Student Number: ______________________________

NATIONAL UNIVERSITY OF SINGAPORE

MA4207 - MATHEMATICAL LOGIC

(Semester 2 : AY2018/2019)

Time allowed : 2 hours 30 minutes

INSTRUCTIONS TO CANDIDATES

1. **Write down your matriculation/student number clearly in the space provided at the top of this page.** This booklet (and only this booklet) will be collected at the end of the examination.
2. Please write your matriculation/student number only. Do not write your name.
3. This examination paper contains **10** questions (each carrying **6** marks) and comprises **21** printed pages. There are 60 marks in total.
4. Answer **ALL** questions.
5. This is a **CLOSED BOOK** examination.
6. Students can use calculators which do not have any information in the memory when brought to the examination hall. When using a calculator, students should still lay out systematically the various steps in the calculations.

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Question	1	2	3	4	5	6	7	8	9	10	Total
Marks											

Question 1 [6 Marks]
Let $C = \{0, 1, A_1, A_2, A_3, A_1 \wedge A_2, A_1 \wedge A_3, A_2 \wedge A_3\}$ where $\wedge$ is the Boolean connective "and". Let D contain any formula which is either a member of C or consists of several members of C connected by $\vee$, the Boolean connective "or". Say that two members $\alpha, \beta \in D$ are essentially equal iff B^3_α and B^3_β are equal as Boolean functions. Let E be obtained from D by omitting essential duplicates, that is, $E \subseteq D$, E is as large as possible and E does not contain any distinct α, β with $B^3_\alpha = B^3_\beta$. Note that the size of E is equal to the number of Boolean functions represented by the members of D.

What is the size of E?

Give a listing of the members of E; if $\alpha \in D$ then exactly one member β which is essentially equal to α should go into D, but it does not matter which such β is selected.

Solution. Given any $\alpha \in D$, one puts that β into E which is obtained from α by removing all redundant terms in the disjunction. If the formula is $\alpha_1 \vee \alpha_2 \vee \ldots \vee \alpha_n$ then there are distinct α_i, α_j with $\{\alpha_i\} \models \alpha_j$, then one removes α_i from the disjunction and the Boolean function remains the same. In particular one omits from a disjunction containing 1 all terms different from 1, from a disjunction containing a nonzero term the term 0 and from a disjunction containing both A_i and $A_i \wedge A_j$ the term $A_i \wedge A_j$. By this way one gets the following formulas in E:

1. truth-constants and atoms: 0, 1, A_1, A_2, A_3;
2. conjunctive terms only: $A_1 \wedge A_2$, $A_1 \wedge A_3$, $A_2 \wedge A_3$;
3. disjunction of two or three atoms: $A_1 \vee A_2$, $A_1 \vee A_3$, $A_2 \vee A_3$, $A_1 \vee A_2 \vee A_3$;
4. disjunction of a conjunction and an atom: $A_1 \vee (A_2 \wedge A_3)$, $A_2 \vee (A_1 \wedge A_3)$, $A_3 \vee (A_1 \wedge A_2)$;
5. disjunction of two conjunctions: $(A_1 \wedge A_2) \vee (A_1 \wedge A_3)$, $(A_1 \wedge A_2) \vee (A_2 \wedge A_3)$, $(A_1 \wedge A_3) \vee (A_2 \wedge A_3)$;
6. majority-function: $(A_1 \wedge A_2) \vee (A_1 \wedge A_3) \vee (A_2 \wedge A_3)$.

This are in total $5 + 3 + 4 + 3 + 3 + 1 = 19$ formulas and thus 19 Boolean functions. These are all positive functions (made of disjunctions of conjunctions of truth-values and atoms) except for the function $B^3_{A_1 \wedge A_2 \wedge A_3}$.

Question 2 [6 Marks]
Construct a Boolean formula α using two types of connectives, namely exclusive or ($\oplus$) as well as and ($\wedge$), such that α satisfies the following constraint: α is true iff one or two of the atoms A_1, A_2, A_3 are true. The number of $\oplus$ can be arbitrary, but the number of $\wedge$ is limited to three.

