Kolmogorov-Loveland Randomness and Stochasticity

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Abstract. One of the major open problems in the field of effective randomness is whether Martin-Löf randomness is the same as Kolmogorov-Loveland (or KL) randomness, where an infinite binary sequence is KL-random if there is no computable non-monotonic betting strategy that succeeds on the sequence in the sense of having an unbounded gain in the limit while betting successively on bits of the sequence. Our first main result states that every KL-random sequence has arbitrarily dense, easily extractable subsequences that are Martin-Löf random. A key lemma in the proof of this result is that for every effective split of a KL-random sequence at least one of the halves is Martin-Löf random. We show that this splitting property does not characterize KL-randomness by constructing a sequence that is not even computably random such that every effective split yields subsequences that are 2-random, hence are in particular Martin-Löf random.

A sequence X is KL-stochastic if there is no computable non-monotonic selection rule that selects from X an infinite, biased sequence. Our second main result asserts that every KL-stochastic sequence has constructive dimension 1, or equivalently, a sequence cannot be KL-random if it has infinitely many prefixes that can be compressed by a factor of $\alpha < 1$ with respect to prefix-free Kolmogorov complexity. This improves on a result by Muchnik, who has shown a similar implication where the premise requires that such compressible prefixes can be found effectively.

1 Introduction

In 1998, Muchnik, Semenov, and Uspensky [11] combined non-monotonic selection rules in the sense of Kolmogorov and Loveland with the concept of computable betting strategies. The resulting concept of *non-monotonic betting strategies* is a generalization of the concept of monotonic betting strategies, used by Schnorr to define a randomness notion nowadays known as *computable randomness*. Schnorr's motivation behind this randomness concept was his criticism of Martin-Löf randomness [7] as not being a completely effective notion of randomness, since the sets used in Martin-Löf tests only have to be uniformly *enumerable*.

An infinite binary sequence against which no *computable* non-monotonic betting strategy succeeds is called *Kolmogorov-Loveland random*, or KL-random, for short. Muchnik, Semenov, and Uspensky [11] showed that Martin-Löf randomness implies KL-randomness. Muchnik et al. [11] and others [1] raised the question whether the two concepts are different. This is now a major open problem in the area. A proof that both concepts are the same would give a striking argument against Schnorr's criticism of Martin-Löf randomness.

Most researchers conjecture the notions are different. However, a result of Muchnik [11] indicates that KL-randomness is rather close to Martin-Löf randomness.

Recall that it is possible to characterize Martin-Löf randomness as incompressibility with respect to prefix-free Kolmogorov complexity K: A sequence A is random if and only if there is a constant c such that the K-complexity of the length n prefix $A \upharpoonright n$ of A is at least $n - c$. It follows that a sequence A cannot be Martin-Löf random if there is a function h such that

$$K(A \upharpoonright h(c)) \leq h(c) - c \qquad \text{for every } c. \tag{1}$$

On the other hand, by the result of Muchnik [11] a sequence A cannot be KL-random if (1) holds for a *computable* function h. So, the difference between Martin-Löf randomness and KL-randomness appears, from this viewpoint, rather small. Not being Martin-Löf random means that for any given constant bound there are infinitely many initial segments for which the compressibility exceeds this bound. If, moreover, we are able to detect such initial segments effectively (by means of a computable function), then the sequence cannot even be KL-random.

In this paper we continue the investigations by Muchnik, Semenov, and Uspensky, and give additional evidence that KL-randomness behaves similar to Martin-Löf randomness.

In Section 3 we refine the splitting technique that Muchnik used in order to obtain the result mentioned above. We show that if A is KL-random and Z is a computable, infinite and co-infinite set of natural numbers, either the bits of A whose positions are in Z or the remaining bits form a Martin-Löf random sequence. In fact both do if A is Δ_2^0. Moreover, in that case, for each computable, nondecreasing, and unbounded function g and almost all n, $K(A \upharpoonright n) \geq n - g(n)$.

