NATIONAL UNIVERSITY OF SINGAPORE

MA 4207: Mathematical Logic
Semester 2; AY 2020/2021; Midterm Test

Time Allowed: 60 Minutes (plus 15 Minutes Uploading)

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number on every page of the exam. Do not write your name. Use for every question an own page with student number and question number on the top.
2. This assessment paper consists of FIVE (5) questions and comprises SIX (6) printed pages.
3. Students are required to answer **ALL** questions.
4. Students should answer the questions in the space provided.
5. This is an **OPEN BOOK** assessment. You can use the lecture notes and the slides for the exam by viewing them on the computer; the screen of the computer must be recorded.
6. After the Exam ends (at 15:30 hrs), please first scan and upload the question paper and then upload your screen recording.
7. The solutions should be uploaded as one pdf-file with a file name of the form Studentnumber-midterm.pdf, for example "A01234567X-midterm.pdf".
8. Every question is worth SIX (6) marks. The maximum possible marks are 30.

Let $F = \{f : \mathbb{N} \to \mathbb{N}$ such that $f(0) = 1$, $f(2) = 1$ and for all $n, m \in \mathbb{N}$, $f(n) + 2f(m+1) + f(n+2) = f(m) + 2f(n+1) + f(m+2)\}$. How many functions are in F:

- ☐ None; ☐ One; ☐ Two;
- ☐ $\aleph_0$ (that is, countably many);
- ☐ $2^{\aleph_0}$ (that is, as many as real numbers);
- ☐ Any other number of functions.

Explain the answer (the explanations carry the majority of the marks). Here $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ is the set of natural numbers.

Solution. The right choice is two functions. Namely $f(n) = (n-1)^2$ and $f(n) = 1$. After choosing $f(1)$ and $m = 0$, the recurrence rekation $f(n+2) = f(0)+f(2)-2f(1)+2f(n+1)-f(n)$ determines all further values. Furthermore, the equation shows that the value $f(n+2)-2f(n+1)+f(n)$ is a constant. Then $f(n+1)-f(n)$ is a polynomial of degree 1, as it is the sum of n constants $f(2)-2f(1)+f(0)$ plus $f(1)-f(0)$. Similarly $f(n)$ is obtained by adding up n terms of the polynomial $f(n+1)-f(n)$ of degree 1 plus the constant $f(0)$, thus the function f is a polynomial of degree 2 (or less). As the function f has the same values at $f(0)$ and $f(2)$, it is either the constant 0 or has at 1 a global minimum or has at 1 a global maximum. However, it cannot have a global maximum at 1, as then $f(1) \geq 2$ and $f(n) = f(1) - (f(1)-1)(n-1)^2$ which would take negative values for $n \geq 3$ what is not permitted. $f(1)$ can also not be a negative value, as $f(1)$ is a natural number. Thus $f(1)$ can be either 0 or 1. These two values give the solutions listed above.

In this and further questions, let $\wedge$ denote "and", $\vee$ denote "inclusive or", $\neg$ denote "not", $\rightarrow$ denote "implies", $\leftrightarrow$ denote "logically equal" and $\oplus$ denote "exclusive or". Here 0 and 1 are the truth constants.

How many functions of the form B^2_α exist when α uses the atoms A_1, A_2 and the connectives $\leftrightarrow$ and $\vee$? Compute the number and explain the answer.

Solution. The answer is $2^3 = 8$ functions. The reason is that $\leftrightarrow$ and $\vee$ first allow to produce the constant 1 function by the formula $A_1 \leftrightarrow A_1$ and the function $B^2_{A_1 \wedge A_2}$ by the formula α being $A_1 \leftrightarrow (A_2 \leftrightarrow (A_1 \vee A_2))$. The formula $\alpha \leftrightarrow (A_1 \wedge A_2)$ produces the negation of α if at least one of the inputs is 0. This allows to make functions which map exactly one of the input tuples $(0,0),(1,0),(0,1)$ to 1: $B_{(A_1 \vee A_2) \leftrightarrow (A_1 \wedge A_2)}$, $B_{A_1 \leftrightarrow (A_1 \wedge A_2)}$ and $B_{A_2 \leftrightarrow (A_1 \wedge A_2)}$. Now one can connect any collection of these three functions with "$\vee$" in order to build a function which takes on $(0,0),(1,0),(0,1)$ three prescribed values a, b, c. However, $(1,1)$ is always mapped to 1, as both allowed connectives $\leftrightarrow$, $\vee$ as well as the derived $\wedge$ output 1 when all inputs are 1. Thus one can realise all functions which take on $(0,0),(1,0),(0,1),(1,1)$ the values $a, b, c, 1$ respectively and as a, b, c can be freely chosen from $\{0,1\}$, there are $2^3 = 8$ functions.

Let $\leftrightarrow, \oplus, \wedge, \vee$ be logical connectives and $0, 1/3, 2/3, 1$ be the truth-constants. Recall the following rules to evaluate connectives in $\{0, 1/3, 2/3, 1\}$-valued fuzzy logic:

$$r \wedge s = \min\{r, s\},\ r \vee s = \max\{r, s\},\ r \rightarrow s = \min\{1, 1 + s - r\},$$
$$r \oplus s = \min\{r + s, 2 - r - s\},\ r \leftrightarrow s = \min\{1 + r - s, 1 + s - r\}.$$

For which of the following functions is there a formula using the atoms A_1, A_2 which uses only $\neg, \oplus, \wedge$ but no truth-constants and not $\leftrightarrow$ to express the following functions in Fuzzy Logic?

