NATIONAL UNIVERSITY OF SINGAPORE

MA 4207: Mathematical Logic

Semester 2; AY 2022/2023; Midterm Test

Time Allowed: 60 Minutes

INSTRUCTIONS TO CANDIDATES

1. Please write your Student Number on every page of the exam. Do not write your name. Use for every question an own page with student number and question number on the top.
2. This assessment paper consists of FIVE (5) questions and comprises ELEVEN (11) printed pages.
3. Students are required to answer **ALL** questions.
4. Students should answer the questions in the space provided.
5. This is an **CLOSED BOOK** assessment with **HELPSHEET**.
6. The questions are worth FIVE (5) or SIX (6) marks, as indicated. The maximum possible marks are 28.

STUDENT NO: ______________________________

This portion is for examiner's use only

Question	Marks	Remarks
Question 1 [5 Marks]:		
Question 2 [5 Marks]:		
Question 3 [6 Marks]:		
Question 4 [6 Marks]:		
Question 5 [6 Marks]:		
Total:		

Question 1 [5 marks] **MA 4207 – Solutions**

Throughout this exam, let $\wedge$ denote "and", $\vee$ denote "inclusive or", $\neg$ denote "not", $\rightarrow$ denote "implies", $\leftrightarrow$ denote "logically equal" and $\oplus$ denote "exclusive or". Let 0 and 1 are the truth constants. Let S be a set of formulas such that for all $\alpha, \beta \in S$, also the formula $\alpha \wedge \beta$ is in S. Furthermore, assume that there is a formula γ such that γ is not a tautology and $S \models \gamma$. Here $S \models \gamma$ means that every truth-assignement σ which makes all formulas in S true also makes the formula γ true. What is the smallest size of a subset $T \subseteq S$ with $T \models \gamma$?

☐ 0, ☐ 1, ☐ 2, ☐ 3,
☐ Finite but one cannot find out how large,
☐ Infinite and Countable, ☐ Uncountable.

Give reasons for the answer.

Solution. The correct size of T is 1. By the Compactness Theorem, there is a finite subset U of S such that $U \models \gamma$. So denote the elements of U with $\alpha_1, \alpha_2, \ldots, \alpha_n$ where n is some natural number. By the property that the conjunction of two members of S is in S, it follows that also each formula $\beta_m = \alpha_1 \wedge \alpha_2 \wedge \ldots \wedge \alpha_m$ is in S. Now β_n is the conjunction of all members of U and is in S, thus $U \models \beta_n$ and $\{\beta_n\} \models U$. As $\models$ is transitive, it follows that $\{\beta_n\} \models \gamma$ and that $T = \{\beta_n\}$ is a one-element subset of S with $T \models \gamma$. As γ is not a tautology, $\emptyset \not\models \gamma$. Thus 1 is the smallest size of a subset of S which tautologically implies γ.

Let R be the set of atoms $\{A_1, A_2, \ldots\}$ and S be the set of well-formed formulas over R. Introduce the following equivalence relation on S: α is equivalent to β iff $\alpha \leftrightarrow \beta$ is a tautology, that is, iff every truth-assignment makes $\alpha \leftrightarrow \beta$ true. Determine the number of equivalence classes of S in dependence on the number of atoms which is either a natural number $n \geq 1$ or $\aleph_0$.

Solution. In the case of n atoms with $n \in \mathbb{N}$, there are 2^n possible truth-assignments of the n atoms. Now one assigns to every truth-assignment σ a value $f(\sigma)$ where f is a function from all possible truth-assignments to $\{0, 1\}$. For this one let α be the disjunction of all conjunctions $A_m \leftrightarrow \sigma(A_m)$ for $m = 1, 2, \ldots, n$ where $f(\sigma) = 1$; in other words, α codes the truth-table entries which are mapped to 1 and takes a disjunction of these. As there are 2^{2^n} many mappings from truth-assignments to $\{0, 1\}$, there are at least 2^{2^n} many equivalence classes of formulas α. Furthermore, every formula β defines a mapping f from each truth-assignments σ to the truth-value $\overline{\sigma}(\beta)$, thus this β is equivalent to one of the before constructed α. Thus there are 2^{2^n} equivalence classes of formulas using n atoms.

In the case of $\aleph_0$ atoms, two different atoms A_i and A_j are not equivalent, as there is a σ which makes A_i true and A_j false; this σ then makes also $A_i \leftrightarrow A_j$ false. So there are at least $\aleph_0$ equivalence classes. Note that all these equivalence classes are generated by formulas in S. On the other hand, as there are only $\aleph_0$ well-formed formulas, the overall number of equivalence classes can also not be larger than $\aleph_0$; thus $\aleph_0$ is the correct cardinal of the number of equivalence classes of formulas in S.

The binding order below is that $\neg$ binds more than everything else and $\wedge$ binds more than $\vee$ and these two bind more than $\oplus$ and these three bind more than $\rightarrow$, $\leftrightarrow$. Remove from the following well-formed formulas as many brackets as possible without changing the syntax of the formula and list out the resulting formulas. Furthermore, say for each of the two formulas whether it is (a) an antitautology, (b) satisfiable but not a tautology or (c) a tautology; give reasons for the answers.

1. $((A \rightarrow B) \wedge (B \rightarrow A))$;
2. $((A \oplus (B \oplus C)) \leftrightarrow (C \oplus (\neg(B \oplus (\neg A)))))$.

