holds with "injectively recursively random", "strongly random" and "Kolmogorov random" in place of "Martin-Löf random".*

One might ask whether the characterisation could be obtained using all r.e. sets $A$, that is, whether a Turing degree is incomplete iff for all r.e. sets $A$ in or below this Turing degree and all Martin-Löf random sets $R$ it holds that $A \leq_T A \cap R$. The answer to this question is negative.

**Theorem 5.** *There is a low r.e. set $\tilde{A}$ and a Martin-Löf random set $R$ such that $\tilde{A} \not\leq_T \tilde{A} \cap R$.*

**Proof.** One can use the methods of Figueira, Miller and Nies with the arguments in Theorem 3 to show that there are a low Martin-Löf random set $R$ and a Turing-complete r.e. set $A$ such that also $R \cap A$ is Martin-Löf random and low. By the Sacks splitting theorem [22], $A$ is the disjoint union of two low r.e. sets $A_1$ and $A_2$ and $A \equiv_T A_1 \oplus A_2$. Note that $R \cap A_1$ and $R \cap A_2$ are both Turing reducible to $R \cap A$: given any $x$, one first checks whether $x \in R \cap A$. If not, then $x \notin R \cap A_1$ and $x \notin R \cap A_2$. If so, then one enumerates $A_1$ and $A_2$ in order to find out in which of the sets $R \cap A_1$ and $R \cap A_2$ the element $x$ is; $x$ appears only in one of the sets $A_1$ and $A_2$. Now at least one of the statements $A_1 \not\leq_T R \cap A_1$ and $A_2 \not\leq_T R \cap A_2$ must hold, as otherwise $R \cap A$ has a Turing degree above the one of $A$ and would not be low. So $\tilde{A} \not\leq_T \tilde{A} \cap R$ for a low r.e. set $\tilde{A}$ where $\tilde{A} = A_1$ or $\tilde{A} = A_2$. ∎

## 3 Intersections of Schnorr random and r.e. sets

The next section will give a characterisation of the high degrees which makes use of Theorem 6 below. That theorem and its generalisations are interesting on their own right and so this section is dedicated to the topic of coindifeeerent r.e. sets and Schnorr random sets being a subset of an r.e. nonhypersimple set. The first result deals with intersections of Schnorr random and dense simple sets. Recall that an r.e. set $A$ is dense simple iff for every recursive function $f$ and almost all $n$, the set $\{0, 1, \ldots, f(n)\} - A$ has at most $n$ elements.

**Theorem 6.** *If $A$ is dense simple and $R$ is Schnorr random then $A \cap R$ is Schnorr random.*

**Proof.** Let $A$ be dense simple and assume that $B = A \cap R$ is not Schnorr random. Now it is shown that then $R$ is also not Schnorr random, proving the theorem by contraposition.

As $B$ is not Schnorr random, there is by prior work [9] a martingale $M$ and a recursive function $f$ such that, for infinitely many $n$, $M(B(0)B(1)\ldots B(f(n))) > 8^n$. As $A$ is dense simple, one can, for almost all $n$, find a stage $s(n)$ for which $\{0, 1, \ldots, f(n)\} - A_{s(n)}$ has at most $n$ elements; for the remaining finitely many $n$ one just takes for $s(n)$ some default value. The function $s$ is recursive.

Now one defines a martingale $M_\eta$ with a parameter $\eta \in \{0,1\}^n$ such that $M_\eta(\sigma)$ is $M(\sigma@\eta) \cdot 4^{-n}$ where one let $x_0, x_1, \ldots, x_{n-1}$ be the first $n$ nonelements of $A_{s(n)}$ and then defines that $\sigma@\eta(x)$ is $\eta(k)$ if $x \in dom(\sigma) \cap \{x_k\}$ and $\sigma@\eta(x) = \sigma(x)$ if $x \in dom(\sigma) - \{x_0, x_1, \ldots, x_{n-1}\}$. The sum $M^*$ of all the $M_\eta$ with $\eta \in \{0,1\}^*$ is a recursive martingale. Now there are infinitely many $n$ where $M(B(0)B(1)\ldots B(f(n))) > 8^n + 2$ and $|\{0, 1, \ldots, f(n)\} - A_{s(n)}| \leq n$; for these $n$ it holds that there is $\eta \in \{0,1\}^n$ for which $M_\eta(R(0)R(1)\ldots R(f(n)))$ coincides with $4^{-n}M(B(0)B(1)\ldots B(f(n)))$ as $R$ and $B$ differ below $f(n)$ only on the complement of $A_{s(n)}$ and $\eta$ tells how to replace the bits of $R$ on this complement in order to get $B$. Hence $M^*(R(0)R(1)\ldots R(f(n))) \geq M_\eta(R(0)R(1)\ldots R(f(n))) \geq 2^n \geq n$ and $M^*$ Schnorr succeeds on $R$; so $R$ can only be Schnorr random if $B$ is Schnorr random as well. ∎