Solution. The formula is

$$A_1 \oplus A_2 \oplus A_3 \oplus (A_1 \wedge A_2) \oplus (A_1 \wedge A_3) \oplus (A_2 \wedge A_3)$$

and if none or all of the atoms are true then either none or all six terms connected by $\oplus$ are 1 and therefore the output is 0; if exactly one atom is 1 then the conjunctions are all 0 and one out of three atoms is 1 and the output is 1; if exactly two atoms are 1, say A_1 and A_2, then two of the three atom-terms and one of the three conjunctions (namely $A_1 \wedge A_2$) are 1 and therefore, three out of six terms are 1 and the exclusive or then produces a 1 (as it is an odd number of terms). This is, however, not the best solution which was found by one student and which uses only one $\wedge$:

$$(A_1 \oplus A_2) \oplus ((A_1 \oplus A_3) \wedge (A_2 \oplus A_3)).$$

If A_1 differs from A_2 then $A_1 \oplus A_2$ is 1 while the two terms $A_1 \oplus A_3$ and $A_2 \oplus A_3$ differ so that their conjunction is 0. As $1 \oplus 0 = 1$, the formula is statisfied. If $A_1 = A_2$ then the formula is 1 iff A_3 differs from A_1 and thus also from A_2; so if $A_3 \neq A_1$ then the formula is $0 \oplus (1 \wedge 1)$ which is 1 else the formula is $0 \oplus (0 \wedge 0)$ which is 0.

Question 3 [6 Marks]
Consider the following set S of formulas in fuzzy logic with values $0, 1/3, 2/3, 1$:

$$S = \{A_i \to A_j : i, j \in \mathbb{N} \text{ with } 1 \leq j < i\}.$$

Here $\overline{\nu}(A_2 \to A_1) = \min\{1, 1 + \nu(A_1) - \nu(A_2)\}$ and ν satisfies S iff for all $\alpha \in S$, $\overline{\nu}(\alpha) = 1$.

Which truth-assignments $\nu : \{A_1, A_2, \ldots\} \to \{0, 1/3, 2/3, 1\}$ satisfy S? Characterise these truth-assignments.

Solution. A truth-assignment satisfies the set S iff ν is monotonically decreasing, that is, iff $\nu(A_{k+1}) \leq \nu(A_k)$ for all k. If this is the case then whenever $j < i$ then $\nu(A_j) \geq \nu(A_i)$ and $\overline{\nu}(A_i \to A_j) = 1$. So $\nu \models S$. If this is not the case then there is a k with $\nu(A_{k+1}) > \nu(A_k)$ and $\overline{\nu}(A_{k+1} \to A_k) = 1 + \nu(A_k) - \nu(A_{k+1}) < 1$, so $\nu \not\models S$.

Question 4 [6 Marks]
Assume that a structure $(\mathbb{Z}, \circ, 0)$ is given by defining the binary operation $\circ$ as follows:

$$x \circ y = (y + 1).$$

Here $+$ and 0 and 1 are defined as usual on $\mathbb{Z}$. Consider the following formulas:

1. $\forall x\, \forall y\, \forall z\, [x \circ (y \circ z) = (x \circ y) \circ z]$;
2. $\forall y\, \exists x\, [x \circ y = 0]$;
3. $\forall x\, \exists y\, [x \circ y = 0]$.

For each formula, (a) say in words what the formula says, (b) say whether the structure satisfies the formula and (c) give a short reason for the answer in (b).

Solution. 1. The first formula says that the binary operation is associative and that thus the structure is a semigroup. The structure does not satisfy this: Given x, y, z, $((x \circ y) \circ z) = (y + 1) \circ z = z + 1$, $x \circ (y \circ z) = x \circ (z + 1) = z + 2$.

2. The second formula says that one can "invert" from the left in the sense that for every right operand y one can find an x such that $x \circ y = 0$. Here "invert" might be a bit rough terminology, as this structure has no neutral element and 0 is an element like all others. Nevertheless, if $y = 1$, then $x \circ y = y + 1$ and therefore the second formula is not true.

3. The third formula says that one can "invert" from the right, again in a rough sense. Indeed, for any z, if $y = z - 1$ then $x \circ y = z$. This is in particular true for $z = 0$ and thus $x \circ -1 = 0$.