We construct counterexamples that show that two of the implications mentioned in the preceding paragraph cannot be extended to equivalences. First, there is a sequence that is not computably random all whose "parts" in the sense above (i.e., which can be obtained through a computable splitting) are Martin-Löf random. Second, there is a sequence A that is not even stochastic such that for all g as above and almost all n, $K(A \upharpoonright n) \geq n - g(n)$; moreover, the sequence A can be chosen to be left-r.e. if viewed as the binary expansion of a real.

In the last two sections we consider KL-stochasticity. A sequence is KL-stochastic if there is no computable non-monotonic selection rule that selects from the given sequence a sequence that is biased in the sense that the frequencies of 0's and 1's do not converge to 1/2. First we give a more direct construction of a KL-stochastic sequence that is not even weakly 1-random. Next we consider constructive dimension. Muchnik [11] demonstrates, by an argument similar to his proof that a sequence A cannot be KL-random if there is a computable function that satisfies (1), that a sequence A cannot be KL-stochastic if there is a computable, unbounded function h and a rational $\alpha < 1$ such that

$$K(A \restriction h(i)) \leq \alpha h(i) \qquad \text{for every } i, \tag{2}$$

i.e., if we can effectively find arbitrarily long prefixes of A that can be compressed by a factor of α in the sense that the prefix-free Kolmogorov complexity of the prefix is at most α times the length of the prefix. Theorem 22 below states that KL-stochastic sequences have constructive dimension 1. This is equivalent to the assertion that in the second mentioned result of Muchnik it is not necessary to require that the function h be computable, i.e., it suffices to require the mere existence of arbitrarily long prefixes of A that can be compressed by a factor of α.

In the remainder of the introduction we gather some notation that will be used throughout the text. Unless explicitly stated otherwise, the term *sequence* refers to an infinite binary sequence and a *class* is a set of sequences. Sequences are denoted by capital letters like $A, B, \ldots, R, S, \ldots$.

We will often deal with generalized joins and splittings. Assume that Z is an infinite and co-infinite set of natural numbers. The *Z-join* $A_0 \oplus_Z A_1$ of sequences A_0 and A_1 is the result of merging the sequences using Z as a guide. Formally,

$$A_0 \oplus_Z A_1(n) = \begin{cases} A_0(|\overline{Z} \cap \{0, \ldots, n-1\}|) & \text{if } Z(n) = 0, \\ A_1(|Z \cap \{0, \ldots, n-1\}|) & \text{if } Z(n) = 1. \end{cases}$$

On the other hand, given a sequence A and a set $Z \subseteq \omega$ one can obtain a new sequence (string) $A \restriction_Z$ by picking the positions that are in Z. Let p_Z denote the principal function of Z, i.e. $\mathrm{p}_Z(n)$ is the $(n+1)$st element of Z (where this is undefined if no such element exists). Formally,

$$A \restriction_Z (n) = A(\mathrm{p}_Z(n)), \qquad \text{where } \mathrm{p}_Z(n) = \mu x[|Z \cap \{0, \ldots, x\}| \geq n+1].$$

If Z is infinite, $A \restriction_Z$ will yield a new infinite sequence, otherwise we define $A \restriction_Z$ to be the string of length $|Z|$ extracted from A via Z. Note that this notation is consistent with the usual notation of initial segments in the sense that $A \restriction_n = A \restriction_{\{0,\ldots,n-1\}}$. Observe that $A = A_0 \oplus_Z A_1$ if and only if $A \restriction_Z = A_1$ and $A \restriction_{\overline{Z}} = A_0$.

Due to space considerations, several proofs are omitted. These proofs can be found in the full version of this paper [10].

2 Random and stochastic sequences

In this section, we give a brief and informal review of the concepts of effective randomness and stochasticity that are used in the following, for further details and formal definitions we refer to the surveys and monographs cited in the bibliography [11, 1, 5, 9, 19].