1. $B^2_{A_1 \vee A_2}$, the "or" of two inputs;
2. $B^2_{2/3}$, the constant 2/3 function;
3. $B^2_{A_1 \leftrightarrow A_2}$, the function evaluating the equivalence.

Put brackets where needed in order to avoid misunderstandings. Explain the answers and either provide the formulas whenever they exist or write why they do not exist.

Solution.

1. The formula $\alpha = \neg(\neg A_1 \wedge \neg A_2)$ is already okay, as De Morgan's Law also holds with min, max in place of $\wedge$, $\vee$ in Fuzzy Logic. So $B^2_\alpha = B^2_{A_1 \vee A_2}$.
2. For the constant 2/3 function there does not exist a formula without using truth-constants, as fuzzy logic returns always either 0 or 1 if a formula is fed with classic truth-values 0 or 1 on all atoms.
3. The formula $\beta = (A_1 \oplus \neg A_2)$ returns for truth-values c, d of the atoms A_1, A_2, respectively, the value $\min\{c + 1 - d, d + 1 - c\}$ which is the same value as $A_1 \leftrightarrow A_2$.

Construct a set S of sentential formulas satisfying Fuzzy Logic such that the number of truth-assignments ν satisfying S is countable in the case that $Q = \{0, 1\}$ and uncountable in the case that $Q = \{0, 1/3, 2/3, 1\}$. Truth-constants are not allowed, but the connectives $\vee$, $\wedge$, $\oplus$, $\leftrightarrow$, $\rightarrow$, $\neg$ are all allowed and defined as in the previous question. The atoms allowed are all atomes A_k with $k = 1, 2, \ldots$; so there are countably many atoms.

Solution. Let $S = \{A_{k+2} \rightarrow A_k, A_{k+1} \rightarrow A_k, A_k = A_{k+2} \oplus A_{k+1} : k \geq 2, k \text{ even}\}$.

In the classic case of $\{0, 1\}$, for even k, if $\nu(A_k) = 0$ then $\nu(A_{k+1}) = 0$ and $\nu(A_{k+2} = 0$; if $\nu(A_k) = 1$, then exactly one of $\nu(A_{k+1}), \nu(A_{k+2})$ is 1 and the other one is 0. There are only finitely many atoms 1 iff either $A_2 = 0$ or there is an odd $k + 1 \geq 2$ with $A_{k+1} = 1$. $\nu(A_1)$ can be freely chosen. So one can see that either $\nu(A_k) = 1$ for all even k and $\nu(A_k) = 0$ for all odd $k \geq 3$ or there are only finitely many atoms where ν is not 0 and thus there are only finitely many ν which satisfy all formulas in S.

In fuzzy logic with $Q = \{0, 1/3, 2/3, 1\}$, $2/3 \oplus 0 = 2/3$ and $2/3 \oplus 2/3 = 2/3$. Thus one can choose $\nu(A_{2k}) = 2/3$ and then freely choose $\nu(A_{2k+1}) \in \{0, 2/3\}$ without violating the above constraints in the set S. It follows that in this way one can choose uncountably many ν which make S true.

Assume that (X, f) is a model of a base set X with a unary function f. Furthermore, one writes $f^0(x)$ for the identity x and $f^{k+1}(x)$ for $f(f^k(x))$ where $k \in \mathbb{N}$ is a constant; this is just a convention to make the formulas more readable and is not part of the official logical language, in particular one cannot quantify over this k. Now consider the following axiom:

$$\forall x\,[f^{30}(x) = x].$$

In other words, X is partitioned into cycles whose length divides 30 and where the elements in the cycle are moved along the cycle by f. How many models of size 4 and of size 6 does this axiom have?

Solution. Note that f is one-one and surjective by the fact that each element, after 30 times applying f, is mapped to itself. This also explains the statement in the task that X is partitioned into cycles.

In the following let (a, b, c, d, e) say that the domain consits of a 1-cycles, b 2-cycles, c 3-cycles, d 5-cycles and e 6-cycles. As 4 is not a factor of 30, there cannot be any 4-cycles. Furthermore if two models have the same parameters (a, b, c, d, e), one can bijectively map cycles of length k to cycles of length k and then extends this to a model-isomorphism. Thus equal parameters describe the same model. If they have different parameters, they cannot be isomorphic. Thus the number of models of given size up to 9 can be described by listing all tuples (a, b, c, d, e) such that the weighted sum $a + 2b + 3c + 5d + 6e$ gives the prescribed size. From size 10 onwards there can also be 10-cycles, from size 15 onwards there can be 15-cycles and from size 30 onwards there can be 30-cycles.

For the case $|X| = 4$, $d = 0, e = 0$ for cardinality reasons. Thus only the following is possible: $(4, 0, 0, 0, 0)$, $(2, 1, 0, 0, 0)$, $(0, 2, 0, 0, 0)$, $(1, 0, 1, 0, 0)$; note that all models satisfy $a + 2b + 3c + 5d + 6e = 4$ and if this sum overshoots or undershoots, the parameters cannot define a model. In particular, for $|X| = 4$, there are four models.

If $|X| = 6$ the adjusted formula is $a+2b+3c+5d+6e = 6$. This is satisfied by the following models: $(6, 0, 0, 0, 0)$, $(4, 1, 0, 0, 0)$, $(2, 2, 0, 0, 0)$, $(0, 3, 0, 0, 0)$, $(3, 0, 1, 0, 0)$, $(0, 0, 2, 0, 0)$, $(1, 1, 1, 0, 0)$, $(1, 0, 0, 1, 0)$ and $(0, 0, 0, 0, 1)$. This gives 9 models (X, f) with $|X| = 6$.

END OF PAPER