Midterm Examination
MA 4207: Mathematical Logic

Monday 23 March 2015, Duration 45 minutes

Matriculation Number: ________________

Rules
This test carries 20 marks and consists of 5 questions. Each questions carries 4 marks; full marks for a correct solution; a partial solution can get a partial credit. In all questions, the logical language includes equality.

Question 1 [4 marks].
For how many values of the free variable x is the formula $\exists y\,[x = y * y + 2 * y + 2]$ true in the finite model $(\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}, +, *, =)$ of arithmetics modulo 10?

Solution. One can transform the body of the existentially quantified formula to $(x - 1) = (y + 1) * (y + 1)$. So the question is that how many numbers of the form $x - 1$ are, modulo 10, square numbers. As this is only a rotation by 1, one might just ask how many numbers are, modulo 10, square numbers. These squares are the last digits of the two-digit squares $0, 1, 4, 9, 16, 25, 36, 49, 64, 81$ and thus the digits $0, 1, 4, 5, 6, 9$. So the correct answer is **6**.

If one wants to compute them directly, without setting back onto the well-known squares, one could also just make modulo 10 the table of the possible terms $y * y + 2 * y + 2$ and get $0, 1, 2, 5, 6, 7$ which are also 6 entries.

Question 2 [4 marks].
Let $\alpha(a, b, c, d, e) = 1$ iff exactly 0, 2 or 5 inputs are 1 and let $\beta() = 0$ (constant 0). Can these formulas be used to express the following three formulas: $a \rightarrow b$ (implication), $a \vee b$ (or) and $a \oplus b$ (exclusive or)? For each of these three functions, either give the formula (which should only use α and β) or say why it does not work.

Solution. Note that here it is asked to express the three formulas by using nested expressions of the functions α and β and using, possibly repeated, the inputs a, b besides the nullary β.

The formula $a \rightarrow b$ is false iff a is 1 and b is 0. Thus $\alpha(a, a, a, b, b)$ is equivalent to $a \rightarrow b$.

The formula $a \vee b$ could be represented by $\alpha(\neg a, \neg a, \neg b, \neg b, 0)$ where for 0 one uses $\beta()$ and for $\neg a$ one uses $\alpha(a, \beta(), \beta(), \beta(), \beta())$ and for $\neg b$ one uses $\alpha(b, \beta(), \beta(), \beta(), \beta())$. So the overall formula is $\alpha(\alpha(a, \beta(), \beta(), \beta(), \beta()), \alpha(a, \beta(), \beta(), \beta(), \beta()), \alpha(b, \beta(), \beta(), \beta(), \beta()), \alpha(b, \beta(), \beta(), \beta(), \beta()), \beta())$.

The formula $a \oplus b$ is $\alpha(1, a, b, 0, 0)$, as this formula is only true when exactly one of a, b is 1. This formula expressed as $\alpha(\alpha(\beta(), \beta(), \beta(), \beta(), \beta()), a, b, \beta(), \beta())$.

Question 3 [4 marks].
Assume $(A, \circ, =)$ satisfies the following axioms:

- $\forall x, y\, [(x \circ y = x) \vee (x \circ y = y)]$;
- $\forall x, y\, [x \circ y = y \circ x]$;
- $\forall x, y, z\, [x \circ (y \circ z) = (x \circ y) \circ z]$.

Make a model $(A, \circ, =)$ of these axioms such that A has 3 elements. Up to isomorphism, how many three-element models $(A, \circ, =)$ of these axioms do exist?

Solution. The axioms say that the operation $\circ$ has always map inputs x, y to one of the elements x, y, it has to be commutative and it has to be associative. An example of such an operation is the maximum-operation: the maximum of x, y is always either x or y and it does not matter in which order the maximum is taken. Furthermore, $\max\{\max\{x, y\}, z\} = \max\{x, \max\{y, z\}\}$, that is the maximum is associative. So $(\{0, 1, 2\}, \max, =)$ would be a model for the given axioms.

Now one would ask how many models of three elements are there. First assume that $a \circ b = a$ and $a \circ c = a$ and assume that $b \circ c = b$. Then the mapping $2 \mapsto a$, $1 \mapsto b, 0 \mapsto c$ is a bijection which preserves the operation in the structure, thus an isomorphism. As $a \circ a = a$, a satisfies $a \circ y = a$ for all y. Up to renaming the members of $\{a, b, c\}$, all cases where there is an x with $\forall y\, [x \circ y = x]$ are the same as this one, that is, isomorphic to this case.

