National University of Singapore
MA4207 Mathematical Logic

Semester II (2019–2020)

Time allowed: 2 hours 30 minutes

---

**INSTRUCTIONS TO CANDIDATES**

1. Get ready a signed copy of the Exam declaration form for this exam.
2. Use A4 paper and pen (blue or black ink) to write your answers.
3. Write your student number clearly on the top left of every page of the exam. Do not write your name.
4. Write on one side of the paper only. Write the question number and page number on the top right corner of each page.
5. This examination paper contains TWELVE (12) questions and comprises THIRTEEN (13) pages. Answer **ALL** questions.
6. The total mark for this paper is FORTY (40).
7. This is an OPEN BOOK examination.
8. You may use any calculator. However, you should lay out systematically the various steps in the calculation.
9. Join the Zoom conference and turn on the video setting at all time during the exam. Adjust the camera such that your face and upper body including your hands are captured on Zoom.
10. You may go for a short toilet break (not more than 5 minutes) during the exam.
11. At the end of the exam,
    - scan or take pictures of your work (make sure the images can be read clearly) together with the declaration form;
    - merge all your images into one pdf file (arrange them in the order: Declaration form, Question 1, Question 2, ..., Question 12);
    - name the file by student number (a.k.a. matric number) followed by underscore followed by course-code MA4207, for example, `A1283125Z_MA4207.pdf`.
    - upload your pdf into the LumiNUS folder "Exam Submission".
12. The folder "Exam Submission" will close at 15:50 hrs; no submission will be accepted afterwards, unless there is a valid reason.

---

**Question 1** [3 Marks]
Construct a formula using the atoms $A_1, A_2, A_3, A_4, A_5$ which outputs 0 in the case that at most one atom is 1 and which outputs 1 in the case that at least three atoms are 1; the formula should only use the connectives $\vee$ and $\wedge$ and in total at most ten connectives should be used. In order to save on brackets, it can be assumed that $\wedge$ binds more than $\vee$, so $A_6 \wedge A_7 \vee A_8$ is, when written as a wff, $((A_6 \wedge A_7) \vee A_8)$. Explain the formula.

**Solution.** Call $A_1, A_5$ neighbours and all $A_k, A_{k+1}$ neighbours. If one has three atoms out of the five, then two of them are neighbours. So the idea is to make the disjunction over each conjunction of a pair of neighbours. There are five pairs of neighbours which will be connected by conjunctions and then the resulting terms will be connected by four disjunctions, giving exactly nine connectives. Furthermore, no single atom being 1 can produce the output 1, as the single atom is conjuncted with another atom being 0. On the other hand, if at least three atoms are 1, then two of them are neighbours which are mapped to 1 by the conjunction and the subsequent disjunctions will preserve the value. When written down in logic, the formula looks as follows:

$$(A_1 \wedge A_2) \vee (A_2 \wedge A_3) \vee (A_3 \wedge A_4) \vee (A_4 \wedge A_5) \vee (A_5 \wedge A_1).$$

Now if one spaces out three atoms being 1, one has to use $A_1, A_3, A_5$ in order to avoid the first four conjunctive terms to be 1, but then the last one becomes 1. Thus if three or more atoms are 1, the result is 1; if at most one atom is 1, the result is 0.

**Question 2** [3 Marks]
Consider the formula
$$\alpha = (A_1 \wedge A_2 \wedge A_3) \oplus (A_4 \vee A_5 \vee A_6 \vee A_7)$$
and calculate how many tuples of the resulting function $B^8_\alpha$ are mapped to 0 and how many are mapped to 1. Explain the solution.