Solution. The minimum number in an informal writing of the formula of brackets are obtained as follows:

1. $(A \rightarrow B) \wedge (B \rightarrow A)$;
2. $A \oplus B \oplus C \leftrightarrow C \oplus \neg(B \oplus \neg A)$.

As $\wedge$ binds strong than $\rightarrow$, the brackets cannot be removed in the first formula. As $\neg$ binds stronger than $\oplus$ the remaining pair of brackets cannot be removed in the second formula.

The first formula is satisfiable by making both atoms 1; however, if one atom takes the truth-value 1 and the other one the truth-value 0, the formula is not satisfied, hence it is not a tautology.

For the second formula, note that negation is the same as taking the exclusive or with 1 and that exclusive or is commutative and associative. Furthermore, $\alpha \oplus \beta \oplus \beta$ is equivalent to α for all formulas α and β. Thus one can modify the formulas as follows: The formula

$$A \oplus B \oplus C \leftrightarrow C \oplus \neg(B \oplus \neg A)$$

is equivalent to

$$A \oplus B \oplus C \leftrightarrow C \oplus 1 \oplus B \oplus A \oplus 1$$

and this is equivalent to

$$A \oplus B \oplus C \leftrightarrow A \oplus B \oplus C \oplus 1 \oplus 1$$

and the so obtained result is equivalent to

$$A \oplus B \oplus C \leftrightarrow A \oplus B \oplus C$$

and the last one is a tautology, that is, always true. Thus the second formula is a tautology which evalutates to 1 on all truth-assignments.

Let $\leftrightarrow, \oplus, \wedge, \vee$ be logical connectives and $Q = \{0, 1/2, 1\}$ be the set of truth-constants. Recall the following rules to evaluate connectives in Q-valued fuzzy logic:

$$r \wedge s = \min\{r, s\},\ r \vee s = \max\{r, s\},\ r \to s = \min\{1, 1 + s - r\},$$
$$r \oplus s = \min\{r + s, 2 - r - s\},\ r \leftrightarrow s = \min\{1 + r - s, 1 + s - r\}.$$

For which truth-values $q \in Q$ is there a formula α_q which always evaluates to q, without using any member of Q as a truth-constant? If the formula α_q exists, provide this formula; if the formula α_q does not exist, provide a proof that it does not exist.

Solution. Let A be an atom, which takes any of the values in Q.

1. 1 is equal to $A \leftrightarrow A$. The equivalence always evaluates to 1 if the expressions on both sides evaluate exactly the same member of Q.

2. 1/2: There is no formula which always evaluates to 1/2. Note that when all atoms are either 0 or 1, each fuzzy logic formula evaluates to 0 or 1: this is obviously true for atoms. If it is true for the formulas α, β, then it is also true for $\alpha \wedge \beta$, $\alpha \vee \beta$, $\alpha \oplus \beta$, $\alpha \to \beta$, $\alpha \leftrightarrow \beta$ and $\neg\alpha$, as the fuzzy connectives are chosen such that, in the case of inputs from $\{0, 1\}$, they also return the same value as in the classic case which is from $\{0, 1\}$. Thus it follows from structural induction that all fuzzy logic formulas evaluate to either 0 or 1 in the case that all inputs are from 0 and 1 and that truth-constants cannot be used.

3. 0: $\neg(A \leftrightarrow A)$ returns the value 0, as $A \leftrightarrow A$ returns the value 1.

Consider First-Order Logic over the structure $(\mathbb{Z}, +, -, \cdot, =, \neq, \leq, 0, 1, 2, \ldots)$ where all natural numbers (written in decimal) are allowed as constants and where one can compare numbers with predicates $=, \neq, \leq$ and where the meaning of all these symbols is the usual one. Furthermore, define $F(x)$ to be the number of $y \geq 1$ such that there exists a $z \in \mathbb{Z}$ with $x = y \cdot z$. In other words, $F(x)$ is the number of positive factors of x. For example, $F(10) = 4$ and $F(16) = 5$. Now let A to be the set of all $x \geq 0$ such that $F(x)$ is finite and even and B to be the set of all $x \geq 0$ such that $F(x)$ is finite and odd.

Give first-order formulas ϕ, ψ using the above structure such that $\phi(x)$ is true iff $x \in A$ and $\psi(x)$ is true iff $x \in B$. These first-order formulas can use $+, -, \cdot$ and any decimal number as constant; however, they cannot use F, as F is not part of the structure.

Solution. If one looks at the examples, $10 = 1 \cdot 10 = 2 \cdot 5 = 5 \cdot 2 = 10 \cdot 1$, in the case of an even number of factors, they come in pairs of distinct factors. On the other hand, $16 = 1 \cdot 16 = 2 \cdot 8 = 4 \cdot 4 = 8 \cdot 2 = 16 \cdot 1$, there is an odd number of factors and thus in the middle pair, both factors are the same, that is, the number is a square number. Indeed, $F(x)$ is odd iff the number is a square number. This property is then exploited in the formula. Furthermore, the case $x = 0$ is a special case, as for all natural numbers y, $0 = y \cdot 0$. Thus 0 has infinitely many factors and therefore $F(0) = \aleph_0$. Therefore one must require that x is at least 1. So one gets the following formulas:

ϕ: $x \in A \Leftrightarrow x \geq 1 \wedge \forall y\, [y \cdot y \neq x]$;
ψ: $x \in B \Leftrightarrow x \geq 1 \wedge \exists y\, [y \cdot y = x]$.

END OF PAPER