So consider the case that there is no x with $x \circ y = x$ for all y. If now $a \circ b = b$ then one can conclude that $b \circ c = c$ and $c \circ a = a$, as $x \circ x = x$. But now $a \circ (b \circ c) = a$ and $(a \circ b) \circ c = c$, so the structure does not satisfy the associativity. Similarly, if $a \circ b = a$ then $b \circ c = b$ and $c \circ a = c$ and again one can show that the structure is not associative. Thus this other case does not occur and all the three-element structures are isomorphic to $(\{0, 1, 2\}, \max, =)$.

Question 4 [4 marks].
Assume the structure $(\mathbb{N}, +, <, 0, 1, P, =)$ together with a predicate P. While the structure itself follows the model of the natural numbers, there is nothing by default fixed about P. Can one choose a formula α such that

$$[(\mathbb{N}, +, <, 0, 1, P, =) \models \alpha] \text{ iff } [Px \text{ is equivalent to "}x \text{ is a square number"}]?$$

If the answer is "yes" then provide such a formula α else explain why such α does not exist.

Solution. The answer is **"yes"**. The reason is that one can use that, by induction, $(x+1)^2$ is the first square number after x^2 and furthermore, $(x+1)^2$ is by $2x+1$ larger than x^2. So the idea is to define α as follows:

$$P(0) \wedge P(1) \wedge \forall x \, \forall y \, [P(x) \wedge P(x+y) \wedge \forall z \, [0 < z \wedge z < y \rightarrow \neg P(x+z)] \rightarrow \\ (P(x+y+y+1+1) \wedge \forall z \, [y < z \wedge z < y+y+2 \rightarrow \neg P(z)])].$$

So the formula α says that $0, 1$ are squares and if $x, x+y$ are neighbouring squares then $x+y$ and $x+2y+2$ are also neighbouring squares. This formula is based on the fact that if $x, x+y$ are neighbouring squares then $y = 2u+1$ for some u and $x = u^2$ and $x+y = (u+1)^2$. Now $(u+2)^2 = u^2+4u+4 = x+2y+2$ is the next square after $(u+1)^2$. Having these properties, one can show that if the model satisfies α then the least two numbers satisfying P are $0, 1$ and by induction, whenever u^2 and $(u+1)^2$ satisfy P, the next number which satisfies P is $(u+2)^2$. Thus $P(x)$ is equivalent to $x = u^2$ for some u.

Note that the condition in the question does not say that the square numbers are definable from addition and order. Indeed, this would be false. The reason is that α contains the predicate P inside the formula. This is not permitted when one defines one set or function in a language where this set or function does not exist. For example, the even numbers are definable s x is even $\Leftrightarrow \exists y \, [x = y + y]$, so this formula does note use the term "even" on the right side of the $\Leftrightarrow$. However, P is used all over in α.

Question 5 [4 marks].
Let the logical language contain equality (=) and one unary (that is, one-place) function symbol f. Make a formula α such that (a) α has a model $(A, f, =)$ and (b) all models $(A, f, =)$ of α satisfy that A is infinite.

Solution. Here it is required that one formula α satisfies both of (a) and (b). The basic idea is to postulate that the function f is surjective and not injective; such a thing cannot exist in finite models. So a formula would be

$$\forall x\, \exists y\, \exists z\, [y \neq z \wedge f(y) = x \wedge f(z) = x]$$

and this formula says that every x is equal to $f(y)$ and to $f(z)$ for two different elements y, z. A model for this would be $(\mathbb{N}, x \mapsto Floor(x/2), =)$ where $Floor(r)$ is the largest integer which is below r, so $Floor(1) = 1$ and $Floor(1.5) = 1$. Another formula would be

$$\forall x\, \forall y\, [x \neq y \rightarrow f(x) \neq f(y)] \wedge \exists x\, \forall y\, [f(y) \neq x]$$

which says that f is injective but not surjective. The natural numbers with the successor-function $x \mapsto x + 1$ would form a model for this formula. Again there are no finite models for this formula.

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