Question 5 [6 Marks]
Let a logical language contain the equality $=$ and exactly two function symbols f, g, but neither constants nor predicate symbols. Furthermore, let f, g be unary functions, that is, functions with one input each. Consider the following two axioms:

1. $\forall x \, \forall y \, [f(x) \neq g(y)]$;
2. $\forall x \, \forall y \, [x \neq y \rightarrow f(x) \neq f(y)]$.

Answer the following questions with respect to the following cardinals: $\kappa = 1, 2, 3, \aleph_0, \aleph_1, \aleph_2$. These are the first three nonzero cardinals plus the first three infinite cardinals.

(a) For which cardinals κ does there exist a model of size κ which makes both axioms true?

(b) For which cardinals κ does there exist a model of size κ which makes the first axiom true?

Provide reasons for both answers.

Solution. For (a), the cardinals are $\kappa = \aleph_0, \aleph_1, \aleph_2$; more precisely all infinite cardinals. If X is an infinite set, then one can choose $f : X \rightarrow X$ such that f is one-one but not surjective, so there is one element $a \in X - range(f)$. Now one let $g(x) = a$ for all x and both axioms are satisfied. For finite κ, the f, g do not exist, as the first axiom implies that f is not surjective (the ranges of f and g are disjoint) and the second axioms implies that f is injective. Such a functions f does not exist from a finite set X to itself.

For (b), the cardinals are $2, 3, \aleph_0, \aleph_1, \aleph_2$, more precisely, all cardinals κ with $\kappa \geq 2$. The reason is that one just requires that the ranges of f and g are disjoint and then, if there are two different elements $0, 1$ and perhaps some more, f could map every element to 0 and g every element to 1. However, if the base set X has exactly one element, then this is in the range of both f and g and therefore $\kappa = 1$ does not qualify.

Question 6 [6 Marks]
Let the logical language contain the equality $=$ and one relation $\equiv$. Furthermore, let $(\mathbb{N}, =, \equiv)$ be the model with $\mathbb{N}$ being the set of natural numbers and $\equiv$ defined being as follows for any $m, n \in \mathbb{N}$:

$$m \equiv n \Leftrightarrow \exists i, j, k\, [n + 1 = 2^k \cdot (2i + 1) \wedge m + 1 = 2^k \cdot (2j + 1)].$$

That is, $m \equiv n$ holds exactly if, in the binary representation, the last nonzero bits of $m + 1$ and $n + 1$ are at the same position. Let T be the theory of this model, that is, the set of all sentences made true by $(\mathbb{N}, \equiv, =)$. Note that T is axiomatisable.

(a) Is T $\aleph_0$-categorical? ☐ Yes, ☐ No.
Explain the answer.

(b) Is T $\aleph_1$-categorical? ☐ Yes, ☐ No.
Explain the answer.

(c) Provide an explicit and consistent list of axioms allowing to deduce all sentences in T.

Solution. (a) Any countable model consists of infinitely many equivalence classes of size $\aleph_0$. If $(A, \equiv, =)$ and $(B, \equiv, =)$ are such models, then one can first make a bijection between the equivalence classes of A and B — this exists as both have countably many equivalence classes. Furthermore, if F maps an equivalence class E of A to an equivalence class $F(E)$ to B, then both E and $F(E)$ are countable and there is a bijection f_E from E to $F(E)$, these bijections f_E can then be unioned up to a bijection between $(A, \equiv, =)$ and $(B, \equiv, =)$. As any two countable models are isomorphic, the structure is $\aleph_0$-categorical and so is its theory.

(b) It is not $\aleph_1$-categorical. Two models of size $\aleph_1$ are $(A, \equiv, =)$ where A consists of $\aleph_0$ many equivalence classes, each of size $\aleph_1$, and $(B, \equiv, =)$ where B consists of $\aleph_1$ many equivalence classes, each of size $\aleph_0$. So both A and B have $\aleph_0 \cdot \aleph_1 = \aleph_1$ many elements. There is no isomorphism from $(A, \equiv, =)$ to $(B, \equiv, =)$, as such an isomorphisms induces a bijection between the equivalence classes and the first model has $\aleph_0$ equivalence classes and the second has $\aleph_1$ equivalence classes and sets of different cardinality cannot have a bijection among them.