Intuitively speaking, a non-monotonic betting strategy defines a process that place bets on bits of a given sequence X. More precisely, the betting strategy determines a sequence of mutually distinct places $n_0, n_1, \ldots$ at which it bets a certain portion of the current capital on the value of the respective bit of X being 0 or 1. (Note that, by betting none of the capital, the betting strategy may always choose to "inspect" the next bit only.) The place n_{i+1} and the bet which is to be placed depends solely on the previously scanned bits $X(n_0)$ through $X(n_i)$. Payoff is fair in the sense that the stake is double in case the guess on the next bit was correct and is lost otherwise. For a betting strategy b that is applied with a certain initial capital c, we write $d_b^A(n)$ for the capital that has been accumulated after the first n bets on the bits of a sequence A while betting according to b; the function d_b is called the corresponding payoff function or martingale.

A non-monotonic betting strategy b *succeeds on* a sequence A if

$$\limsup_{n\to\infty} d_b^A(n) = \infty\,.$$

A sequence A is KL-*random* if there is no partial computable non-monotonic betting strategy that succeeds on A. The concept of KL-randomness remains the same if one uses in its definition computable instead of partial computable non-monotonic betting strategies [9].

One can modify the concept of a betting strategy in that, instead of specifying a bet on every next bit to be scanned, the strategy simply determines whether the next bit should be selected or not. Such a strategy is called a *selection rule*. The sequence selected from X is then the sequence of all bits that are selected, in the order of selection. A sequence X is called stochastic with respect to a given class of admissible selection rules if no selection rule in the class selects from X an infinite sequence that is biased in the sense that the frequencies of 0's and 1's do not converge to 1/2. A sequence is *Kolmogorov-Loveland stochastic* or *KL-stochastic*, for short, if the sequence is stochastic with respect to the class of partial computable non-monotonic selection rules; again, this concept remains the same if one replaces "partial computable" by "computable". A sequence is *Mises-Wald-Church stochastic* or *MWC-stochastic*, for short, if the sequence is stochastic with respect to the class of partial computable monotonic selection rules.

Furthermore, we consider Martin-Löf random sequences [7]. Let $W_0, W_1, ...$ be a standard enumeration of the recursively enumerable sets.

Definition 1. *A* Martin-Löf test *is a uniformly recursively enumerable sequence* $(A_n : n \in \omega)$ *of sets of strings such that for every* n*,*

$$\lambda(\{Y : Y \text{ has a prefix in } A_n\}) \leq 2^{-(n+1)},$$

where λ *denotes Lebesgue measure on Cantor space.*

A sequence $\mathcal{X}$ is covered *by a Martin-Löf test $(A_n\colon n \in \omega)$ if for every n the set A_n contains a prefix of X. A sequence is* Martin-Löf random *if it cannot be covered by any Martin-Löf test.*

A Martin-Löf test is called a *Schnorr test* if the Lebesgue measure of the set $\{Y\colon Y \text{ has a prefix in } A_n\}$ is computable in n (in the usual sense that the measure can be approximated effectively to any given precision strictly larger than 0); a sequence is called *Schnorr-random* if it cannot be covered by a Schnorr test.

Remark 2. Let b be a computable non-monotonic betting strategy that on every sequence scans all places of the sequence. Then there is a monotonic betting strategy that succeeds on every sequence on which b succeeds. This follows from results of Buhrman, Melkebeek, Regan, Sivakumar and Strauss [2].

One can use this fact to infer the following proposition.

Proposition 3. *The class of computably random sequences is closed under computable permutations of the natural numbers.*

3 Splitting properties of KL-random sequences

KL-random sequences bear some properties which make them appear quite "close" to Martin-Löf random sequences. One of them is a splitting property, which stresses the importance of non-monotonicity in betting strategies.

