NATIONAL UNIVERSITY OF SINGAPORE

MA 4207: Mathematical Logic

Semester 2; AY 2018/2019; Midterm Test

Time Allowed: 50 Minutes

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number. Do not write your name.
2. This assessment paper consists of FIVE (5) questions and comprises ELEVEN (11) printed pages.
3. Students are required to answer **ALL** questions.
4. Students should answer the questions in the space provided.
5. This is a **CLOSED BOOK** assessment.
6. It is permitted to use calculators, provided that all memory and programs are erased prior to the assessment; no other material or devices are permitted.
7. Every question is worth FIVE (5) or SIX (6) marks. The maximum possible marks are 28.

STUDENT NO: ________________________________

This portion is for examiner's use only

Question	Marks	Remarks
Question 1:		
Question 2:		
Question 3:		
Question 4:		
Question 5:		
Total:		

In this and further questions, let $\wedge$ denote "and", $\vee$ denote "inclusive or", $\neg$ denote "not", $\rightarrow$ denote "implies", $\leftrightarrow$ denote "logically equal" and $\oplus$ denote "exclusive or".

Provide a formula α which outputs 1 iff the values of the first five atoms A_1, A_2, A_3, A_4, A_5 are not all equal; this formula should use at most seven of the above connectives. Give a short reason why the formula works.

Solution. The formula is

$$(A_1 \oplus A_2) \vee (A_2 \oplus A_3) \vee (A_3 \oplus A_4) \vee (A_4 \oplus A_5).$$

If all A_i and A_j are evaluated to the same value then all the $\oplus$ give the value 0 and the or-connection of these also gives the value 0. Furthermore, if there is an $k \in \{1, 2, 3, 4\}$ such that A_{k+1} has a different truth-value than A_1 then there is a minimal such k and thus A_k and A_{k+1} have a different truth-value for this minimal such k. It follows that $A_k \oplus A_{k+1}$ evaluates to 1 and thus the full formula which or-connects the corresponding four terms evaluates to 1.

Consider the fuzzy logic with values $0, 1/2, 1$ and recall that for any truth-assignment ν using these three values, $\overline{\nu}(\alpha \oplus \beta) = \min\{\overline{\nu}(\alpha) + \overline{\nu}(\beta), 2 - \overline{\nu}(\alpha) - \overline{\nu}(\beta)\}$. Furthermore, ν satisfies a set S of formulas iff $\overline{\nu}(\alpha) = 1$ for all $\alpha \in S$.

Construct a set S of formulas which can use the truth-values as constants, can use the atoms and can use the connective $\oplus$ (exclusive or) such that S is satisfied by exactly those ν which take on all atoms the same truth-value.

Solution. S consists of all formulas of the form $(A_k \oplus (A_h \oplus 1))$. Note that this formula is, in fuzzy logic, equivalent to $A_k \leftrightarrow A_h$ as is verified by the following connection: Let $p = \nu(A_k)$ and $q = \nu(A_h)$. Then $\overline{\nu}(A_h \oplus 1) = 1 - q$ and $\overline{\nu}(A_k \oplus (A_h \oplus 1)) = \min\{p + 1 - q, 2 - p - (1 - q)\} = \min\{1 - (q - p), 1 + (q - p)\}$. The latter is 1 iff $p = q$: If $q - p < 0$ then the formula takes the value $1 + (q - p)$, if $q - p > 0$ then the formula takes the value $1 - (q - p)$. In both cases, the value is strictly below 1. Thus, $\overline{\nu}(A_k \oplus (A_h \oplus 1)) = 1$ iff $\nu(A_k) = \nu(A_h)$. So if ν takes on all atoms the same truth-value q then ν satisfies every formula $(A_k \oplus (A_h \oplus 1))$ in S else ν does not satisfy those formulas $(A_k \oplus (A_h \oplus 1))$ in S where $\nu(A_k) \neq \nu(A_h)$. Thus S satisfies the required properties.

For $\kappa = \{0, 1, 2, 3, \aleph_0, 2^{\aleph_0}\}$, is there an infinite set S_κ of formulas such that

(a) Every finite subset of S_κ is satisfied by $2^{\aleph_0}$ many ν and

(b) The full set S_κ is satisfied by exactly κ many ν?

List for each of the above κ either a set S_κ satisfying (a) and (b) or say why such a set S_κ does not exist. Here $\aleph_0$ is the number of elements of $\mathbb{N}$ and $2^{\aleph_0}$ is the number of elements of $\mathbb{R}$.

Solution. For $\kappa > 0$, there are many solutions, but only one sample solution is given in each case. The atoms to be used are the A_k with $k \geq 1$. Note that A_0 does not exist.

$\kappa = 0$: There is no S_0, as the Compactness Theorem says that S_0 does not exist: If each finite subset is made true by some ν, so is S_0 itself and so there is at least one ν satisfying S_0.

$\kappa = 1$: One can simply take $S_1 = \{A_k : k = 1, 2, \ldots\}$. For a satisfiable set S_κ of formulas, each finite subset is made true by a satisfying ν which can be arbitrarily changed on the infinitely many atoms which do not occur in the finite subset, thus for all satisfiable S_κ, the condition on the uncountably many ν satisfying a given finite subset is always true. Furthermore, S_1 itself is made true only by the ν with $\nu(A_k) = 1$ for all k.

$\kappa = 2$: One can take $S_2 = \{A_k : k \geq 2\}$. Now S_2 is satisfied true by two ν: That one which maps A_1 to 0 and all others to 1 and that one which maps all A_k to 1.

