Student Number: ______________________

NATIONAL UNIVERSITY OF SINGAPORE

MA4207 - MATHEMATICAL LOGIC

(Semester 2 : AY2017/2018)

Time allowed : 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.
2. Please write your matriculation/student number only. Do not write your name.
3. This examination paper contains **10** questions (each carrying **6** marks) and comprises **21** printed pages.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** examination.
6. Students can use calculators which do not have any information in the memory when brought to the examination hall. When using a calculator, students should still lay out systematically the various steps in the calculations.

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Question	1	2	3	4	5	6	7	8	9	10	Total
Marks											

Question 1 [6 Marks]
Recall that in Fuzzy Logic, one defines the truth-values of logical connectives as follows:

1. $\nu(q) = q$ for $q \in Q$;
2. $\nu(\alpha \wedge \beta) = \min\{\nu(\alpha), \nu(\beta)\}$;
3. $\nu(\alpha \vee \beta) = \max\{\nu(\alpha), \nu(\beta)\}$;
4. $\nu(\neg\alpha) = 1 - \nu(\alpha)$;
5. $\nu(\alpha \oplus \beta) = \min\{\nu(\alpha) + \nu(\beta), 2 - \nu(\alpha) - \nu(\beta)\}$.

Here $Q = \{0, 1/2, 1\}$ is the permitted set of truth-values. Fill out the truth-table for the formula

$$\alpha = ((A_1 \wedge (A_2 \oplus (\neg A_2))) \vee (A_2 \wedge (A_1 \vee (\neg A_1))))$$

given in the below table:

A_1	A_2	α
0	0	
0	1/2	
0	1	
1/2	0	
1/2	1/2	
1/2	1	
1	0	
1	1/2	
1	1	

Solution. The truth-table is as follows:

A_1	A_2	α
0	0	0
0	1/2	1/2
0	1	1
1/2	0	1/2
1/2	1/2	1/2
1/2	1	1/2
1	0	1
1	1/2	1
1	1	1

Question 2 [6 Marks]
In a circuit, each gate can supply its output to arbitrarily many gates on a higher level, but and-gates and or-gates can have only two and not-gates only one input. Construct a circuit with at most six and-gates, three or-gates and six not-gates, which computes the function

$$x_1, x_2, x_3, x_4 \mapsto x_1 \oplus x_2 \oplus x_3 \oplus x_4.$$

Solution. The solution is given in the following ascii graphic.

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               z = or(and(y1,not(y2)),and(not(y1),y2))
                     |                            |
 y1 = or(and(x1,not(x2)),and(x2,not(x1)))    y2 = or(and(x3,not(x4)),and(not(x3),x4))
     |               |                                |             |
    x1              x2                               x3            x4

where the three subblocks each follow the inscribed formula in the same
way, so for the top block, it is

                            z
                            |
                           or
                          /   \
                       and     and
                      /  |      |  \
                    not  |      |  not
                     |   |      |   |
                    y1   y2    y1   y2

Inputs are sometimes listed several times in order to indicate repetition
of their usage.
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Question 3 [6 Marks]
Consider the following set S of formulas in sentential logic:

$$S = \{A_{2k+1} \vee A_{2k+2}, \neg A_{2k+2} \vee \neg A_{2k+3} : k \in \mathbb{N}\}.$$

Determine how many truth-assignments ν to the atoms exist which make all formulas in S true. This number is either a number in $\mathbb{N}$ or $\aleph_0$ (countable) or $2^{\aleph_0}$ (cardinal of the real numbers).

Solution. The right value is $\aleph_0$. For a proof, consider the truth-value function μ which has $\mu(A_{2k+1}) = 1$ and $\mu(A_{2k+2}) = 0$ for all k. It is obvious that μ makes all formulas true in S. Furthermore, if a further ν does the same, but is different from μ, there is a least index ℓ with $\mu(A_\ell) \neq \nu(A_\ell)$. Now, $\nu(A_{\ell+1})$ must be $\mu(A_\ell)$, as otherwise the formula which says that one of A_ℓ and $A_{\ell+1}$ have the truth-value $\mu(A_\ell)$ is not satisfied and this formula is in S. Thus $\nu(A_{\ell+1}) \neq \mu_{A_\ell}$ as well and this continues for all larger ℓ and so ν is equal to μ up to ℓ and from then onwards different from μ. One can furthermore see, that for each ℓ such a ν exists and makes all formulas in S true, thus the number of ν which make all formulas in S true is countable.