**Solution.** The correct solution is that 212 tuples are mapped to 1 and 44 tuples are mapped to 0.

A formula of the form $\beta \oplus \gamma$ is mapped to 1 if either $\beta$ is mapped to 1 and $\gamma$ mapped to 0 or $\beta$ is mapped to 0 and $\gamma$ is mapped to 1. So let $\beta$ be the conjunction of the first three atoms and $\gamma$ be the disjunction of the last four atoms. $\beta$ maps one tuple to 1 and seven tuples to 0; $\gamma$ maps fifteen tuples to 1 and one to 0. Note that both operate on different sets of variables, so when considering the first seven variables, one has that $\beta \wedge \neg\gamma$ maps one tuple to 1 and $\neg\beta \wedge \gamma$ maps $7 \cdot 15 = 105$ tuples to 1, giving 106 tuples in total. $128 - 106 = 22$ tuples are mapped to 0. However, there is an eighth variable taken into account for $B^8_\alpha$, this refers to the value of $A_8$ and has no effect on the value of $B^8_\alpha$; however, it doubles the number of tuples. Thus as a result, 212 tuples are mapped to 1 and 44 tuples are mapped to 0.

**Question 3** [3 Marks]
Prove either verbally or by a truth-table method that the following formula is a tautology:

$$((((A_1 \wedge A_2) \wedge A_3) \rightarrow A_4) \leftrightarrow (A_1 \rightarrow (A_2 \rightarrow (A_3 \rightarrow A_4)))).$$

**Solution.** Let $\alpha, \beta$ refer to the two halves of the formula before and after the $\leftrightarrow$.

If $A_4$ is 1 then $\alpha, \beta$ are both of the form $\gamma \rightarrow 1$ and such formulas are always true, so $\alpha \leftrightarrow \beta$ is true and the formula is satisfied.

Now assume that $A_4 = 0$. If at least one of $A_1, A_2, A_3$ is 0 then $\alpha$ is 1, as it is of the form $0 \rightarrow 0$. The formula $\beta$ is either of the form $\delta \rightarrow \gamma \rightarrow (0 \rightarrow 0)$ and true or of the form $\delta \rightarrow (0 \rightarrow (1 \rightarrow 0))$ what is the same as $\delta \rightarrow (0 \rightarrow 0)$ and true or of the form $0 \rightarrow (1 \rightarrow (1 \rightarrow 0))$ what is $0 \rightarrow (1 \rightarrow 0)$ what is $0 \rightarrow 0$ what is 1, so again also $\beta$ is true.

If $A_1, A_2, A_3$ are 1 and $A_4$ is 0, then $\alpha$ is of the form $1 \rightarrow 0$ and false, but also $\beta$ is of the form $1 \rightarrow (1 \rightarrow (1 \rightarrow 0))$ what simplifies to $1 \rightarrow (1 \rightarrow 0)$ what again simplies to $1 \rightarrow 0$ and 0, so both sides are 0 and $0 \leftrightarrow 0$ is true.

Thus in all cases, $\alpha \leftrightarrow \beta$ is true and thus the formula is a tautology.

**Question 4** [3 Marks]
Consider the following set $S$ of formulas in fuzzy logic with values $0, 1/2, 1$ and atoms $A_1, A_2, \ldots$:

$$S = \{A_k \leftrightarrow (A_{2k} \oplus A_{2k+1}), A_{2k+1} \rightarrow A_{2k}, A_{2k} \rightarrow A_k : k = 1, 2, 3, \ldots\} \cup \{A_1\}.$$

Here

$$\begin{aligned} \overline{\nu}(\alpha \rightarrow \beta) &= \min\{1, 1 + \overline{\nu}(\beta) - \overline{\nu}(\alpha)\}, \\ \overline{\nu}(\alpha \leftrightarrow \beta) &= \min\{1 + \overline{\nu}(\alpha) - \overline{\nu}(\beta), 1 + \overline{\nu}(\beta) - \overline{\nu}(\alpha)\}, \\ \overline{\nu}(\alpha \oplus \beta) &= \min\{\overline{\nu}(\alpha) + \overline{\nu}(\beta), 2 - \overline{\nu}(\beta) - \overline{\nu}(\alpha)\}. \end{aligned}$$

Which truth-assignments $\nu : \{A_1, A_2, \ldots\} \rightarrow \{0, 1/2, 1\}$ satisfy $S$, that is, satisfy $\overline{\nu}(\alpha) = 1$ for all $\alpha \in S$? Describe (perhaps in dependence of a parameter) all the $\nu$ which make all formulas in $S$ to have the value 1.

