NATIONAL UNIVERSITY OF SINGAPORE

MA 4207: Mathematical Logic

Semester 2; AY 2017/2018; Midterm Test

Time Allowed: 50 Minutes

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number. Do not write your name.
2. This assessment paper consists of FIVE (5) questions and comprises ELEVEN (11) printed pages.
3. Students are required to answer **ALL** questions.
4. Students should answer the questions in the space provided.
5. This is a **CLOSED BOOK** assessment.
6. It is permitted to use calculators, provided that all memory and programs are erased prior to the assessment; no other material or devices are permitted.
7. Every question is worth FIVE (5) or SIX (6) marks. The maximum possible marks are 28.

STUDENT NO: ______________________________

This portion is for examiner's use only

Question	Marks	Remarks
Question 1:		
Question 2:		
Question 3:		
Question 4:		
Question 5:		
Total:		

In this and further questions, let $\wedge$ denote "and", $\vee$ denote "inclusive or", $\neg$ denote "not", $\rightarrow$ denote "implies", $\leftrightarrow$ denote "logically equal" and $\oplus$ denote "exclusive or". Using these, one defines the following Boolean functions:

1. $f_1(x_1, x_2, x_3) = ((x_1 \vee x_2) \wedge \neg x_3)$;
2. $f_2(x_1, x_2, x_3) = (((x_1 \oplus x_2) \wedge (x_2 \oplus x_3)) \wedge (x_1 \oplus x_3))$;
3. $f_3(x_1, x_2, x_3) = ((\neg x_1) \vee x_2)$;
4. $f_4(x_1, x_2, x_3) = (x_1 \oplus ((\neg x_3) \vee x_3))$.

Is the set $X = \{f_1\}$ complete? ☐ Yes, ☐ No.

Is the set $Y = \{f_2, f_3\}$ complete? ☐ Yes, ☐ No.

Is the set $Z = \{f_3, f_4\}$ complete? ☐ Yes, ☐ No.

Give reasons for the answers below.

Recall that a set of functions is complete when all Boolean functions of two inputs of some other complete set can be generated by putting together these functions and using the inputs x_1 and x_2. An example of such a complete set of functions is given by the functions $x_1, x_2 \mapsto x_1 \wedge x_2$, $x_1, x_2 \mapsto x_1 \vee x_2$ and $x_1, x_2 \mapsto \neg x_1$.

Solution. It is sufficient to show that complete sets of functions generate $\vee$ and $\neg$, as all other Boolean functions with two inputs can be put together from these.

The set $X = \{f_1\}$ is not complete, as $f(0, 0, 0) = 0$ and so one cannot realise $\neg$ or 1 using X.

The set $Y = \{f_2, f_3\}$ is complete as $f_2(x_1, x_2, x_3) = 0$ for all input combintations (as there are two equal ones x_i, x_j among the inputs and $x_i \oplus x_j = 0$ and this 0 is connected with $\wedge$ with other terms). Furthermore, $f_3(x_1, x_2, x_3)$ is the same as $x_1 \rightarrow x_2$. Now one can set $\neg x_1$ as $x_1 \rightarrow 0$ which is $f_3(x_1, f_2(x_1, x_1, x_1), x_1)$ and $x_1 \vee x_2$ as $f_3(\neg(x_1), x_2, x_2)$ which is $f_3(f_3(x_1, f_2(x_1, x_1, x_1), x_1), x_2, x_2)$.

The set $Z = \{f_3, f_4\}$ is complete, as $\neg x_1 = f_4(x_1, x_1, x_1)$ and and $x_1 \vee x_2 = f_3(\neg x_1, x_2, x_2) = f_3(f_4(x_1, x_1, x_1), x_2, x_2)$.

Construct an infinite set S of formulas in sentential logic such that there are exactly countably many truth-assignments ν which make all formulas in S true. Recall that countably many means as many as there are members of $\mathbb{N}$ and this corresponds, in the language of set theory, to the cardinal $\aleph_0$. Note that the number of all truth-assignments is uncountable. The formulas in S can use the atoms $A_1, A_2, \ldots$ and the usual logical connectives $(\neg, \vee, \wedge, \rightarrow, \leftrightarrow, \oplus)$.