(c) The following axioms allow to derive all members of T, as they have the following five statements, where the last two are given for all $n \in \mathbb{N}$.

1. $\equiv$ is reflexive: $\forall x\, [x \equiv x]$.
2. $\equiv$ is symmetric: $\forall x\, \forall y\, [x \equiv y \rightarrow y \equiv x]$.
3. $\equiv$ is transitive: $\forall x\, \forall y\, \forall z\, [x \equiv y \rightarrow y \equiv z \rightarrow x \equiv z]$.
4. Every equivalence class contains at least $n+1$ elements: $\forall x_1\, \forall x_2\, \ldots \forall x_n\, \exists y\, [y \equiv x_1 \wedge y \neq x_1 \wedge y \neq x_2 \wedge \ldots \wedge y \neq x_n]$.
5. There are at least $n + 1$ different equivalence classes: $\forall x_1\, \forall x_2\, \ldots \forall x_n\, \exists y\, [y \not\equiv x_1 \wedge y \not\equiv x_2 \wedge \ldots \wedge y \not\equiv x_n]$.

Question 7 [6 Marks]
Recall the axioms in Λ:

(1) Tautologies — that is, one takes a tautology from sentential logic (a formula which is true whatever truth-value one chooses for the atoms) and then one replaces consistently each atom A_k by some wff α_k of first-order logic;

(2) Axioms of the form $\forall x\,[\alpha] \to \alpha^x_t$ where the substitution α^x_t is permitted, that is, the term t contains only variables which are free at all places where x occurs free in α;

(3) Axioms of the form $\forall x\,[\alpha \to \beta] \to \forall x\,[\alpha] \to \forall x\,[\beta]$;

(4) Axioms of the form $\alpha \to \forall x\,[\alpha]$ where x does not occur free in α;

(5) Axioms about equality to be described below;

(6) Axioms about equality to be described below;

(7) Universally quantified versions of the above.

Describe the axioms of type 5 and 6 and explain which variables and which type of formulas can be used as building blocks for them and give for each of the two axioms an example.

Solution. The axioms of the fifth type are simple all equalities of the form $x = x$ where x is any variable. The axioms of the sixth type are of the form $x = y \to (\alpha \to \beta)$ where x, y are variables and α, β are both atomic formulas and β is obtained from α by interchanging some of the occurrences of the two variables.

Examples of the formulas are $x = x$ and $y = z \to P(y) \to P(z)$ where P is some unary predicate.

Question 8 [6 Marks]
Assume that the logical language contains a constant c, a unary function f and equality $=$. Consider the following statement:

$$\exists x\,[f(x) = f(c)].$$

Rewrite this statement using only a universal quantifier and then prove the resulting statement formally, that is, show how the statement can be proven from the empty precondition using the axioms from Λ and Modus Ponens.

Solution. The statement is to be written as

$$\neg\forall x\,[f(x) \neq f(c)]$$

and one uses that $(A_1 \to \neg A_2) \to (A_2 \to \neg A_1)$ is a tautology in sentential logic.

1. $\emptyset \vdash \forall x\,[f(x) \neq f(c)] \to f(c) \neq f(c)$ (Axiom 2);
2. $\emptyset \vdash (\forall x\,[f(x) \neq f(c)] \to f(c) \neq f(c)) \to (f(c) = f(c) \to \neg\forall x\,[f(x) \neq f(c)]$ (Axiom 1);
3. $\emptyset \vdash f(c) = f(c) \to \neg\forall x\,[f(x) \neq f(c)]$ (Modus Ponens);
4. $\emptyset \vdash \forall y\,[y = y]$ (Axiom 5);
5. $\emptyset \vdash \forall y\,[y = y] \to f(c) = f(c)$ (Axiom 2);
6. $\emptyset \vdash f(c) = f(c)$ (Modus Ponens);
7. $\emptyset \vdash \neg\forall x\,[f(x) \neq f(c)]$ (Modus Ponens);
8. $\emptyset \vdash \exists x\,[f(x) = f(c)]$ (Rewriting of Quantifier).