Proposition 4. *Let Z be a computable, infinite and co-infinite set of natural numbers, and let $A = A_0 \oplus_Z A_1$. Then A is KL-random if and only if*

$$A_0 \textit{ is } KL^{A_1}\textit{-random} \quad \textit{and} \quad A_1 \textit{ is } KL^{A_0}\textit{-random}. \tag{3}$$

Theorem 5. *Let Z be a computable, infinite and co-infinite set of natural numbers. If the sequence $A = A_0 \oplus_Z A_1$ is KL-random, then at least one of A_0 and A_1 is Martin-Löf random.*

Proof. Suppose neither A_0 nor A_1 is Martin-Löf random. Then there are Martin-Löf tests $(U_n^0\colon n \in \omega)$ and $(U_n^1\colon n \in \omega)$ with $U_n^i = \{\sigma_{n,0}^i, \sigma_{n,1}^i, \dots\}$, that cover A_0 and A_1, respectively.

Define functions f_0, f_1 by $f_i(n) = \mu k\ \sigma_{n,k}^i \sqsubset A_i$. Obviously there must be an $i \in \{0,1\}$ such that there are infinitely many m for which $f_i(m) \geq f_{1-i}(m)$. We define a new Martin-Löf test $\{V_n\}$ by $V_n = \bigcup_{m>n} \bigcup_{k=0}^{f_i(m)} [\sigma_{n,k}^{1-i}]$. Then $\{V_n\}$ is a Schnorr test relative to the oracle A_i (a Schnorr^{A_i}-test) and covers A_{1-i}, so A_{1-i} is not Schnorr^{A_i}-random. Since KL-randomness implies Schnorr-randomness (for relativized versions, too), it follows that A_{1-i} is not KL^{A_i}-random, contradicting Theorem 4. $\square$

An interesting consequence of (the relativized form of) Theorem 5 is stated in Theorem 8; in the proof of this theorem we will use Remark 6, due to van

Lambalgen [20] (also see [4] for a proof). For KL-randomness, the closest one can presently come to van Lambalgen's result is Proposition 4. Note the subtle difference: in the case of Martin-Löf randomness, one merely needs A_0 to be random, not random relative to A_1.

Remark 6. Let Z be a computable, infinite and co-infinite set of natural numbers. The sequence $A = A_0 \oplus_Z A_1$ is Martin-Löf random if and only if A_0 is Martin-Löf random and A_1 is Martin-Löf random relative to A_0. (Furthermore, this equivalence remains true if we replace Martin-Löf randomness by Martin-Löf randomness relative to some oracle.)

Definition 7. *A set Z has* density α *if* $\lim_{m\to\infty} \frac{|Z\cap\{0,\dots,m-1\}|}{m} = \alpha$.

Theorem 8. *Let R be a KL-random sequence and let $\alpha < 1$ be a rational. Then there is a computable set Z of density at least α such that $R\restriction_Z$ is Martin-Löf random.*

Proof. For a start, we fix some notation for successive splits of the natural numbers. Let $\{\mathrm{N}_w\}_{w\in\{0,1\}^*}$ be a uniformly computable family of sets of natural numbers such that for all w,

$$(i)\ \mathrm{N}_\varepsilon = \omega\ , \qquad (ii)\ \mathrm{N}_w = \mathrm{N}_{w0}\dot{\cup}\mathrm{N}_{w1}\ , \qquad (iii)\ \mathrm{N}_w \text{ has density } \frac{1}{2^{|w|}}\ ,$$

where $\dot{\cup}$ denotes disjoint union.

By (iii), for any word w the complement $\overline{\mathrm{N}_w}$ of N_w has density $1 - 1/2^{|w|}$, thus it suffices to show that there are words $w_1 \sqsubseteq w_2 \sqsubseteq \dots$ such that for all i,

$$(iv)\ |w_i| = i \ \text{ and } \ (v)\ R_i = R\restriction_{\overline{\mathrm{N}_{w_i}}} \text{ is Martin-Löf random}\ .$$

The w_i are defined inductively. For a start, observe that by Theorem 5 for $r_1 = 0$ or for $r_1 = 1$ the sequence $R\restriction_{\mathrm{N}_{r_1}}$ is Martin-Löf random; pick r_1 such that the latter is true and let $w_1 = 1 - r_1$. For $i > 1$, let w_i be defined as follows. By Proposition 4 the sequence $R\restriction_{\mathrm{N}_{w_i}}$ is KL-random relative to R_{i-1}, hence by (ii) and by a relativized version of Theorem 5, for $r_i = 0$ or for $r_i = 1$ the sequence $R\restriction_{\mathrm{N}_{wr_i}}$ is Martin-Löf random relative to R_i; pick r_i such the latter is true and let $w_i = w(1 - r_i)$.