**Solution.** As $A_{2k} \rightarrow A_k$, $\nu(A_{2k}) \leq \nu(A_k)$. Furthermore $\nu(A_{2k+1}) \leq \nu(A_{2k})$. If $\nu(A_k) = 0$ then $\nu(A_{2k}) = 0$ and $\nu(A_{2k+1}) = 0$; if $\nu(A_k) = 1/2$ then $\nu(A_{2k}) = 1/2$ and $\nu(A_{2k+1}) = 0$; if $\nu(A_k) = 1$ then either $\nu(A_{2k}) = 1/2$ and $\nu(A_{2k+1}) = 1/2$ or $\nu(A_{2k}) = 1$ and $\nu(A_{2k+1}) = 0$. Thus one has the following solutions: $\nu_0(A_{2^i}) = 1$ and $\nu_0(A_h) = 0$ for all other $h$. If $k > 0$ then $\nu_k(A_{2^i}) = 1$ for all $i < k$, $\nu_k(A_{2^{k+j}}) = 1/2$ and $\nu_k(A_{2^{k+j}+2^j}) = 1/2$ for all $j$ and $\nu_k(A_h) = 0$ for all other $h$. So there are countably many $\nu$ which make all formulas in $S$ true.

**Question 5** [4 Marks]
Assume that $S$ is in first-order logic over the logical language with $f$, $+$ and $<$ consisting of the following formulas:

1. $\forall x \, \forall y \, [x < y \rightarrow f(x) < f(y)]$;
2. $\forall x \, \forall y \, [f(x+y) = f(x) + f(y)]$.

Let $\mathbb{Z}$ be the integers and $+, <$ have the usual meaning on the integers. Explain what each of the formulas says and characterise the functions $f : \mathbb{Z} \rightarrow \mathbb{Z}$ which satisfy the above axioms and determine how many of these $f$ exist (finitely, countably or uncountably many).

**Solution.** The first formula says that the function $f$ is order-preserving, that is, if $x$ is properly smaller than $y$ the same holds for $f(x)$ compared to $f(y)$. The second formula says that $f$ is a group homomorphism.

The function $f$ must map 0 to 0, as $f(0 + y) = f(0) + f(y)$ for all $y$. Furthermore, $f(1) > f(0) = 0$. So $f(1)$ is positive. Now one can show by induction that $f(x) = x \cdot f(1)$. Thus the function $f$ has to be the multiplication of the input with a positive integer constant.

It is also easy to see that all such functions satisfy both formulas: So let $f(x) = c \cdot x$ where $c \in \{1, 2, 3, \ldots\}$. If $x < y$ then $c \cdot x < c \cdot y$, so the first formula is satisfied. Furthermore, $f(x + y) = c \cdot (x + y) = c \cdot x + c \cdot y = f(x) + f(y)$, so the second formula is satisfied.

In particular, as there are countably many positive integers $c$, there are countably many such functions.

**Question 6** [3 Marks]
Let a logical language contain the equality $=$ and one unary function symbols $f$ and two constants $a, b$. Furthermore, let $f^1(x)$ abbreviate $f(x)$, $f^2(x) = f(f(x))$ and $f^{n+1}(x) = f(f^n(x))$. So if one says that $f^3(x) \neq x$ is in $S$ then this means that the formula $f(f(f(x))) \neq x$ is in $S$. Now consider the set $S$ containing the following formulas:

1. $\forall x\, \forall y\, [f(x) = f(y) \rightarrow x = y]$;
2. $\forall x\, \exists y\, [x = f(y)]$;
3. for all $n \geq 1$ the formula $\forall x\, [x \neq f^n(x)]$;
4. for all $n \geq 1$ the formulas $f^n(a) \neq b$ and $f^n(b) \neq a$;
5. $a \neq b$.