Question 4 [6 Marks]
Assume that a structure $(A, \circ, e)$ of at least two elements with one constant e is given by defining the binary operation $\circ$ as follows:

$$x \circ y = \begin{cases} x & \text{if } y = e; \\ y & \text{if } y \neq e. \end{cases}$$

Now consider the following three formulas:

1. $\forall x\, \forall y\, \forall z\, [x \circ (y \circ z) = (x \circ y) \circ z]$;
2. $\forall x\, [x \circ e = x \wedge e \circ x = x]$;
3. $\forall x\, \forall y\, \exists z\, [x \circ z = y]$.

For each formula, (a) say in words what the formula says, (b) say whether the structure satisfies the formula and (c) give a short reason for the answer in (b).

Solution. 1. The first formula says that the binary operation is associative and that thus the structure is a semigroup. Indeed, one can see from the definition that $x \circ y$ is always the right-most of these two elements which is not e; in the case that both are e, it is e. One can furthermore see that, independently on how one puts brackets into $x \circ y \circ z$, if all of x, y, z are e then $x \circ y \circ z = e$ else $x \circ y \circ z$ is the rightmost of these three elements which is not e: A detailed case-distinction is the following: If $z \neq e$ then $(x \circ y) \circ z = z$ and $x \circ (y \circ z) = x \circ z = z$; if $z = e$ and $y \neq e$ then $(x \circ y) \circ z = y \circ z = y$ and $x \circ (y \circ z) = x \circ y = y$; if $y = e$ and $z = e$ then $(x \circ y) \circ z = x \circ y = x$ and $x \circ (y \circ z) = x \circ y = x$. Thus the formula is satisfied.

2. The second formula says that e is the neutral element. Indeed, $x \circ e = x$ for all x and $e \circ x$ depends on x being e or not; if $x = e$ then the result is the first operand e which is correct and if $x \neq e$ then the result is the second operand which is x and which is also correct. So the second formula is also satisfied and the structure is a monoid.

3. The third formula says that for all x, y there is a z such that $x \circ z = y$. Indeed, for all $y \neq e$ this is correct, as $x \circ y = y$ by definition, so $z = y$ can be taken. However, if $x \neq e$ and $y = e$ then there is a problem: If $z = e$ then $x \circ z = x \neq e$ and if $z \neq e$ then $x \circ z = z \neq e$. So the statement is false for this structure and the structure is not a group.

Note that any structure satisfying all three axioms is a group and that all groups satisfy all three axioms. To see this, one observes that the third axiom enforces that for every x there is a x' such that $x \circ x' = e$. Now for x' there is a further x'' such that $x' \circ x'' = e$. If one now looks at $x \circ (x' \circ x'')$, it is $x \circ e$ and thus x; if one looks at $(x \circ x') \circ x''$, it is $e \circ x''$ and thus x''. By associativity, $x = x''$. Thus $x' \circ x = e$ and x' is the inverse of x from both sides. This observation is for information only, students are not required to observe that a structure satisfies all three axioms iff it is a group.

Question 5 [6 Marks]
Let a logical language contain equality and exactly one function symbol f, but neither constants nor predicate symbols. Furthermore, let f be a unary function, that is, a function with one input. Construct a formula α such that the following holds:

> For each cardinal κ there is a structure (A, f) and a value assignment s with $(A, f), s \models \alpha$ iff either $\kappa = 3n + 1$ for some natural number n or $\kappa \geq \aleph_0$.

Explain why the formula α provided is correct.

Solution. One possibility to choose f is the following:

$$\exists x \, \forall y \, [(y = x \leftrightarrow f(y) = y) \wedge f(f(f(y))) = y].$$

This formula says that one element is mapped by f to itself while all other elements are part of some three-cycle consisting of elements $z, f(z), f(f(z))$ where f then maps $f(f(z))$ back to z. Thus if κ is finite, then $\kappa = 3n + 1$ for some n and it is easy to see that the formula is correct. If κ is infinite, then $3\kappa + 1 = \kappa$ by cardinal arithmetics and so the union of κ three-cycles and a one-cycle has cardinality κ. Thus all infinite cardinals are also represented.

Question 6 [6 Marks]
Let the logical language contain = and countably many constants $c_0, c_1, \ldots$ (for each $k \in \mathbb{N}$ one c_k) and let $S = \{c_i \neq c_j : i, j \in \mathbb{N} \wedge i \neq j\}$, that is, S says that all constants are different. Let T be the set of theorems of S, that is, the set of all sentences α such that $S \vdash \alpha$. Answer the following questions and give reasons for the answers.

1. Is S $\aleph_0$-categorical?
2. Is S $\aleph_1$-categorical?
3. Is T decidable?

Solution. 1. The set S is not $\aleph_0$-categorical, as one can, for each $n \leq \aleph_0$, take A_n to be the set consisting of the values of all constants and n further elements which are different from all constants. If $n, m \leq \aleph_0$ are different cardinals, then A_n and A_m are not isomorphic, as an isomorphism must map the value of each constant c_k in A_n to the value of the constant c_k in A_m and then match bijectively the remaining n / m elements which are different from all constants, what is impossible, as $n \neq m$ (as cardinals).