One can generalise the result using the notions of indifference. A set $I$ is called indifferent for Schnorr randomness iff every two sets $P, R$ with the symmetric difference of $P$ and $R$ being a subset of $I$ satisfy that either both $P$ and $Q$ or none of them is Schnorr random. The previous result can be strengthened as follows.

**Theorem 7.** *Let $A$ be an r.e. and coinfinite set. Then $A$ is dense simple iff the complement of $A$ is indifferent for Schnorr randomness.*

**Proof.** If $A$ is dense simple then one can adjust the proof from the previous theorem. That is, assume that a martingale $M$ Schnorr succeeds on a set $P$ and that the symmetric difference of $P$ and $R$ is disjoint to $A$. Then one can construct a martingale $M^*$ as in the previous theorem such that $M^*$ Schnorr succeeds on $R$.

For the converse direction assume that $A$ is not dense simple. Then there is an increasing recursive function such that, for infinitely many $n$, there are more then $n$ nonelements of $A$ below $f(n)$. Now, one can make a recursive sequence $a_0, a_1, \ldots$ such that $a_{n+1} = a_n \cdot 3n \cdot 4^n \cdot f(a_n)$.

There are infinitely many $n$ there is an $m$ between $a_n$ and $a_{n+1}$ such that there $m$ nonelements of $A$ below $f(m)$. There is a $c \in \{0,1,2\}$ such that infinitely of these $n$ are of the form $3k+c+1$. Now let $b_0 = 0$ and $b_k = a_{3k+c}$ for $k > 0$. It follows that for infinitely many $k$ there are more than $b_k \cdot 4^k \cdot 2k$ nonelements of $A$ between $b_k$ and $b_{k+1}$. Given a Schnorr random set $R$, one makes now a set $P$ inductively for $k = 0, 1, \ldots$ such that $P$ coincides with $R$ on $A$ and, for all $k$ where there are more then $2k \cdot 4^k$ nonelements of $A$ between $b_k$ and $b_{k+1}$, one chooses $P$ on these nonelements such that

$$(\sum_{x=b_k, b_k+1, \ldots, b_{k+1}-1} P(x) \cdot 2^{x \ mod \ 2k}) \ mod \ 4^k - 1 = 0$$

where the function $x \ mod \ y$ denotes the remainder when dividing $x$ by $y$. Note if there are $4^k$ nonelements of $A$ between $b_k$ and $b_{k+1}$ all having the same remainder $\ell$ modulo $2k$, one can then take

$$r = 4^k - 1 - (\sum_{x=b_k, b_k+1, \ldots, b_{k+1}-1} (A \cap R)(x) \cdot 2^{x \ mod \ 2k}) \ mod \ 4^k - 1$$

and furthermore choose $r'$ such that $r = 2^\ell r' \ mod \ 4^k - 1$ and then let $P(x) = 1$ for $r'$ numbers $x$ in $\{b_k, b_k+1, \ldots, b_{k+1}-1\} - A$ which have the remainder $\ell$ modulo $2k$; for the other $x$ where $P(x)$ is not yet defined and $x < b_{k+1}$, one takes $P(x) = 0$. Hence the resulting sum vanishes modulo $4^k - 1$.