Provide a countable model for this structure where every element is definable (the constants $a, b$ can be used).

How many countable models do exist for this structure (up to isomorphism)? In particular, is the structure $\aleph_0$-categorical? Explain the answer.

Is the structure $\aleph_1$-categorical? Explain the answer.

**Solution.** An explicit model is by taking the integers, letting $a = 0$, $b = 1$ and $f(x) = x + 2$. 0 and 1 are equal to the constants $a, b$ and therefore definable in the model. For every $n > 0$, the number $x = -2n$ can be defined by the formula $f^n(x) = a$, the number $x = -2n + 1$ can be defined by the formula $f^n(x) = b$, the number $x = +2n$ can be defined by the formula $x = f^n(a)$ and the number $x = +2n + 1$ can be defined by the formula $x = f^n(b)$.

There are countably many countable models. The above model consists of two $\mathbb{Z}$-chains, one given by the even and one given by the odd numbers. For every $m > 2$, one can with $a, b$ being $0, 1$, respectively, create a model with $m$ $\mathbb{Z}$-chains by having that $f(x) = x + m$. In these, only the members of the chains through $a$ and $b$ are definable, the others not. Furthermore, there is a model with countably infinitely many $\mathbb{Z}$-chains. These are all models up to isormophism. So there are $\aleph_0$ many countable models up to isomorphism.

The structure is not $\aleph_0$-categorical, as there are various countable models. However, it is $\aleph_1$-categorical, as the only model, up to isomorphism, is the collection of $\aleph_1$ many $\mathbb{Z}$-chains out of which one contains the constant $a$ and another one contains the constant $b$.

**Question 7** [4 Marks]
Given $k$, construct an Abelian semigroup having $k+1$ definable elements and 3 nondefinable elements.

**Solution.** Let $A = \{0, 1, \ldots, k+3\}$ and for $x, y \in A$, let $x \circ y = \max\{x, y, 3\}$. The associativity and commutativity of the operation $\circ$ follows from that of the maximum-operation. So $x \circ y \circ z = \max\{3, x, y, z\}$ independently of the order of $x, y, z$ and one can see that bracketing does not have any efect. $0, 1, 2$ are not in the range of $\circ$ and for every $x \in \{0, 1, 2\}$ and every $y$, $x \circ y = y \circ x = \max\{3, y\}$. So there is no property of $\circ$ which allows to distinguish these three elements. However, the $k+1$ numbers $3, 4, \ldots, k, k+1, k+2, k+3$ are definable, as they are ordered by $x \leq y \leftrightarrow x \circ y = y$ and furthermore they are those numbers which are in the range of $\circ$. More precisely, if $y = 3 + z$, $y$ is that element for which there are exactly $z$ different elements below $y$ in the range of $\circ$ with respect to the ordering defined above. This can be formalised in a first-order formula. A more concrete way to define the elements inductively is as follows. One defines 3 as the unique element $x$ satisfying a formula $\psi(x)$ given as

$$x \circ x = x \wedge \forall y\, [x \circ (y \circ y) = y \circ y].$$

If now an element $x$ is defined using some formula $\psi(x)$ and thus in the range of $\circ$, then the next element $x+1$ is the unique element $z$ defined by

$$\exists x\, \forall y\, [\psi(x) \wedge x \neq z \wedge x \circ z = z \wedge (x \circ (y \circ y) = y \circ y \to x \neq y \circ y \to z \circ (y \circ y) = y \circ y)].$$

This allows to define all elements from 3 onwards by induction.