2. The set S is $\aleph_1$-categorical, as the only model of size $\aleph_1$ consists of the values of the constants plus $\aleph_1$ many elements which are different from all constants.

3. All models of S are infinite, as there are infinitely many constants and the formulas in S assert that they represent different values. Note that by the Łoś-Vaught Test, S is complete, that is, for all sentences α, either $S \vdash \alpha$ or $S \vdash \neg\alpha$. Furthermore, S is consistent, as S has models. As S is clearly recursively enumerable and as the logical language is reasonable (only equality and countably many constant symbols and nothing else), one can furthermore see, that the set T of all sentences which can be deduced from S is recursively enumerable. For every sentence α, one can enumerate T until either α or $\neg\alpha$ appears in T and one knows that the other of these two formulas will never appear, thus T is even decidable.

Question 7 [6 Marks]
Recall the axioms in Λ:

(1) Tautologies;

(2) Axioms of the form $\forall x\,[\alpha] \to \alpha^x_t$ where the substitution α^x_t is permitted;

(3) Axioms of the form $\forall x\,[\alpha \to \beta] \to \forall x\,[\alpha] \to \forall x\,[\beta]$;

(4) Axioms of the form $\alpha \to \forall x\,[\alpha]$ where x does not occur free in α;

(5) Axioms of the form $x = x$;

(6) Axioms of the form $x = y \to \alpha \to \beta$ where α, β are primitive formulas and β is obtained from α by exchanging some (but not necessarily all) x and y;

(7) Universally quantified versions of the above.

To explain these axioms, do the following:

1. Provide an example of a tautology formed of formulas α and β and write in a few words, what a tautology formed from subformulas is;
2. Explain when a substitution is permitted and say whether $(\neg\forall y\,[x = y])^x_y$ is permitted and give reasons for the answer;
3. Say why is it in the fourth axiom required that x does not occur free in α.

Solution. 1. Tautologies in sentential logic are well-formed formulas which are true for every truth-assignment ν to the atoms. Given now a tautology with atoms $A_1, \ldots, A_k$ and any k formulas $\gamma_1, \ldots, \gamma_k$, one can obtain a tautology for Axiom 1 from these subformulas $\gamma_1, \ldots, \gamma_k$ by consistently replacing each occurrence of an atom A_ℓ by γ_ℓ. In the case of two given subformulas α, β, examples of tautologies built from these are $\alpha \to \beta \to \alpha$ and $\alpha \to \neg\alpha \to \beta$.

2. A substitution α^x_t is permitted if at every place where x is replaced by t in the formula, it does not happen that any of the variables occurring in t is inside the range of a quantifier in that location. For example, in the formula $(\neg\forall y\,[x = y])^x_y$ the x, when replaced by y, falls into the range of $\forall y$. While the formula before the substitution was true in all models with at least two elements (independent of the value of x), the new formula $\neg\forall y\,[y = y]$ is never true, that is, unsatisfiable; thus the substitution would create an axiom which is not valid: $\forall x\,[\neg\forall y\,[x = y]] \to \neg\forall y\,[y = y]$. To avoid non-valid axioms, one created the notion of permitted substitutions and only those can be used in Axiom 2.

3. The requirement in the fourth axiom is needed so that the axiom is valid in all models. Assume that α is true for some values of x but not all and x is free in α. If $\mathfrak{A}, s \models \alpha$ by $s(x)$ being one of the values for which α is true, then $\mathfrak{A}, s \not\models \forall x\,[\alpha]$ and therefore $\alpha \to \forall x\,[\alpha]$ is not a valid formula; a dependence of α on x can, however, only happen when x occurs free in α.

Question 8 [6 Marks]
Let the logical language contain one predicate symbol P and one constant c. Use the axioms from Λ and Modus Ponens and nothing else to prove the following:

$$\{\forall x\,\forall y\,[P(x) \to P(y)], P(z)\} \vdash \forall y\,[P(y)].$$

Solution. Let $S = \{\forall x\,\forall y\,[P(x) \to P(y)], P(z)\}$.

1. $S \vdash \forall x\,\forall y\,[P(x) \to P(y)]$ (Copy);
2. $S \vdash \forall x\,\forall y\,[P(x) \to P(y)] \to \forall y\,[P(z) \to P(y)]$ (Axiom 2);
3. $S \vdash \forall y\,[P(z) \to P(y)]$ (Modus Ponens);
4. $S \vdash \forall y\,[P(z) \to P(y)] \to \forall y\,[P(z)] \to \forall y\,[P(y)]$ (Axiom 3);
5. $S \vdash \forall y\,[P(z)] \to \forall y\,[P(y)]$ (Modus Ponens);
6. $S \vdash P(z) \to \forall y\,[P(z)]$ (Axiom 4);
7. $S \vdash P(z)$ (Copy);
8. $S \vdash \forall y\,[P(z)]$ (Modus Ponens);
9. $S \vdash \forall y\,[P(y)]$ (Modus Ponens).