Now (iv) follows for all i by an easy induction argument, using van Lambalgen's result from Remark 6. $\square$

The functions f_i in the proof of Theorem 5 can be viewed as a modulus for a certain type of approximation to the sequences under consideration. The technique of comparing two given moduli can also be applied to other types of moduli, e.g., to a modulus of convergence of an effectively approximable sequence.

Theorem 9. *Let Z be a computable, infinite and co-infinite set of natural numbers and let $A = A_0 \oplus_Z A_1$ be KL-random where A_1 is in Δ^0_2. Then A_0 is Martin-Löf random.*

By applying Theorem 9 to the set Z and its complement, the following Corollary is immediate.

Corollary 10. *Let Z be a computable, infinite and co-infinite set of natural numbers and let $A = A_0 \oplus_Z A_1$ be KL-random and Δ^0_2. Then A_0 and A_1 are both Martin-Löf random.*

The next example shows that splitting properties like the one considered in Corollary 10 do not necessarily imply Martin-Löf randomness.

Theorem 11. *There is a sequence A which is not computably random such that for each computable infinite and co-infinite set V, $A{\restriction}_V$ is 2-random, i.e. is Martin-Löf random relative to $\emptyset'$.*

A function g is an *order* if g is computable, nondecreasing, and unbounded.

Corollary 12. *Suppose A is in Δ^0_2 and is KL-random. Then for each order g and almost all n, $K(A \restriction n) \geq n - g(n)$.*

Remark 13. In the full version of this article it will be shown that there is a left-r.e. real A which is not MWC-stochastic, but satisfies $K(A \restriction n) \geq^+ n - g(n)$ for each order g and almost all n. Thus even for left-r.e. reals, the conclusion of Corollary 12 is not equivalent to Martin-Löf randomness.

4 Kolmogorov-Loveland Stochasticity

There are two standard techniques for constructing KL-random sequences. The first one is a probabilistic construction due to van Lambalgen [20]. The second one is to construct directly a Martin-Löf random sequence, e.g., by diagonalizing against a universal left-computable martingale. Theorem 14 is demonstrated by a further technique that allows to construct KL-stochastic sequences with certain additional properties that could not be achieved by the mentioned standard methods.

A sequence X is *weakly 1-random* (also called *Kurtz-random*) if X is contained in every r.e. open class of uniform measure 1. Note that Schnorr randomness implies weak 1-randomness, but not conversely.

Theorem 14. *There is a non-empty Π^0_1 class $\mathcal{P}$ of KL-stochastic sequences such that no $X \in \mathcal{P}$ is weakly 1-random.*

The proof of Theorem 14 is omitted due to space considerations. By the usual basis theorems [12], the following corollary is immediate.

Corollary 15. *There is a left-r.e., not weakly 1-random KL-stochastic sequence. There is a low, not weakly 1-random, KL-stochastic sequence. There is a not weakly 1-random KL-stochastic sequence that is of hyperimmune-free degree.*

5 The dimension of KL-stochastic sequences

There exists an interesting connection between the asymptotic complexity of sequences and Hausdorff dimension. Hausdorff dimension is defined via Hausdorff measures, and similar to Lebesgue measure, one can define effective versions of them. This leads to the concept of *constructive dimension*, first introduced by Lutz [6], which can equivalently defined in terms of prefix-free Kolmogorov complexity K.