**Question 8** [3 Marks]
Let a logical language contain the constants $c_1, c_2, \ldots$ plus equality. Furthermore, consider the following set $S$ of formulas: $S = \{c_i = c_j \rightarrow c_i = c_k : i < j < k\}$. How many models with 8 elements does $S$ have? Explain the answer.

**Solution.** First one has to find the least $j$ so that there is an $i < j$ with $c_i = c_j$. Now $c_i = c_k$ for all $k > j$. Thus if there are two such pairs $(i, j)$, $(i', j')$ with $c_i = c_j$ and $i < j$ as well as $c_{i'} = c_{j'}$ and $i' < j'$, then for all $k > i + j + i' + j'$, $c_i = c_k$ and $c_{i'} = c_k$. Thus $c_i = c_{i'}$ and all constants which are equal to some other constant are equal with each other. Furthermore, it is possible that the constants do not cover all the 8 elements, say that only five of them are equal to constants. So finite models are given by three parameters: (a) the least number $j$ with an $i < j$ such that $c_i = c_j$; (b) the unique $i \in \{1, 2, \ldots, j-1\}$ with $c_i = c_j$; (c) the number of elements in the model which is at least $j - 1$. The latter is equal to 8 by the choice of the question. Now one can choose any $j \in \{2, 3, 4, 5, 6, 7, 8, 9\}$ and furthermore choose the unique $i \in \{1, 2, \ldots, j-1\}$ with $c_i = c_j$; there are 8 choices for $j$ and subsequently $j - 1$ such choices for $i$. So one has $1 + 2 + 3 + \ldots + 8 = 36$ models.

**Question 9** [3 Marks]
Recall that the axiom set $\Lambda$ of the proof calculus consists of six axiom groups plus the rule that all axioms can be quantified. The following formulas are intended to be, perhaps quantified, examples of $\Lambda$'s axioms groups 1–6, respectively. However, three are faulty. Identify these, write what went wrong and provide a corrected example.

Group 1. $\forall x \, \forall y \, [x = z \wedge y = z] \to \forall x \, \forall y \, [x \neq z \wedge y \neq z] \to \forall x \, \forall y \, [x = z \wedge y = z]$

Group 2. $\forall x \, \forall y \, [x + y = z] \to \forall y \, [(y + y) + y = z]$

Group 3. $\forall x \, [P(x) \to Q(x)] \to \forall x \, [P(x)] \to \forall y \, [Q(y)]$

Group 4. $y = 37 \to \forall x \, [y = 37]$

Group 5. $\forall x \, \exists y \, [x = y]$

Group 6. $\forall x \, \forall y \, [x = y \to (R(x, y) \to R(y, y))]$

**Solution.** Out of the six examples, those at 2, 3 and 5 are faulty.

1. $\forall x\, \forall y\, [x = z \wedge y = z] \rightarrow \forall x\, \forall y\, [x \neq z \wedge y \neq z] \rightarrow \forall x\, \forall y\, [x = z \wedge y = z]$:
   This formula is a correct example. One can see it as a formula of the form $A_1 \rightarrow A_2 \rightarrow A_1$ which is a tautology. Now one replaces $A_1$ by $\forall x\, \forall y\, [x = z \wedge y = z]$ and $A_2$ by $\forall x\, \forall y\, [x \neq z \wedge y \neq z]$ in order to get the given formula.

2. $\forall x\, \forall y\, [x + y = z] \rightarrow \forall y\, [(y + y) + y = z]$:
   This formula has the mistake that the quantified variable $x$ from the precondition is replaced by $(y + y)$ what is a term within the range of the quantifier $\forall y$, this is not permitted. A formula which is an Example of Axiom 2 is replacing $x$ by some term only using variables different from $y$: $\forall x\, \forall y\, [x + y = z] \rightarrow \forall y\, [(z + z) + y = z]$.