Now one can make a martingale $M = \sum M_k$ where the submartingale $M_k$ starts with the capital $2^{-k-1}$ and bets only the last $2k-1$ positions before $b_{k+1}$ all capital on the unique value which would be needed in order to make the weighted sum from $b_k$ to $b_{k+1}-1$ of the $P(x)$ to be 0 modulo $4^k - 1$ at $b_{k+1}$. The submartingale achieves the value $2^{k-2}$ in the case that the choice of the values of $P$ to meet this sum was possible and as that can be achieved infinitely often, the martingale $M$ reaches for infinitely many $k$ on the first $b_{k+1}$ bits a capital above $2^{k-2}$. Hence $P$ is not Schnorr random and the complement of $A$ cannot be indifferent for Schnorr randomness. ∎

Note that here the indifference holds in a very strong way. If $A$ is dense simple then the complement of $A$ is indifferent for all Schnorr random sets $R$ and if $A$ is not dense simple then the complement of $A$ is not indifferent for any Schnorr random set $R$. This stands in contrast to the situation of Martin-Löf randomness where for each indifferent set $I$ is only indifferent for some but not all Martin-Löf random sets [8]. The next result shows that a similar result does not hold with respect to intersections, that is, making all bits 0 on the complement of a r.e. set is not the same as changing them in a manner which can be exploited by martingales.

**Example 8.** *There is a r.e. nonhypersimple set $A$ and a recursively random set $R$ such that $R \subseteq A$.*

**Proof.** Let $I_0, I_1, \ldots$ be a recursive partition of the natural numbers into intervals such that $(3 + \min(I_n))^{n^2} < \max(I_n) < \min(I_{n+1})$ for all $n$. Furthermore, one constructs the r.e. set $A$ in stages and $R$ to be a set which is adversary to a given limit-recursive martingale $M$ which will also be constructed in stages. The martingale $M$ consists of two components: One component starts with capital 1/2 and puts half of its capital on $R(x) = 1$ whenever $x \notin A_s$ where $s$ is the last stage where the second component changed in one of the intervals $I_n$ with $\min(I_n) \leq x$. The second component is a sum of submartingales $M_k$ which start with capital $2^{-k-2}$ and abstain from betting on intervals $I_0, I_1, \ldots, I_k$ and which then follow the $k$-th partial-recursive martingale (up to a scaling factor) whenever the $k$-th partial-recursive martingale is defined on all inputs $\sigma$ with $|\sigma|$ being bounded by the maximimum of $I_n$ for the $n$ with $x \in I_n$. Furthermore, $R_s$ is the set adversary to the $s$-th approximation of $M$, that is, the $s$-th approximation of $M$ does not go above 1 on $R_s$.

Initially $A_s = 0$ and the first component bets on all inputs. The approximations to the final values of an $M_k$ change only below $\max(I_n)$ iff there is an $m$ with $k < m \leq n$ such that the $k$-th partial-recursive martingale becomes defined on all inputs $\sigma$ with $|\sigma| \in I_m$. That happens at most $n$ times. So the sum of the $M_k$ changes less than $n^2$ times below $\max(I_n)$. When such a change happens at stage $s$ then the first component becomes updated to bet only outside $A_s$ and then $R_s$ is updated accordingly. Then, at stage $s+1$, $A_{s+2} = A_s \cup R_{s+1}$. Clearly $A$ is an r.e. set.

Each time when $A_{s+2} \cap I_n \neq A_{s+1} \cap I_n$, both components of the martingale had just been updated and so also the set $R_s$ to $R_{s+1}$. This happens less than $n^2$ times. One shows now that whenever such an update happens for the $m$-th time ($m = 1, 2, \ldots$) then there are at least $(3 + \min(I_n))^{n^2-m}$ nonelements of $A_s$ below $\max(I_n)$. This is due to the fact that the first component of the martingale bets for each $x$ outside $A_s$ always half of its capital on the value of $R(x)$ to be 1 and as the martingale does not go above 1, the bet must fail in at least a third of the bets, as otherwise the capital reaches $1/4 \cdot 3^{2u/3}/2^u$ after $u$ bets and for all sufficiently large $u$ this value is above 1. So, for almost all $n$, $A$ has at least $\min(I_n) + 3$ nonelements below $\max(I_n)$ and therefore $I_n \not\subseteq A$ for almost all $n$. It follows that $A$ is not hypersimple.

Furthermore, at each update of $R_s$, the corresponding elements are moved into $A_{s+2}$ and therefore $R \subseteq A$. As the martingale $M^*$ subsumes all recursive martingale with some scaling and with some initial betting ignored, it follows that no recursive martingale succeeds on $R$ and $R$ is recursively random. ▮

## 4 Characterising the high r.e. degrees

The main result of this section is now obtained in two steps.