Theorem 16. *The constructive dimension* $\dim_1 A$ *of a sequence* A *is given by*

$$\dim_1 A = \liminf_{n\to\infty} \frac{K(A\restriction_n)}{n}. \tag{4}$$

Note that C, plain Kolmogorov complexity, and K differ by at most $\log(|x|)$, so in Theorem 16 one can replace K by C. Theorem 16 was proven in the presented form by Mayordomo [8], but much of it was already implicit in earlier work by Ryabko [14, 15], Staiger [17, 18], and Cai and Hartmanis [3]. For more on constructive dimension see Reimann [13].

Muchnik [11] refuted a conjecture by Kolmogorov (who asserted that there exists a KL-stochastic sequence A such that $K(A\restriction_n) = \mathrm{O}(\log n)$) by showing that, if A is KL-stochastic, then $\limsup_{n\to\infty} K(A\restriction_n)/n = 1$. In the following, we are going to strengthen this result by showing that $\dim_1 A = 1$ for any KL-stochastic sequence A.

This relates to a result of Ryabko [16], who observed that the probabilistic argument for the construction of KL-stochastic sequences yields with probability 1 a sequence that has constructive dimension 1.

The proof of Theorem 22 bears some similarities to the proof of Theorem 8, where it has been shown that any KL-random sequence has arbitrarily dense subsequences that are Martin-Löf random. We will need the following Proposition, which is a slightly generalized version of a corresponding result by Muchnik et al. [11]. The proof of the proposition is omitted.

Proposition 17. *For any rational* $\alpha < 1$ *there is a natural numbers* k_α *and a rational* $\varepsilon_\alpha > 0$ *such that the following holds. Given an index for a computable martingale* d *with initial capital* 1*, we can effectively find indices for computable monotonic selection rules* $s_1, \dots, s_{2k_\alpha}$ *such that for all words* w *where*

$$d(w) \geq 2^{(1-\alpha)|w|} \tag{5}$$

there is an index i *such that the selection rule* s_i *selects from* w *a finite sequence of length at least* $\varepsilon_\alpha|w|$ *such that the ratio of* 0*'s and the ratio of* 1*'s in this finite sequence differ by at least* ε_α.

Definition 18. *Let* α *be a rational. A word* w *is called* α-compressible *if* $\mathrm{K}(w) \leq \alpha|w|$.

Remark 19. Given a rational $\alpha < 1$ and a finite set D of α-compressible words, we can effectively find an index for a computable martingale d with initial capital 1 such that for all $w \in D$ we have $d(w) \geq 2^{(1-\alpha)|w|}$.

For a proof, let d_w be the martingale that starts with initial capital $2^{-\alpha|w|}$ and plays a doubling strategy along w, i.e., always bets all its capital on the next bit being the same as the corresponding bit of w; then we have in particular $d_w(w) = 2^{(1-\alpha)|w|}$.

Let d be the sum of the martingales d_w over all words $w \in D$, i.e., betting according to d amounts to playing in parallel all martingales d_w where $w \in D$. Obviously $d(v) \geq d_w(v)$ for all words v and all $w \in D$, so it remains to show that the initial capital of d does not exceed 1. The latter follows because every $w \in D$ is α-compressible, i.e., can be coded by a prefix-free code of length at most $\alpha|w|$, hence the sum of $2^{-\alpha|w|}$ over all $w \in D$ is at most 1.

Lemma 20. *Let $A = A_1 \oplus A_2$ be KL-stochastic. Then one of the sequences A_1 and A_2 has constructive dimension* 1.

Proof. For a proof by contradiction, assume that the consequence of the lemma is false, i.e., that there is some rational number $\alpha_0 < 1$ such that A_1 and A_2 both have constructive dimension of at most α_0. Pick rational numbers α_1 and α such that $\alpha_0 < \alpha_1 < \alpha < 1$. By Theorem 16, for $r = 1, 2$, there are arbitrarily large prefixes w of A_r that are α_1-compressible, i.e., $\mathrm{K}(w) \leq \alpha_1|w|$. We argue next that for any m there are arbitrarily large intervals I with $\min I = m$ such that the restriction w of A_r to I is α-compressible.