Solution. There are many solutions. One of the easiest one is the following: $S = \{A_i \rightarrow A_j : 1 \leq i \leq j\}$. Now let $\nu_0(A_i) = 0$ for all atoms A_i; this ν_0 satisfies all formulas in S. Furthermore, for each $i > 0$, there is exactly one ν_i satisfying for all j that $\nu_i(A_j) = 1 \Leftrightarrow i \leq j$ and this ν_i makes also all formulas in S true. There are no other truth-assignments making all formulas in S true, as any further truth-assignment μ satisfies $\mu(A_i) = 1$ and $\mu(A_j) = 0$ for some i, j with $i < j$; hence the formula $A_i \rightarrow A_j$ is in S and $\overline{\mu}(A_i \rightarrow A_j) = 0$. Thus the countable set $\{\nu_0, \nu_1, \ldots\}$ is exactly the set of all truth-assignments which make all formulas in S true.

Let $S = \{A_i \oplus A_j \oplus A_k \oplus A_h : 1 \leq i < j < k < h \leq 7\}$. What is the size of the largest subset T of S which is satisfiable? Give the number of formulas in T and explain how to determine it.

Solution. For a way towards the solution, let ν be a truth-assignment which makes the maximal number of formulas in S true. Let m be the number of atoms which are 1 and k be the number of atoms which are 0. Now for $\alpha \in S$, $\overline{\nu}(\alpha) = 1$ iff exactly an odd number of the atoms in it are evaluated to 1. Thus the number of atoms evaluated to 1 is either one or three. Now one can look how many formulas in S can be chosen to have this amount of atoms among the m true ones and the other atoms among the k false ones and one obtains the formula

$$f(m,k) = (m \cdot (m-1) \cdot (m-2) \cdot k + m \cdot k \cdot (k-1) \cdot (k-2))/6$$

and due to symmetry, one needs only to look at the following values: $f(4,3)$, $f(5,2)$, $f(6,1)$, $f(7,0)$. When tabling out f for these values, one gets the following table:

m	k	$m \cdot (m-1) \cdot (m-2) \cdot k$	$m \cdot k \cdot (k-1) \cdot (k-2)$	sum	$f(m,k)$
4	3	72	24	96	16
5	2	120	0	120	20
6	1	120	0	120	20
7	0	0	0	0	0

So if ν makes 1 or 2 or 5 or 6 atoms 1 and the others 0 then 20 of the formulas in S are satisfied. If ν makes 3 or 4 atoms 1 then 16 of the formulas in S are satisfied. If ν makes 0 or 7 atoms 1 then no formula in S is satisfied. Hence one can satisfy at most 20 of the formulas in S and the largest cardinality of a satisfiable subset of S is 20. One choice of such a subset is $\{A_1 \oplus A_i \oplus A_j \oplus A_k : 2 \leq i < j < k \leq 7\}$ and the ν satisfying all of these formula is given by $\nu(A_1) = 1$ and $\nu(A_h) = 0$ whenever $h \neq 1$.

Recall the following from Question 1: let $\wedge$ denote "and", $\vee$ denote "inclusive or", $\neg$ denote "not", $\rightarrow$ denote "implies", $\leftrightarrow$ denote "logically equal" and $\oplus$ denote "exclusive or". Furthermore, let a language in first-order logic contain one function f and two unary predicates *Country* and *Colour* and constants *Argentina*, *Singapore*, *USA*, *Vietnam* as well as *Blue*, *Red*, *White*, *Yellow*. The domain X of the structure contains countries and colours. Consider the following axioms in first-order logic on the structure where the universal quantifier $\forall$ and the existential quantifier $\exists$ always range over X and f tries to pick for each country one colour of the flag in a way specified below:

1. $\forall x\,[Country(x) \oplus Colour(x)]$;
2. $Country(Argentina) \wedge Country(Singapore) \wedge Country(USA) \wedge Country(Vietnam) \wedge Argentina \neq Singapore \wedge Argentina \neq USA \wedge Argentina \neq Vietnam \wedge Singapore \neq USA \wedge Singapore \neq Vietnam \wedge USA \neq Vietnam$;
3. $\forall x\,[Colour(x) \rightarrow (x = Blue \vee x = Red \vee x = White \vee x = Yellow)]$;
4. $\forall x\,[Colour(f(x)) \wedge f(f(x)) = f(x)]$;
5. $\forall x\,\forall y\,[Country(x) \wedge Country(y) \wedge f(x) = f(y) \rightarrow x = y]$;
6. $f(Argentina) = Blue \vee f(Argentina) = White \vee f(Argentina) = Yellow$;
7. $f(Singapore) = Red \vee f(Singapore) = White$;
8. $f(USA) = Blue \vee f(USA) = Red \vee f(USA) = White$;
9. $f(Vietnam) = Red \vee f(Vietnam) = Yellow$.