3. $\forall x\, [P(x) \rightarrow Q(x)] \rightarrow \forall x\, [P(x)] \rightarrow \forall y\, [Q(y)]$: The principle form of this axiom is indeed Axiom 3, but the last part is quantifying over the wrong variable, namely over $y$ instead of $x$. Though this formula is valid, it should be as follows: $\forall x\, [P(x) \rightarrow Q(x)] \rightarrow \forall x\, [P(x)] \rightarrow \forall x\, [Q(x)]$.

4. $y = 37 \rightarrow \forall x\, [y = 37]$:
   This formula of Axiom 4 is correct, note that Axiom 4 requires that the variable quantified (here $x$) does not occur free in the formula. So it more or less says that one can add a quantifier over an irrelevant variable without losing correctness.

5. $\forall x\, \exists y\, [x = y]$:
   The formulas by Axiom 5 are $x = x$ and quantified versions thereof. Furthermore, all quantifiers must be universal. So $\forall x\, \forall y\, [x = x]$ and also $\forall x\, \forall y\, [y = y]$ would be correct.

6. $\forall x\, \forall y\, [x = y \rightarrow (R(x, y) \rightarrow R(y, y))]$:
   This formula is correct. Axiom 6 says that if $x = y$ then one can obtain the third term of the chain implication from the second term by interchanging some of the equal variables; here the first term must be an equality between variables and the second an atomic formula, so a predicate or an equation of two terms.

**Question 10** [3 Marks]
Assume that the logical language contains a unary predicate $P$, a constant $c$ and equality $=$. Consider the following statement:

$$\exists x\,[\neg(P(x) \rightarrow P(c))] \rightarrow \exists x\,[x \neq c].$$

Rewrite the statement and prove it using the axioms of $\Lambda$ and Modus Ponens only.

**Solution.** First the statement is rewritten as

$$\neg\forall x\,[(P(x) \rightarrow P(c))] \rightarrow \neg\forall x\,[x = c].$$

Now one derives it as follows.

1. $\emptyset \vdash \forall y\,\forall x\,[x = y \rightarrow P(x) \rightarrow P(y)]$ (Axiom 6);
2. $\emptyset \vdash \forall y\,\forall x\,[x = y \rightarrow P(x) \rightarrow P(y)] \rightarrow \forall x\,[x = c \rightarrow P(x) \rightarrow P(c)]$ (Axiom 2);
3. $\emptyset \vdash \forall x\,[x = c \rightarrow P(x) \rightarrow P(c)]$ (Modus Ponens);
4. $\emptyset \vdash \forall x\,[x = c \rightarrow P(x) \rightarrow P(c)] \rightarrow \forall x\,[x = c] \rightarrow \forall x\,[P(x) \rightarrow P(c)]$ (Axiom 3);
5. $\emptyset \vdash \forall x\,[x = c] \rightarrow \forall x\,[P(x) \rightarrow P(c)]$ (Modus Ponens);
6. $\emptyset \vdash (\forall x\,[x = c] \rightarrow \forall x\,[P(x) \rightarrow P(c)]) \rightarrow (\neg\forall x\,[P(x) \rightarrow P(c)] \rightarrow \neg\forall x\,[x = c])$ (Axiom 1 version of Contraposition);
7. $\emptyset \vdash \neg\forall x\,[P(x) \rightarrow P(c)] \rightarrow \neg\forall x\,[x = c]$ (Modus Ponens);
8. $\emptyset \vdash \exists x\,[\neg(P(x) \rightarrow P(c))] \rightarrow \exists x\,[x \neq c]$ (Rewriting by rules of Existential Quantifier).