Let $w_0, w_1, \ldots$ be an effective enumeration of all α-compressible words w. For the scope of this proof, say a word w is a *subword of X at m* if

$$w = X(m)X(m+1)\ldots X(m+|w|-1)\ .$$

Let ε_α be the constant from Proposition 17.

Claim 1. For $r = 1, 2$, the function g_r defined by

$$g_r(m) = \min\{i\colon w_i \text{ is a subword of } A_r \text{ at } m \text{ and } |w_i| > \frac{2}{\varepsilon_\alpha^2} m\}$$

is total.

Proof. There are infinitely many α_1-compressible prefixes v of A_r. Given any such prefix of length at least m, let u and w be the words such that $v = uw$ and $|u| = m$. Then we have

$$\mathrm{K}(v) \leq^+ \mathrm{K}(w) + 2\log m \leq \alpha_1|v| + 2\log m = \alpha|w| \left(\frac{\alpha_1}{\alpha}\frac{|v|}{|w|} + \frac{2\log m}{\alpha|w|}\right)\ ,$$

where the expression in brackets goes to $\alpha_1/\alpha < 1$ when the length of w goes to infinity. As a consequence, we have $\mathrm{K}(w) \leq \alpha|w|$ for all such words w that are long enough, hence by assumption on A for any m and t there is a word w_i and an index i as required in the definition of $g_r(m)$. □

Let $m_0 = 0$ and for all $t > 0$, let

$$m_{t+1} = m_t + \max\{|w_i| \colon i \le \max\{g_1(m_t), g_2(m_t)\}\}\ .$$

In the following, we assume that there are infinitely many t where

$$g_1(m_t) \le g_2(m_t)\ ; \tag{6}$$

we omit the essentially identical considerations for the symmetric case where there are infinitely many t such that $g_1(m_t) \ge g_2(m_t)$. Let

$$D_t = \{w_0, w_1, \ldots, w_{g_2(m_t)}\}$$

Claim 2. There are infinitely many t such that some word in D_t is a subword of A_1 at m_t.

Proof. By definition of $g_1(m_t)$, the word $w_{g_1(m_t)}$ is a subword of A_1 at m_t, where this word is in D_t for each of the infinitely many t such that $g_1(m_t)$ is less than or equal to $g_2(m_t)$. □

Claim 3. Given D_t and m_t, we can compute an index for a monotonic computable selection rules $s(t)$ that scans only bits of the form

$$A_1(m_t), A_1(m_t + 1), \ldots, A_1(m_{t+1} - 1)$$

of A such that for infinitely many t the selection rule $s(t)$ selects from these bits a finite sequence of length at least $2m_t/\varepsilon_\alpha$ where the ratios of 0's and of 1's in this finite sequence differ by at least ε_α.

Proof. By Proposition 17 and Remark 19, from the set D_t we can compute indices for monotonic computable selection rules $s_1, \ldots, s_{2k_\alpha}$ such that for each $w \in D_t$ there is an index i such that the selection rule s_i selects from w a finite sequence of length at least $\varepsilon_\alpha |w|$ such that the ratio of 0's and 1's in this finite sequence differ by at least ε_α. Any word $w \in D_t$ has length of at least $2m_t/\varepsilon_t^2$, hence the selected finite sequence has length of at least $2m_t/\varepsilon_\alpha$. Furthermore, by Claim 2, there are infinitely many t such that some $w \in D_t$ is a subword of A_1 at m_t, and among the corresponding indices i some index i_0 between 1 and $2k_\alpha$ must appear infinitely often. So it suffices to let for any t the selection rule $s(t)$ be equal to the i_0th selection rule from the list of selection rules computed from D_t. □

Now we construct an non-monotonic computable selection rule s that witnesses that A is not KL-stochastic. The selection rule s works in stages $t = 0, 1, \ldots$ and scans during stage t the bits of A that correspond to bits of the form

$$A_1(y) \quad \text{and} \quad A_2(y), \qquad \text{where } m_t \le y < m_{t+1}\ .$$