**Theorem 9.** *If a coretraceable set $A$ is bounded by a non-high r.e. degree and $R$ is Schnorr random then $A \leq_T A \cap B$.*

**Proof.** Let such an $A$ be given and let $A_s$, $U$, $I_p$, $t_0, t_1, \ldots$, $J_p$ as in Remark 1. Now let $R$ be an arbitrary Schnorr random set.

As in Theorem 2, note that there are infinitely many $p \in dom(U)$ such that $a_{m,t_{p+U(p)}} = a_m$ for all $m \leq \max(I_p)$. Furthermore, as the Turing degree of $A$ is bounded by a non-high r.e. set, there is a recursive function $h$ such that, for infinitely many $p$, $U(p)$ halts in time $h(p)$ and $U(p) < h(p)$ and $\max(J_p) < h(p)$ and $J_p \cap A = \emptyset$. The set $S = \{p \in dom(U) : U(p)$ halts in $h(p)$ steps and $U(p) < h(p)$ and $\max(J_p) < h(p)\}$ is recursive.

Now let $g^{A \cap R}(n)$ be the first $p \in S$ found such that $p \geq n$ and $J_p \cap (A \cap R) = \emptyset$. Then one can define that $n \in A$ iff the least element $a$ of $J_{g^{A\cap R}(n)}$ has the property that the retraceing function starting from $a$ only provides elements different from $n$.

Now assume by way of contradiction that this Turing reduction fails for infinitely many $n$. Then there are infinitely many $p \in S$ such that $J_p \subseteq A$ and $J_p \cap R = \emptyset$. Now one can define a martingale $M$ to be the sum of submartingales $M_p$ with the following property. Each submartingale $M_p$ starts with the initial value $2^{-p-1}$ and, in the case that $p \in S$, the submartingale bets for all members of $x \in J_p$ all capital on $x \notin R$; for $x \notin J_p$, $M_p$ abstains from betting and preserves its capital. If $p \notin S$ then the submartingale $M_p$ remains constant at $2^{-p-1}$. The starting value of the overall martingale is $1 = \sum_p 2^{-p-1}$; furthermore, some submartingales remain constant all the time and therefore the overall sum is always positive, although some of the submartingales take on some inputs the value 0. As there are now infinitely many $p \in S$ with $J_p \cap R = \emptyset$, the submartingales $M_p$ for these $p$ start with $2^{-p-1}$ and make $2|p|$ correct bets where they double their capital so that they reach $2^{p-1}$ after $h(|p|)$ steps so that the Schnorr condition is violated: $M(R(0)R(1)\ldots R(h(p))) > p$ for infinitely many $p$. This contradicts to $R$ being Schnorr random and therefore $A \leq_T A \cap R$. ∎

**Theorem 10.** *Every r.e. high Turing degree contains an r.e. coretraceable set $A$ such that $A \not\leq_T A \cap R$ for some Schnorr random set $R$.*

**Proof.** By assumption, the Turing degree contains a dense simple and coretraceable set $\tilde{A}$. Furthermore, there is also a high r.e. Turing degree strictly below the one of $\tilde{A}$ [20, Exercise XI.1.13 (b)] and this degree also contains a dense simple and coretraceable set $B$ as well as a Schnorr random set $R$. Furthermore, let $\tilde{A}_s$ and $B_s$ be coretraceable enumerations of $\tilde{A}$ and $B$, respectively. Let $b_{m,s}$ be the $m$-th nonmember of $B_s$ with $b_{m,0}$ being the least one. Now one defines

$$A = \{x : \exists m \leq x \exists s > x[x \in B_s \vee (m \in \tilde{A}_s \wedge x = b_{m,s})]\}$$

and it is directly clear that $A$ is an r.e. superset of $B$. Furthermore, if $x \notin A$ then one can choose an $m \leq s$ and $s > x$ such that $x = b_{m,s}$. Now, due to the update algorithm of the $b_{m,s}$ and $B_s$, either $x = b_m$ or $x \in B$; as the latter is excluded one knows that $x = b_m$ and one can compute $m, b_0, b_1, \ldots, b_m$ from $x$. Furthermore, as $\tilde{A}$ is coretraceable and $x \notin A$ one knows that $m \notin \tilde{A}$ and one can compute from $m$ for which $n < m$ it holds that $n \in \tilde{A}$. Hence one can compute from $x$ for which $n$ for $n = 0, 1, \ldots, m$ the value $b_n$ is in $A$. Hence $A$ is coretraceable. Note that these arguments also give the equation

$$A = B \cup \{b_m : m \in \tilde{A}\}$$

as well as $\tilde{A} \leq_T A$. The other direction $A \leq_T \tilde{A}$ also holds as $B \leq_T \tilde{A}$ by choice and one can compute the sequence $b_0, b_1, \ldots$ relative to $\tilde{A}$ and check whether $x \in A$ by checking if either $x \in B$ or $x = b_m$ for some $m \in \tilde{A}$ with $m \leq x$.