How many models, up to isomorphism, are there of this structure? ___________

Is the number of countries always four? ☐ Yes, ☐ No.

Is the number of colours always four? ☐ Yes, ☐ No.

Give short reasons for the answers.

Solution. The first axiom states that every object in the structure is either a country or a colour but not both. The second axiom says that the constants *Argentina*, *Singapore*, *USA* and *Vietnam* are countries and are all pairwise distinct. The third axioms states that every colour is either Blue or Red or White or Yellow. The fourth axioms states that the range of f are the colours and that f is the identity when restricted to its range. Furthermore, the countries are mapped in a one-one way to the colours and hence, as there are at least four countries, there are also at least four colours; however, as each colour is equal to one of four constants, there are **exactly four countries** and **exactly four colours** and f restricted to the countries is a bijection from the countries to the colours. The constants for the colours are therefore also pairwise different.

Evaluating the value-conditions on f gives the following models; here only the values for f on the countries are listed, not those on the colours, as f maps each colour to itself.

Country	Colour for Model 1	for Model 2	for Model 3	for Model 4
Argentina	*Blue*	*Blue*	*White*	*Yellow*
Singapore	*Red*	*White*	*Red*	*White*
USA	*White*	*Red*	*Blue*	*Blue*
Vietnam	*Yellow*	*Yellow*	*Yellow*	*Red*

This list of models is exhaustive and so there are exactly **four models**.

Question 5 [6 marks]

Write axioms in a logical language with equality, one binary operation $+$ and constants $0, 1, 2$ to define all structures $(X, +, 0, 1, 2)$ with the following properties: All constants are different and every element equals one of the constants and the addition $+$ (which can be written the usual way, no need to make it a function) is commutative and associative and 0 is the neutral element of $+$. Furthermore, list out at least two models of these axioms by giving the corresponding tables for $+$.

Model 1:

+	0	1	2
0			
1			
2			

Model 2:

+	0	1	2
0			
1			
2			

Write the axioms of the structure below here; when needed, draw a new table as above.

Solution. The axioms are as follows:

1. $0 \neq 1 \wedge 0 \neq 2 \wedge 1 \neq 2$;
2. $\forall x\,[x = 0 \vee x = 1 \vee x = 2]$;
3. $\forall x\,\forall y\,\forall z\,[x + (y + z) = (x + y) + z]$;
4. $\forall x\,\forall y\,[x + y = y + x]$;
5. $\forall x\,[x + 0 = x]$.

The first two state that there are exactly the three elements given by the constants, the third axioms states that $+$ is associative, the fourth axiom that $+$ is commutative and the fifth axiom that 0 is the neutral element. The following five are models for associative and commutative monoids on $\{0, 1, 2\}$ with 0 being the neutral element:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	2

+	0	1	2
0	0	1	2
1	1	1	1
2	2	1	1

+	0	1	2
0	0	1	2
1	1	2	2
2	2	2	2

The first model is addition modulo 3 and the next two models are given by the maximum operation where either the constant 2 or the constant 1 is used for the largest element; note that the neutral element 0 is the smallest element when taken as the neutral element for the maximum operation. For the addition modulo 3, exchanging the roles of 1 and 2 does not lead to a new model. The last two models just take a fixed value a when none of the operands is 0; here a can arbitrarily be 1 or 2. Two students found the following additional models:

+	0	1	2
0	0	1	2
1	1	2	1
2	2	1	2

+	0	1	2
0	0	1	2
1	1	1	2
2	2	2	1

These models assign to a sum $a_1 + a_2 + \ldots + a_n$ the value 0 iff all a_i are 0. In the case that not all members of the sum are 0, they assign the value c if there is an odd number of c and the value $3 - c$ if there is an even number of c. Here c can be either 1 or 2. The first of these two models has $c = 1$ and the second has $c = 2$.

END OF PAPER