Question 9 [6 Marks]
Provide the statements of the Generalisation Theorem and Deduction Theorem. Then prove the below formula using the following methods: the Generalisation Theorem, any direction of the Deduction theorem, the axioms of Λ and Modus Ponens. The formula to be proven is a version of the "Principle of alphabetic variants":

$$\emptyset \vdash \forall x\,[\neg(x+1=0)] \to \forall y\,[\neg(y+1=0)].$$

Here $+$ is a binary operation and $0, 1$ are constants.

Solution. In the following, let S be any set of wffs and let α, β be wffs. The Generalisation Theorem says the following:

If a variable x does not occur freely in any formula in S and if $S \vdash \alpha$ then $S \vdash \forall x\,[\alpha]$.

The Deduction Theorem says the following:

$S \cup \{\alpha\} \vdash \beta$ iff $S \vdash \alpha \to \beta$.

Both directions of the Deduction Theorem can be used in proofs.

The derivation is the following:

1. $\emptyset \vdash \forall x\,[x+1 \neq 0] \to y+1 \neq 0$ (Axiom 2);
2. $\{\forall x\,[x+1 \neq 0]\} \vdash y+1 \neq 0$ (Deduction Theorem);
3. $\{\forall x\,[x+1 \neq 0]\} \vdash \forall y\,[y+1 \neq 0]$ (Generalisation Theorem);
4. $\emptyset \vdash \forall x\,[x+1 \neq 0] \to \forall y\,[y+1 \neq 0]$ (Deduction Theorem).

Question 10 [6 Marks]
Consider the following statement:

$$\neg\forall x\, \forall y\, [P(x) \rightarrow P(y)] \rightarrow \neg\forall x\, \forall y\, [x = y].$$

Here P is a unary predicates. Say informally what the formula means and then prove the statement formally. The following methods are allowed: Using all axioms of Λ, Modus Ponens, Deduction Theorem (any direction) and Generalisation Theorem.

Solution. Note that $\neg(P(x) \rightarrow P(y))$ is $P(x) \wedge \neg P(y)$ and by inserting double negations at the right places and utilising the definition of $\exists$, the formula says

$$\exists x\, \exists y\, [P(x) \wedge \neg P(y)] \rightarrow \exists x\, \exists y\, [x \neq y].$$

In words: "If P does not take on all members of a structure the same truth-value then the structure has at least two elements." This is clearly a valid statement. Now a formal derivation is given.

1. $\emptyset \vdash x = y \rightarrow P(x) \rightarrow P(y)$ (Axiom 6);
2. $\emptyset \vdash \forall y\, [x = y \rightarrow P(x) \rightarrow P(y)]$ (Generalisation Theorem);
3. $\emptyset \vdash \forall y\, [x = y \rightarrow P(x) \rightarrow P(y)] \rightarrow \forall y\, [x = y] \rightarrow \forall y\, [P(x) \rightarrow P(y)]$ (Axiom 3);
4. $\emptyset \vdash \forall y\, [x = y] \rightarrow \forall y\, [P(x) \rightarrow P(y)]$ (Modus Ponens);
5. $\emptyset \vdash \forall x\, [\forall y\, [x = y] \rightarrow \forall y\, [P(x) \rightarrow P(y)]]$ (Generalisation Theorem);
6. $\emptyset \vdash \forall x\, [\forall y\, [x = y] \rightarrow \forall y\, [P(x) \rightarrow P(y)]] \rightarrow \forall x\, \forall y\, [x = y] \rightarrow \forall x\, \forall y\, [P(x) \rightarrow P(y)]$ (Axiom 3);
7. $\emptyset \vdash \forall x\, \forall y\, [x = y] \rightarrow \forall x\, \forall y\, [P(x) \rightarrow P(y)]$ (Modus Ponens);
8. $\emptyset \vdash (\forall x\, \forall y\, [x = y] \rightarrow \forall x\, \forall y\, [P(x) \rightarrow P(y)]) \rightarrow (\neg\forall x\, \forall y\, [P(x) \rightarrow P(y)] \rightarrow \neg\forall x\, \forall y\, [x = y])$ (Axiom 1, Law of Contraposition);
9. $\emptyset \vdash \neg\forall x\, \forall y\, [P(x) \rightarrow P(y)] \rightarrow \neg\forall x\, \forall y\, [x = y]$ (Modus Ponens).