By Theorem 6, $R \cap B$ is Schnorr random. Furthermore, $R \cap B$ is Turing reducible to $B$. Due to the equation $(R \cap B) \cap A = R \cap (A \cap B) = R \cap B$, the intersection of the set $A$ with the Schnorr random $R \cap B$ has a Turing degree strictly below $A$ and this shows that the high r.e.

Turing degrees differ from the nonhigh ones concerning with respect to the Turing degrees of intersections of r.e. coretraceable sets with Schnorr random ones. ▮

**Corollary 11.** *An r.e. Turing degree is high iff it contains or bounds a coretraceable r.e. set $A$ such that $A \not\leq_T R \cap A$ for some Schnorr random set $R$.*

It is unknown whether this characterisation carries over to recursive randomness. But one gets the corresponding charactersiation if one considers just coretraceable sets in place of coretraceable r.e. sets.

**Theorem 12.** *An r.e. Turing degree is high iff it contains a coretraceable set $A$ such that $A \not\leq_T A \cap R$ for some recursively random set $R$.*

**Proof.** The part for non-high r.e. degrees follows from Theorem 9 and the fact that every recursively random set is also Schnorr random. So assume that the given r.e. Turing degree is high and that $\tilde{A}$ is a given coretraceable set in it.

Now there is a recursively random set $R <_T \tilde{A}$; $R$ can be found as there is a high r.e. Turing degree strictly below the one of $\tilde{A}$. Now define inductively $b_0, b_1, \ldots$ such that $b_0$ is the minimal nonelement of $R$ and $b_{n+1}$ is the minimal number of the form $2^{b_n} \cdot (2m+1)$ which is not in $R$; this number must exist as a random set is bihyperimmune. The construction is recursive in $R$ and let $B$ be the complement of $\{b_0, b_1, \ldots\}$. Furthermore, $B$ is coretraceable as $b_n = \max\{k : 2^k$ divides $b_{n+1}\}$. Now let $A = B \cup \{b_n : n \in \tilde{A}\}$. Clearly $A \leq_T \tilde{A}$. Furthermore, for each $b_n \notin A$ one can compute from $b_n$ the values $n, b_0, b_1, \ldots, b_n$ and then using the fact that $\tilde{A}$ is coretraceable decide which $m \leq n$ are in $\tilde{A}$. Hence one can determine which of the $b_m$ with $m \leq n$ are outside $A$ and so $A$ is coretraceable. Furthermore, the above arguments also show that $\tilde{A} \leq_T A$. Now $R = R \cap B = R \cap A$ and therefore $A \not\leq_T A \cap R$. ▮

## 5 Discussion

The starting point for the present work was that $A \leq_T A \cap R$ whenever $A$ is an incomplete r.e. and coretraceable set and $R$ is Martin-Löf random. One can generalise this result to various randomness notions and get that an r.e. Turing degree is non-high iff for every r.e. coretraceable set $A$ in or below the Turing degree and every Schnorr random set $R$, $A$ is Turing reducible to $A \cap R$. It remains open whether the same characterisation can be done with recursively random sets, Theorem 12 quantifies only over all coretraceable sets of the given r.e. degree and not over all coretraceable r.e. sets of the given r.e. degree.

**Open Problem 13.** *Does every r.e. high Turing degree contain an r.e. and coretraceable set $A$ such that $A \not\leq_T A \cap R$ for some recursively random set $R$?*

One can also connect the randomness notions to immunity. A slight modification of above proofs gives that an r.e. set $A$ the following characterisations: $A$ is Turing incomplete iff every Martin-Löf random set is immune relative to $A$; $A$ is nonhigh iff every Schnorr random set is immune relative to $A$; $A$ is nonrecursive iff every Kurtz random set is immune relative to $A$. Furthermore, for strong randomness, no such dichtinomy holds as every strongly random set is immune relative to the halting problem.