**Question 11** [4 Marks]
Let $f$ be a unary function symbol and $a, b$ two constants. Now make a proof only using axioms from $\Lambda$, copying from $S = \{\forall x\,[f(x) = a \rightarrow f(x) \neq b]\}$ and Modus Ponens for the following statement:

$$\{\forall x\,[f(x) = a \rightarrow f(x) \neq b]\} \vdash \forall x\,[f(x) = b] \rightarrow \forall x\,[f(x) \neq a].$$

**Solution.** The proof is as follows, where $\alpha$ is the formula $\forall x\,[f(x) = a \rightarrow f(x) \neq b]$.

1. $\{\alpha\} \vdash \forall x\,[(f(x) = a \rightarrow f(x) \neq b) \rightarrow (f(x) = b \rightarrow f(x) \neq a)]$ (Axioms 1,7, see below);
2. $\{\alpha\} \vdash \forall x\,[(f(x) = a \rightarrow f(x) \neq b) \rightarrow (f(x) = b \rightarrow f(x) \neq a)] \rightarrow \forall x\,[f(x) = a \rightarrow f(x) \neq b] \rightarrow \forall x\,[f(x) = b \rightarrow f(x) \neq a]$ (Axioms 3);
3. $\{\alpha\} \vdash \forall x\,[f(x) = a \rightarrow f(x) \neq b] \rightarrow \forall x\,[f(x) = b \rightarrow f(x) \neq a]$ (Modus Ponens);
4. $\{\alpha\} \vdash \forall x\,[f(x) = a \rightarrow f(x) \neq b]$ (Copy);
5. $\{\alpha\} \vdash \forall x\,[f(x) = b \rightarrow f(x) \neq a]$ (Modus Ponens);
6. $\{\alpha\} \vdash \forall x\,[f(x) = b \rightarrow f(x) \neq a] \rightarrow \forall x\,[f(x) = b] \rightarrow \forall x\,[f(x) \neq a]$ (Axiom 3);
7. $\{\alpha\} \vdash \forall x\,[f(x) = b] \rightarrow \forall x\,[f(x) \neq a]$ (Modus Ponens).

Here the tautology used in Axiom 1 in the first formula is $(A_1 \rightarrow \neg A_2) \rightarrow (A_2 \rightarrow \neg A_1)$ and one replaces $A_1$ by $f(x) = a$ and $A_2$ by $f(x) = b$. The so achieved formula is quantified by $\forall x$, what is allowed by Axiom 7.

**Question 12** [4 Marks]
Let the logical language contain the equality $=$ and a unary function $f$. Let

$$S = \{\forall x\,[f(f(x)) = x], \forall x\,[f(f(f(x))) = x]\}.$$

Give a formal proof for

$$S \vdash \forall x\,[x = f(x)]$$

where this formal proof can use the Axioms of $\Lambda$, Modus Ponens, statements in $S$, the Generalisation Theorem and the Deduction Theorem.

**Solution.** The proof goes as follows:

1. $S \vdash \forall x\,[f(f(x)) = x] \to f(f(f(x))) = f(x)$ (Axiom 2);
2. $S \vdash \forall x\,[f(f(x)) = x]$ (Copy);
3. $S \vdash f(f(f(x))) = f(x)$ (Modus Ponens);
4. $S \vdash \forall y\,[y = x \to y = f(x) \to x = f(x)]$ (Axiom 6);
5. $S \vdash \forall y\,[y = x \to y = f(x) \to x = f(x)] \to (f(f(f(x))) = x \to f(f(f(x))) = f(x) \to x = f(x))$ (Axiom 2);
6. $S \vdash f(f(f(x))) = x \to f(f(f(x))) = f(x) \to x = f(x)$ (Modus Ponens);
7. $S \vdash f(f(f(x))) = f(x) \to x = f(x)$ (Modus Ponens);
8. $S \vdash \forall x\,[f(f(f(x))) = x]$ (Copy);
9. $S \vdash \forall x\,[f(f(f(x))) = x] \to f(f(f(x))) = x$ (Axiom 2);
10. $S \vdash f(f(f(x))) = x$ (Modus Ponens);
11. $S \vdash f(f(f(x))) = f(x) \to x = f(x)$ (Modus Ponens);
12. $S \vdash x = f(x)$ (Modus Ponens);
13. $S \vdash \forall x\,[x = f(x)]$ (Generalisation Theorem).