At the beginning of stage t, the value of $g_2(m_t)$ and the set D_t is computed as follows. Successively for $i = 0, 1, \ldots$, check whether w_i is a subword of A_2 at m_t by scanning all the bits

$$A_2(m_t), \ldots, A_2(m_t + |w_i| - 1)$$

of A that have not been scanned so far, until eventually the index i equal to $g_2(m_t)$ is found, i.e., until we find some minimum i such that w_i is a subword of A_2 at m_t. Observe that by definition of m_{t+1}, the index i is found while scanning only bits of the form $A_2(y)$ where $y < m_{t+1}$. Next the selection rule s scans and selects the bits $A_1(m_t), A_1(m_t+1), \ldots$ according to the selection rule s_{i_0} as in Claim 3; recall that this selection rule can be computed from D_t. Finally, stage t is concluded by computing m_{t+1} from $g_1(t)$ and $g_2(t)$, where $g_1(t)$ is obtained like $g_2(t)$, i.e., in particular, the computation of m_{t+1} only requires to scan bits of the form $A_r(y)$ where $y < m_{t+1}$.

By Claim 2 there are infinitely many t such that some $w \in D_t$ is a subword of A_1 at m_t. By choice of $s(t)$ and definition of s, for each such t the selection rule s selects during stage t a finite sequence of length at least $2m_t/\varepsilon_\alpha$ where the ratios of 0's and 1's in this finite sequence differ by at least ε_α. Consequently, the at most m_t bits of A that might have been selected by s before stage t are at most a fraction of $\varepsilon_\alpha/2$ of the bits selected during stage t, hence with respect to all the bits selected up to stage t the ratios of 0's and 1's differ by at least $\varepsilon_\alpha/2$. This contradicts the fact that A is KL-stochastic, hence our assumption that A_1 and A_2 both have constructive dimension strictly less than 1 is wrong. □

Lemma 21. *If $Z \subseteq \omega$ is computable, infinite, co-infinite, with density $\delta = \delta_Z$. Then it holds for any sequences A, B,*

$$\dim_1 B \oplus_Z A \geq \delta \dim_1 A + (1-\delta) \dim_1^A B. \tag{7}$$

The proof of Lemma 21 is omitted due to space considerations.

Theorem 22. *If R is KL-stochastic, then* $\dim_1 R = 1$.

Proof. The proof is rather similar to the proof of Theorem 8, in particular, we use the notation N_w from there. It suffices to show that there are words $w_1 \sqsubseteq w_2 \sqsubseteq \ldots$ such that for all i, we have $|w_i| = i$ and

$$\dim_1 R_i = 1 \,, \qquad \text{where } R_i = R\restriction_{\overline{\mathrm{N}_{w_i}}} \,;$$

the theorem then follows by Lemma 21 and because for any word w, the set $\overline{\mathrm{N}_w}$ has density $1 - 1/2^{|w|}$.

The w_i are defined inductively. For a start, observe that by Lemma 20 for $r_1 = 0$ or for $r_1 = 1$ the sequence $R\restriction_{\mathrm{N}_r}$ has constructive dimension 1; pick r_1 such that the latter is true and let $w_1 = 1 - r_1$. For $i > 1$, let w_i be defined as follows. By an argument similar to the proof of Proposition 4, the sequence $R\restriction_{\mathrm{N}_w}$ is KL-stochastic relative to R_{i-1}, hence by a relativized version of Lemma 20, for $r_i = 0$ or for $r_i = 1$ the sequence $R\restriction_{\mathrm{N}_{wr}}$ has constructive dimension 1 relative to R_w; pick r_i such the latter is true and let $w_i = w(1 - r_i)$.

It remains to show by induction on i that all the sequences R_i have constructive dimension 1. For $i = 1$, this is true by construction, while the induction step follows according to the choice of the w_i and due to Lemma 21 by an argument similar to the corresponding part of the proof of Theorem 8; details are left to the reader. □

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