Question 9 [6 Marks]
Let the logical language contain the unary predicates P and Q and let

$$S = \{\forall x\, \forall y\, [P(x) \rightarrow Q(y)], \neg Q(z)\}.$$

Prove formally the statement

$$S \vdash \neg P(z)$$

using Modus Ponens, the formulas in S and the axioms of Λ.

Solution. The proof is as follows:

1. $S \vdash \forall x\, \forall y\, [P(x) \rightarrow Q(y)]$ (Copy);
2. $S \vdash \forall x\, \forall y\, [P(x) \rightarrow Q(y)] \rightarrow \forall y\, [P(z) \rightarrow Q(y)]$ (Axiom 2);
3. $S \vdash \forall y\, [P(z) \rightarrow Q(y)]$ (Modus Ponens);
4. $S \vdash \forall y\, [P(z) \rightarrow Q(y)] \rightarrow P(z) \rightarrow Q(z)$ (Axiom 2);
5. $S \vdash P(z) \rightarrow Q(z)$ (Modus Ponens);
6. $S \vdash \neg Q(z) \rightarrow (P(z) \rightarrow Q(z)) \rightarrow \neg P(z)$ (Axiom 1);
7. $S \vdash \neg Q(z)$ (Copy);
8. $S \vdash (P(z) \rightarrow Q(z)) \rightarrow \neg P(z)$ (Modus Ponens);
9. $S \vdash \neg P(z)$ (Modus Ponens).

Here the Axiom 1 above can be used, as the tautology is equivalent to $\neg Q(z) \wedge (P(z) \rightarrow Q(z)) \rightarrow \neg P(z)$, that is, if $\neg Q(z)$ holds and $P(z) \rightarrow Q(z)$ holds, then due to $\neg Q(z)$ both preconditions can only be true when $P(z)$ is false and thus $\neg P(z)$ holds. Therefore this rule is a tautology.

Question 10 [6 Marks]
Let the logical language contain the equality $=$ and two constants a, b. Let

$$S = \{\forall x\,[x \neq a \to x = b], \neg\forall x\,[x = b]\}.$$

Give a formal proof for

$$S \vdash a \neq b$$

where this formal proof can use the Axioms of Λ, Modus Ponens, statements in S, the Generalisation Theorem, the Deduction Theorem and the Principle of Contraposition.

Solution. The proof goes as follows:

1. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall x\,[x \neq a \to x = b]$ (Copy);
2. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall x\,[x \neq a \to x = b] \to (x \neq a \to x = b)$ (Axiom 2);
3. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash (x \neq a \to x = b)$ (Modus Ponens);
4. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall u\,\forall v\,\forall w\,[v = w \to u = v \to u = w]$ (Axiom 6,7);
5. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall u\,\forall v\,\forall w\,[v = w \to u = v \to u = w] \to \forall v\,\forall w\,[v = w \to x = v \to x = w]$ (Axiom 2);
6. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall v\,\forall w\,[v = w \to x = v \to x = w]$ (Modus Ponens);
7. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall v\,\forall w\,[v = w \to x = v \to x = w] \to \forall w\,[a = w \to x = a \to x = w]$ (Axiom 2);
8. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall w\,[a = w \to x = a \to x = w]$ (Modus Ponens);
9. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall w\,[a = w \to x = a \to x = w] \to a = b \to x = a \to x = b$ (Axiom 2);
10. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash a = b \to x = a \to x = b$ (Modus Ponens);
11. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash a = b$ (Copy);
12. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash x = a \to x = b$ (Modus Ponens);
13. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash (x = a \to x = b) \to (x \neq a \to x = b) \to x = b$ (Axiom 1);
14. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash (x \neq a \to x = b) \to x = b$ (Modus Ponens);
15. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash\to x = b$ (Modus Ponens);
16. $\{\forall x\,[x \neq a \to x = b], a = b\} \vdash \forall x\,[x = b]$ (Generalisation Theorem);
17. $\{\forall x\,[x \neq a \to x = b], \neg(\forall x\,[x = b])\} \vdash a \neq b$ (Principle of Contraposition).