One might ask whether Martin-Löf randomness can be replaced by another notion and the answer is "yes": A natural generalisation of the results is the following. If $W$ is effectively simple [25] (or, more generally, $W$ is effectively coimmune) and $A$ is coretraceable and of r.e. Turing incomplete degree then $A \leq_T W \cap A$.

## Acknowledgments

The authors want to thank Adam Day and Sanjay Jain for discussions.

## References

1. Lenny Adleman and Manuel Blum. Inductive inference and unsolvability. *The Journal of Symbolic Logic*, 56:891–900, 1991.
2. Marat Arslanov. On some generalizations of a fixed-point theorem. *Soviet Mathematics* (*Iz. VUZ*), *Russian*, 25(5):9–16, 1981, *English translation*, 25(5):1–10, 1981.
3. Laurent Bienvenu, Rupert Hölzl, Thorsten Kräling and Wolfgang Merkle. Separations of non-monotonic randomness notions. Journal of Computation and Logic, to appear; `http://www.math.uni-heidelberg.de/logic/merkle/ps/cca-2009-arxiv.pdf`.
4. Cristian S. Calude. *Information and Randomness - An Algorithmic Perspective.* Second Edition, Springer, Heidelberg, 2002.
5. James C.E. Dekker. A theorem on hypersimple sets. *Proceedings of the American Mathematical Society*, 5:791-796, 1954.
6. James C.E. Dekker and John Myhill. Retraceable sets. *Canadian Journal of Mathematics*, 10:357–373, 1958.
7. Rod Downey, André Nies, Rebecca Weber and Liang Yu. Lowness and $\pi^0_2$ null sets. *The Journal of Symbolic Logic*, 71:1044–1052, 2006.
8. Santiago Figueira, Joseph S. Miller and André Nies. Indifferent sets. *Journal of Logic and Computation* 19:425-443, 2009.
9. Johanna Franklin and Frank Stephan. Schnorr trivial sets and truth-table reducibility. *The Journal of Symbolic Logic*, 75:501–521, 2010.
10. Carl G. Jockusch. Semirecursive sets and positive reducibility. *Transactions of the American Mathematical Society*, 131:420–436, 1968.
11. Alistair H. Lachlan. Complete recursively enumerable sets. *Proceedings of the American Mathematical Society*, 19:99–102, 1968.
12. Ming Li and Paul Vitányi. *An Introduction to Kolmogorov Complexity and Its Applications.* Third Edition, Springer, 2008.
13. Donald Martin. Completeness, the recursion theorem and effectively simple sets. *Proceedings or the American Mathematical Society*, 17:838–842, 1966.

14. Donald Martin. Classes of recursively enumerable sets and degrees of unsolvability. *Zeitschrift für Mathematische Logik und Grundlagen der Mathematik*, 12:295–310, 1966.
15. Per Martin-Löf, The definition of random sequences. *Information and Control*, 9:602–619, 1966.
16. Joseph Miller. Every 2-random real is Kolmogorov random. *The Journal of Symbolic Logic*, 69:907–913, 2004.
17. André Nies. *Computability and Randomness.* Oxford Press, Oxford, 2009.
18. André Nies, Frank Stephan and Sebastian A. Terwijn. Randomness, relativization and Turing degrees. The Journal of Symbolic Logic 70:515–535, 2005.
19. Piergiorgio Odifreddi. *Classical Recursion Theory*, *Studies in Logic and the Foundations of Mathematics*, volume 125. North-Holland, Amsterdam, 1989.
20. Piergiorgio Odifreddi. *Classical Recursion Theory II*, *Studies in Logic and the Foundations of Mathematics*, volume 143. Elsevier, Amsterdam, 1999.
21. Hartley Rogers. *Theory of Recursive Functions and Effective Computability.* McGraw-Hill, New York, 1967.
22. Gerald E. Sacks. On the degrees less than $\mathbf{0}'$. *Annals of Mathematics*, 77:211–231, 1963.
23. Claus Peter Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Springer Lecture Notes in Mathematics*, 1971.
24. Claus Peter Schnorr. Process complexity and effective random tests. *Journal of Computer and System Sciences* 7:376–388, 1973.
25. Raymond M. Smullyan. Effectively simple sets. *Proceedings of the American Mathematical Society*, 15:893–895, 2011.
26. Robert I. Soare. *Recursively enumerable sets and degrees.* Perspectives in Mathematical Logic. Springer-Verlag, Berlin, 1987.