$\kappa = 3$: One can take $S_3 = \{A_1 \vee A_2\} \cup \{A_k : k \geq 3\}$. The three ν which make all atoms true take on all A_k with $k \geq 3$ the value 1 and on A_1, A_2 they take one of the combinations $(0, 1), (1, 1), (1, 0)$ as the truth-values. Other possibilities are not possible.

$\kappa = \aleph_0$: Here one takes $S_\kappa = \{A_{k+1} \to A_k : k \geq 1\}$ and observes that if some A_k takes the value 1 then also all A_h with $h \leq k$ have to take the value 1. Thus the $\aleph_0$ many truth-assignments which make S_κ true are those which either map all atoms to 0 or which map the atoms $A_1, A_2, \ldots, A_k$ to 1 and all others to 0 or which map all atoms to 1. So there are countably many of them.

$\kappa = 2^{\aleph_0}$: In this case, one must take a set S which does only prescribe values for half of the A_k, say $S_\kappa = \{A_k : k \text{ is odd}\}$. Consider all ν with $\nu(A_k) = 1$ for all odd k. Now $\nu \models S_\kappa$ and $\nu \models S'$ for all $S' \subseteq S_\kappa$. As the values $\nu(A_k)$ with k even can be chosen freely, there are $2^{\aleph_0}$ many such ν.

Question 4 [6 marks]

Let a logical language contain the base set X, a unary predicate P, a binary relation E between members of X and three constants a, b, c. Consider the following axioms (written informally and not as wffs for better readability):

1. $a \neq b \wedge a \neq c \wedge b \neq c$;
2. $\forall x\, [P(x) \leftrightarrow (x = a \vee x = b \vee x = c)]$;
3. $\forall x\, \forall y\, [E(x, y) \rightarrow (E(y, x) \wedge (P(x) \leftrightarrow \neg P(y)))]$;
4. $\forall x\, \exists y\, \exists z\, [E(x, y) \wedge E(x, z) \wedge y \neq z]$;
5. $\forall x\, \exists y\, \exists z\, \forall u\, [(P(x) \wedge E(x, u)) \rightarrow (u = y \vee u = z)]$.

How many graphs (up to isomorphism) satisfy the above axioms?

Draw all these graphs. The predicate P does not need to be indicated in the drawing, as $P(x)$ is the same as $x \in \{a, b, c\}$. However, the nodes a, b, c should be named accordingly.

Solution. Let a, b, c denote the nodes of these names and o denote other nodes. Then the following two graphs can be found:

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   a - o           a
  /     \         / \
 o       b       o-b-o
  \     /         \ /
   c - o           c
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Every further graph is isomorphic to these two. Note that edges are only between nodes which satisfy P and nodes which do not satisfy P.

In the case that every node has two neighbours, they must be in a circle where between each of the nodes a, b, c sits one node o which is unique to its neighbours. Given two models of this case, there is an isomorphism between them which maps a to a, b to b, c to c, the node between a, b to the node between a, b, the node between b, c to the node between b, c and the node between c, a to the node between c, a. Thus each two graphs in this case are isomorphic.

In the case that one of the nodes has three neighbours, this node does not satisfy P and is connected to all three nodes satisfying P. Then also there is one further node not satisfying P which is also connected to all members of $\{a, b, c\}$, as these are all connected to exactly two nodes not satisfying P. Again all the graphs of this form are isomorphic, as one maps a to a, b to b, c to c and the remaining nodes are mapped to each other in some one-one way. So again all the graphs in this case are isomorphic.

The graphs of the two cases are not isomorphic with each other, as one has five and one has six nodes. So the overall number of models is two.

Write in first-order logic with one function-symbol f formulas which say the following:

1. f is injective (that is, one-one);
2. There is exactly one x which is not in the range of f;
3. There is an x with $f(f(x)) = x$.

Which of the numbers $0, 1, 2, 3, 4, 5, \aleph_0, \aleph_1, 2^{\aleph_0}$ is the cardinal of the smallest model (X, f) which satisfies these axioms? Explain why the given choice is correct.

Here the cardinal of the model is the size of the base set X. The cardinal $\aleph_0$ is the number of elements of $\mathbb{N}$ and $2^{\aleph_0}$ is the number of elements of $\mathbb{R}$. $\aleph_1$ is the first cardinal larger than $\aleph_0$ and satisfies $\aleph_0 < \aleph_1 \leq 2^{\aleph_0}$.

Solution. The axioms are as follows:

1. $\forall x \, \forall y \, [f(x) = f(y) \rightarrow x = y]$;
2. $\exists x \, \forall y \, \exists z \, [f(y) \neq x \wedge (y \neq x \rightarrow f(z) = y)]$;
3. $\exists x \, [f(f(x)) = x]$.

Let x_0 be the element from the second axiom which is not in the range of f. Now let $x_{n+1} = f(x_n)$ for all natural numbers. Then $x_{n+1} \neq \{x_0, x_1, \ldots, x_n\}$, as x_0 is not in the range of f and $x_1, \ldots, x_n$ are already the image of $x_0, \ldots, x_{n-1}$ and therefore different from $f(x_n)$ by injectivity of f. Thus the set of all x_n is infinite and therefore the cardinal of the smallest model is at least $\aleph_0$.

Furthermore, it is indeed $\aleph_0$ and not larger, as one can satisfy all axioms with the model $(\mathbb{N}, f)$ defined by $f(0) = 1$, $f(1) = 0$ and $f(n+2) = n+3$ for all $n \in \mathbb{N}$. Note that now $x_n = n + 2$ in this model. The elements 0 and 1 are used to satisfy the last axiom and $f(f(0)) = 0$.

END OF PAPER